Extremal (super-)conformal field theories

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[based on work with S. Gukov, C. Keller, G. Moore and H. Ooguri]

Pure gravity in AdS_3

Recently Witten has proposed that pure gravity in AdS_3 has a dual description in terms of a holomorphically factorised extremal self-dual 2d conformal field theory, where the value of the cosmological constant is related to the central charge as

$$c = \frac{3l}{2G} = 24k \ , \qquad \Lambda = -\frac{2}{l^2} \ .$$

[Here k is an integer; thus the allowed values for the cosmological constant are quantised.]

Holomorphic factorisation

Holomorphic factorisation means that the full space of states of the conformal field theory is of the form

$$\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$$

left-movers

right-movers

[In the following concentrate on the left-movers, say.]

Self-dual

Modular invariance requires then that the left-moving character is modular invariant (up to a phase). Because of Verlinde's formula the fusion rules are then trivial, and there is only one representation, namely the vacuum representation.

Such theories are often called self-dual, since the simplest examples are the lattice theories based on even self-dual lattices.

Chiral gravity

Related proposal: instead of assuming holomorphic factorisation, add gravitational Chern-Simons term to the action so that $c_R = 0$. [Li, Song, Strominger]

Naively, $c_R = 0$ means that the right-moving theory is trivial, and hence the resulting theory is a purely left-moving (chiral) theory.

Because of the above argument this chiral theory is then also self-dual.

Extremal CFT

3d gravtiy is perturbatively trivial since there are no gravitational waves. Thus the dual conformal field should also be essentially trivial, i.e. it should essentially just contain Virasoro descendants: extremal CFT.

However, 3d gravity has BTZ black holes; these appear in the dual conformal field theory as states at conformal dimension

$$h > k = \frac{c}{24} \ .$$

Partition function

Thus the chiral partition function of the extremal conformal field theory should be of the form

$$Z = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n} + \mathcal{O}(q) \ .$$

Virasoro descendants

BTZ black holes

j-function

On the other hand, modular invariance implies that we can always express the partition function as

$$Z(\tau) = j(\tau)^k + \sum_{l=1}^{k} a_l j(\tau)^{k-l} \ ,$$

where $j(\tau)$ is the modular invariant function

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots \ .$$

Modular invariance

Thus the negative powers of q fix the partition function uniquely, and hence there is a definite prediction for the partition function of an extremal self-dual conformal field theory.

[Hoehn, Witten]

For example, for k=1, the extremal partition function is

$$j(\tau) - 744 = q^{-1}\left(1 + 196884\, q^2 + 21493760\, q^3 + 864299970\, q^4 + \cdots\right) .$$

This is the partition function of the famous Monster theory.

Higher values of k

Similarly, for k=2, the extremal partition function is

$$Z_{k=2}(\tau) = q^{-2}\left(1 + q^2 + 429875200\, q^3 + 40491909396\, q^4 + \cdots\right) .$$

However, already for k=2 (and indeed for any larger k), it is not known whether there is in fact a consistent conformal field theory giving rise to this character.

[For k=2 and k=3 there are indications based on a genus g=2 analysis that the theories are in fact consistent.]

[Witten, Gaiotto, Yin]

Amusing analogy

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In the following I want to explain why one may doubt that such theories exist. Before I get into this, there is an amusing analogy...

The problem is similar to that of the existence of extremal lattices: an Euclidean even self-dual lattice of dimension d=24k is extremal if all non-trivial vectors x satisfy

$$x^2 \geq 2(k+1) \ .$$

For k=1 the only extremal lattice is the Leech lattice, that is closely related to the Monster CFT.

No-go theorem

For extremal lattices there is actually a no-go-theorem: they can only exist for

$$k \leq 1708 \ .$$

[Mallows & Sloane]

[For larger k the second coefficient of the putative theta series becomes negative.]

Constraints

In order to get insight into the problem of extremal CFTs need to analyse constraints that go beyond modular invariance --- obviously a CFT has to satisfy many more consistency conditions!

On the other hand, we only know the partition function of the putative theory. So a natural idea is to study the modular differential equation that is satisfied by the partition function.

Modular differential equation

Every rational conformal field theory possesses a modular differential equation that annihilates all characters of the theory. [Mathur,Mukhi,Sen]

The origin of this differential equation can be understood as follows. Rationality implies that the vacuum Verma module has a null vector, e.g.

$$\mathcal{N} = \left(L_{-4} - \frac{5}{3}L_{-2}^2\right)\Omega \qquad (c = -22/5)$$

for the Yang-Lee model.

Character relation

It then follows that for each representation

$$\mathrm{Tr}_{\mathcal{H}_j}\left(V_0(\mathcal{N})\, q^{L_0-\frac{c}{24}}\right) = 0 \ .$$

Using standard conformal field theory techniques
this relation can be turned into a differential equation in q

$$\left[\left(q\frac{d}{dq} - \frac{1}{6}E_2(q)\right) q\frac{d}{dq} - \frac{11}{3600}E_4(q)\right] \chi_j(q) = 0 \ .$$

This is the modular differential equation!

Zhu's analysis

More generally, if the theory is C2 finite, one can always find a relation in the vacuum representation of the form

[Zhu]

$$L^{s}_{[-2]}\Omega + \sum_{j=0}^{s} g_j(q) L^{s-j}_{[-2]}\Omega \in O_q(\mathcal{H}) \ .$$

can be converted into differential equation

vanishes in torus amplitude

Modular equation

In fact, the resulting modular differential equation is always modular covariant, i.e. of the form

$$\left[D^s + \sum_{r=0}^{s-2} f_{s-r}(q)\, D^r \right] \chi_j(q) = 0 \ ,$$

where f_m is a modular form of weight 2m, and

$$D^r = cod_{(2r-2)} \cdots cod_{(2)}\, cod_{(0)} \ ,$$

with

$$cod_{(s)} = q\frac{d}{dq} - \frac{s}{12} E_2(q) \ .$$

Converse direction

Conversely, we have proven (assuming C2-finiteness and semisimplicity of Zhu's algebra), that a modular differential equation implies that in the vacuum representation there is a relation of the form [MRG, Keller]

$$(L_{[-2]})^s\Omega + \sum_{i=0}^{s-1} g_i(q)(L_{[-2]})^i\Omega = \sum_l f_l(q) d^l_{[-h(d^l)+1]}e^l$$
$$+ \sum_j h_j(q) \underbrace{\left(a^j_{[-h(a^j)-1]}b^j + \sum_{k\geq 2}(2k-1)G_{2k}(q)\, a^j_{[2k-h(a^j)-1]}b^j\right)}_{O_q(\mathcal{H})}$$

Strategy of argument

Step 1: modular differential equation comes from

$$(L_{[-2]})^s\Omega + \sum_{i=0}^{s-1} g_i(q)(L_{[-2]})^i\Omega = \sum_l f_l(q) d^l_{[-h(d^l)+1]} e^l$$
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Here $g_i(q)$ are modular forms of appropriate weight.

Strategy of argument

Step 2: Consider leading term of this vector as $q \rightarrow 0$.

This must vanish inside trace over highest weight states.

Action of modes on highest weight states is described by Zhu's algebra

$$A(\mathcal{H}_0) = \mathcal{H}_0 / O_{[1,1]}(\mathcal{H}_0)$$

Strategy of argument

Step 3: Semisimplicity of Zhu's algebra implies that leading term must therefore be commutator in Zhu's algebra. This implies that it equals $q \longrightarrow 0$ limit of

$$(L_{[-2]})^s\Omega + \sum_{i=0}^{s-1} g_i(q)(L_{[-2]})^i\Omega = \sum_l f_l(q) d^l_{[-h(d^l)+1]} e^l$$

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Strategy of argument

Step 4: Iterate argument order by order in q.

C2-finiteness guarantees that only finitely many terms appear; then one can estimate growth property and show that $f_l(q)$ and $h_j(q)$ are analytic in q (at least for q small).

$$(L_{[-2]})^s\Omega + \sum_{i=0}^{s-1} g_i(q)(L_{[-2]})^i\Omega = \sum_l f_l(q) d^l_{[-h(d^l)+1]} e^l$$
$$+ \sum_j h_j(q) \underbrace{\left(a^j_{[-h(a^j)-1]} b^j + \sum_{k\geq 2} (2k-1) G_{2k}(q)\, a^j_{[2k-h(a^j)-1]} b^j \right)}_{O_q(\mathcal{H})}$$

Converse direction

Thus modular differential equation implies that in the vacuum representation there is a relation of the form

[MRG, Keller]

$$(L_{[-2]})^s\Omega + \sum_{i=0}^{s-1} g_i(q)(L_{[-2]})^i\Omega = \sum_l f_l(q) d^l_{[-h(d^l)+1]}e^l$$
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Here $f_l(q)$ and $h_j(q)$ are analytic functions of q (at least for small q), while $g_i(q)$ are modular forms.

Null vector

In particular, this implies that the vacuum representation has a non-trivial null vector, and given the analytic structure, it is natural to believe that it arises at conformal dimension $h=2s$. (This is where our argument is not water-tight... However, no counterexample is known to this claim.) [cf Gaiotto]

If this is true, then we get a non-trivial constraint on the extremal self-dual theories.

Counting

It is not difficult to see (by counting modular forms) that any self-dual theory at c=24 k satisfies (generically) a modular differential equation of order

$$s = \sqrt{12k} + \mathcal{O}(1) \ .$$

[Need self-duality here: only one character to consider!]

In turn, this suggests that the vacuum representation has a null-vector at conformal dimension

$$h \simeq 2\sqrt{12k} < k \qquad \text{if } k \geq 42.$$

Contradiction

But the vacuum representation of the extremal theory consists just of Virasoro descendants up to level k, and hence does not have any null-vectors at such low conformal weights.

This suggests that the extremal partition functions cannot come from consistent conformal field theories, except for some small values of c $(k \leq 42)$. [MRG]

[More generally, any self-dual theory should have a Virasoro-primary at conformal weight $h = 2\sqrt{12k}$.]

N=2 susy version

Incidentally, this result also ties in with what one can prove for the N=2 analogue of this question.

The N=2 theory has more structure, and thus we can consider not just the partition function, but also the elliptic genus:

$$Z(q,y) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0-\frac{c}{24}}\, y^{J_0}(-1)^F\right) \ .$$

This allows us to keep track of the U(1) charge as well.

BTZ black holes

The analogue of the bosonic quantisation condition is to consider theories with

$$c = \frac{3l}{2G} = 6m \ , m \in \mathbb{Z} \ .$$

As before, they are perturbatively trivial, but there exist BTZ black holes whose mass and charge (translated into conformal field theory quantities) satisfy

$$4mL_0 - J_0^2 \geq 0 \ .$$

[Cvetic & Larsen]

Extremal ansatz

The natural analogue for the elliptic genus of an extremal superconformal field theories is then:

$$Z(q,y) = q^{-\frac{m}{4}}\,(1-q)\prod_{n=1}^{\infty}\frac{(1-yq^{n+1/2})\,(1-y^{-1}q^{n+1/2})}{(1-q^n)^2} + \text{non-polar}\ .$$

N=2 superconformal descendants of vacuum

states with $4mL_0 - J_0^2 \geq 0$.

[This is the formula in NS sector; by spectral flow can deduce formula for R sector.]

Weak Jacobi forms

The elliptic genus also has got good modular properties. In fact, it defines what is called a weak Jacobi form of weight w=0:

$$\phi\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big) = (c\tau+d)^w e^{2\pi i m\frac{cz^2}{c\tau+d}}\phi(\tau,z) \ ,$$

where

$$q = e^{2\pi i\tau}\ , \quad y = e^{2\pi i z}\ , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \ .$$

Weak Jacobi form

It also satisfies the other properties of a weak Jacobi form:

[Kawai et. al.]

$$\phi(\tau, z + \ell\tau + \ell') = e^{-2\pi i m(\ell^2\tau + 2\ell z)}\phi(\tau, z) \quad \ell, \ell' \in \mathbb{Z} \ ,$$

(spectral flow)

and

$$\phi(\tau, z) = \sum_{n\geq 0, \ell\in\mathbb{Z}} c(n, \ell)\, q^n\, y^l \ .$$

[This is the version of the formula in the R sector; from now on we work in R sector.]

Polar terms

In analogy with the bosonic case one can show that a weak Jacobi form is uniquely determined by its `polar' coefficients, i.e. by those terms

$$q^n y^l \quad \text{with} \quad 4mn - l^2 < 0 \ .$$

The coefficients of these terms are determined from the super-Virasoro descendants of the vacuum --- thus the extremal ansatz specifies the elliptic genus uniquely!

Properties

The spectral flow invariance implies that the coefficients are uniquely determined by those satisfying

$$|l| \leq m \ .$$

For even weight (w=0) we have furthermore

$$c(n,l) = c(n,-l) \ ,$$

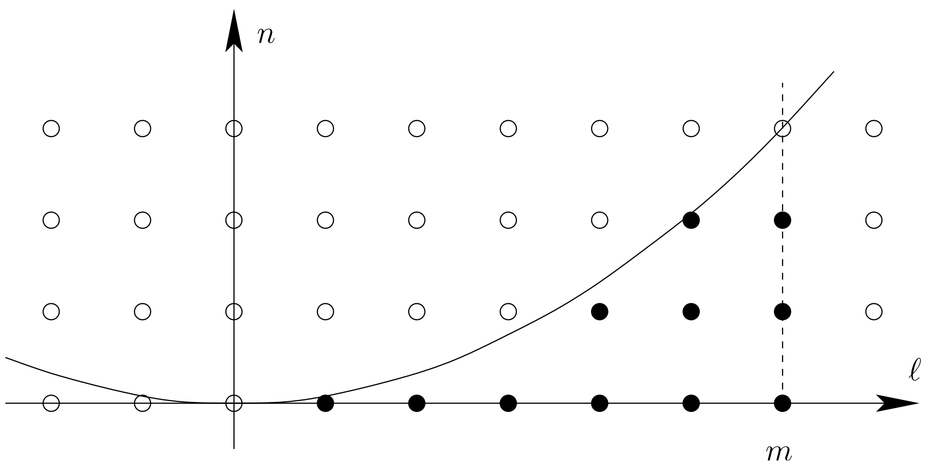
and hence we can restrict ourselves to

$$0 < l \leq m \ .$$

Polar region

Thus it follows that every weak Jacobi form is uniquely specified by the coefficients in the polar region

$$P^{(m)} = \left\{ (l,n) : 1 \le l \le m, 0 \le n, p = 4mn - l^2 < 0 \right\} \ .$$



Number of polar terms

It is not difficult to count the number of terms in the polar region, and one finds that

$$|P^{(m)}| = \frac{m^2}{12} + \frac{5m}{8} + \mathcal{O}(\sqrt{m}) \ .$$

Weak Jacobi forms

On the other hand, the space of weak Jacobi forms of even weight is freely generated by

$$\begin{aligned}
\phi_{0,1} &= y + 10 + y^{-1} + \mathcal{O}(q) \\
\phi_{-2,1} &= y - 2 + y^{-1} + \mathcal{O}(q) \\
E_4 &= 1 + 120q + \mathcal{O}(q^2) \\
E_6 &= 1 - 504q + \mathcal{O}(q^2)\ .
\end{aligned}$$

[Eichler, Zagier]

$[\phi_{w,m}$ has weight w and index m.$]$

Weight zero

At weight zero and index m the space of weak Jacobi forms is thus spanned by

$$(\phi_{-2,1})^{2a+3b}\,(\phi_{0,1})^{m-2a-3b}\,E_4^a\,E_6^b\ ,$$

where a and b are non-negative integers such that

$$2a+3b\leq m\ .$$

Counting weak Jacobi forms

Thus the number of weak Jacobi forms of index m is

$$\dim(\mathcal{J}_{0,m}) = \frac{m^2}{12} + \frac{m}{2} + \mathcal{O}(1) \ .$$

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But recall that the number of polar terms is

$$|P^{(m)}| = \frac{m^2}{12} + \frac{5m}{8} + \mathcal{O}(\sqrt{m}) \ .$$

Obstructions

Thus there are in general more polar terms than weak Jacobi forms.

Polar terms determine weak Jacobi form uniquely.

BUT not every collection of polar terms can be completed into weak Jacobi form.

[MRG, Gukov, Keller, Moore, Ooguri]

Extremal SCFTs

So what about the extremal SCFTs?

One finds by direct computation that the extremal elliptic genus can be completed into a weak Jacobi form for

m=1, 2, 3, 4, 5, 7, 8, 11, 13,

but not for any other m up to 400.

Extremal SCFTs

We have also found an analytic argument that shows that there can only be finitely many m for which the extremal ansatz can be completed into a weak Jacobi form, and we have strong arguments to suggest that above list is already complete.

[MRG, Gukov, Keller, Moore, Ooguri]

Sketch of argument

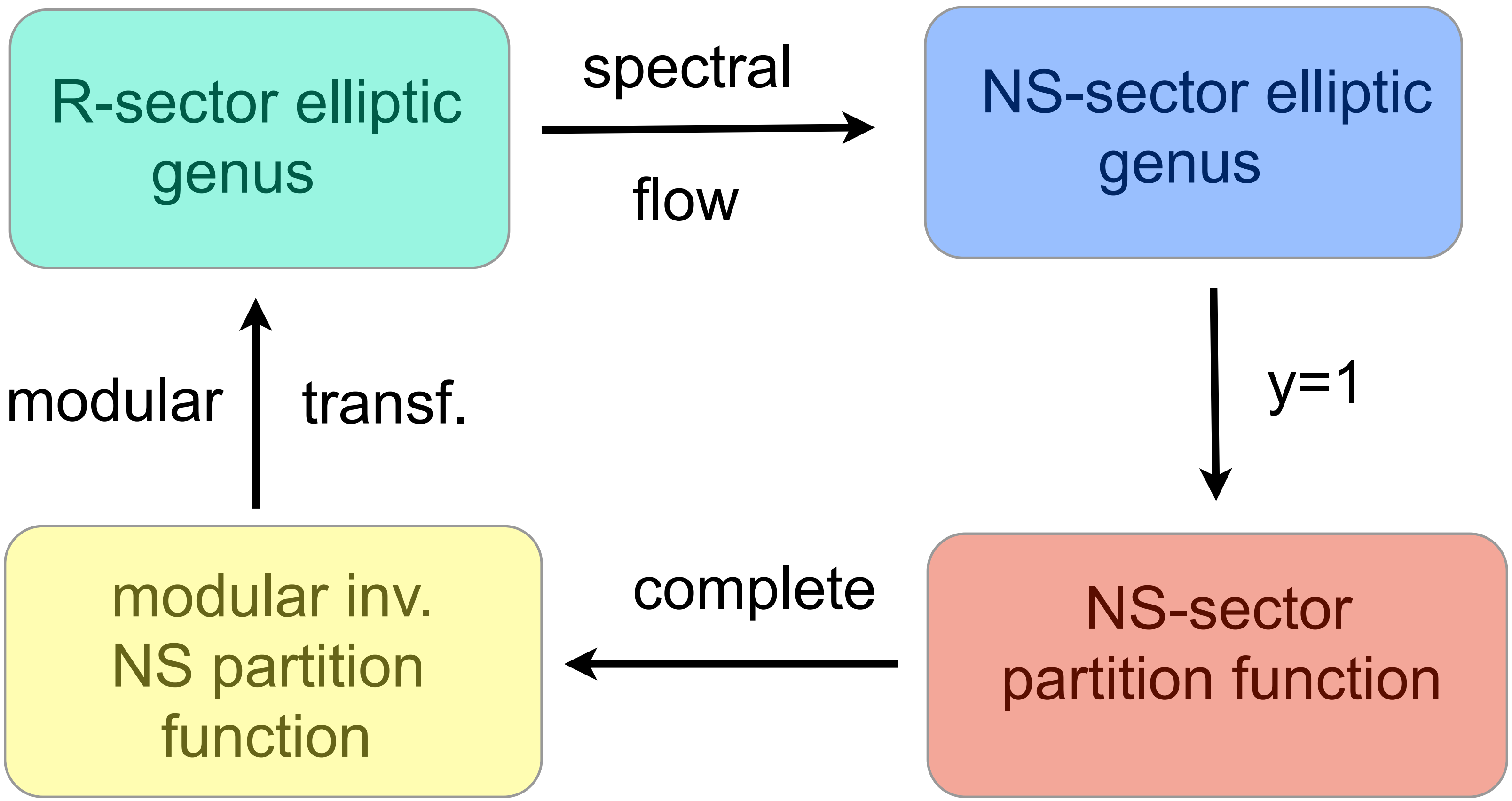
One can show that the non-polar coefficient c(0,0) is uniquely determined by the polar coefficients c(0,l).

[Gritsenko]

Using this formula one finds that for the extremal SCFT at c=6m we have

$$
\begin{aligned}
\#\ \mathrm{R\ ground\ states} &= \sum_l c(0,l)(-1)^l \\
&= 12m - 2\big(1-(-1)^m\big)\ .
\end{aligned}
$$

Sketch of argument



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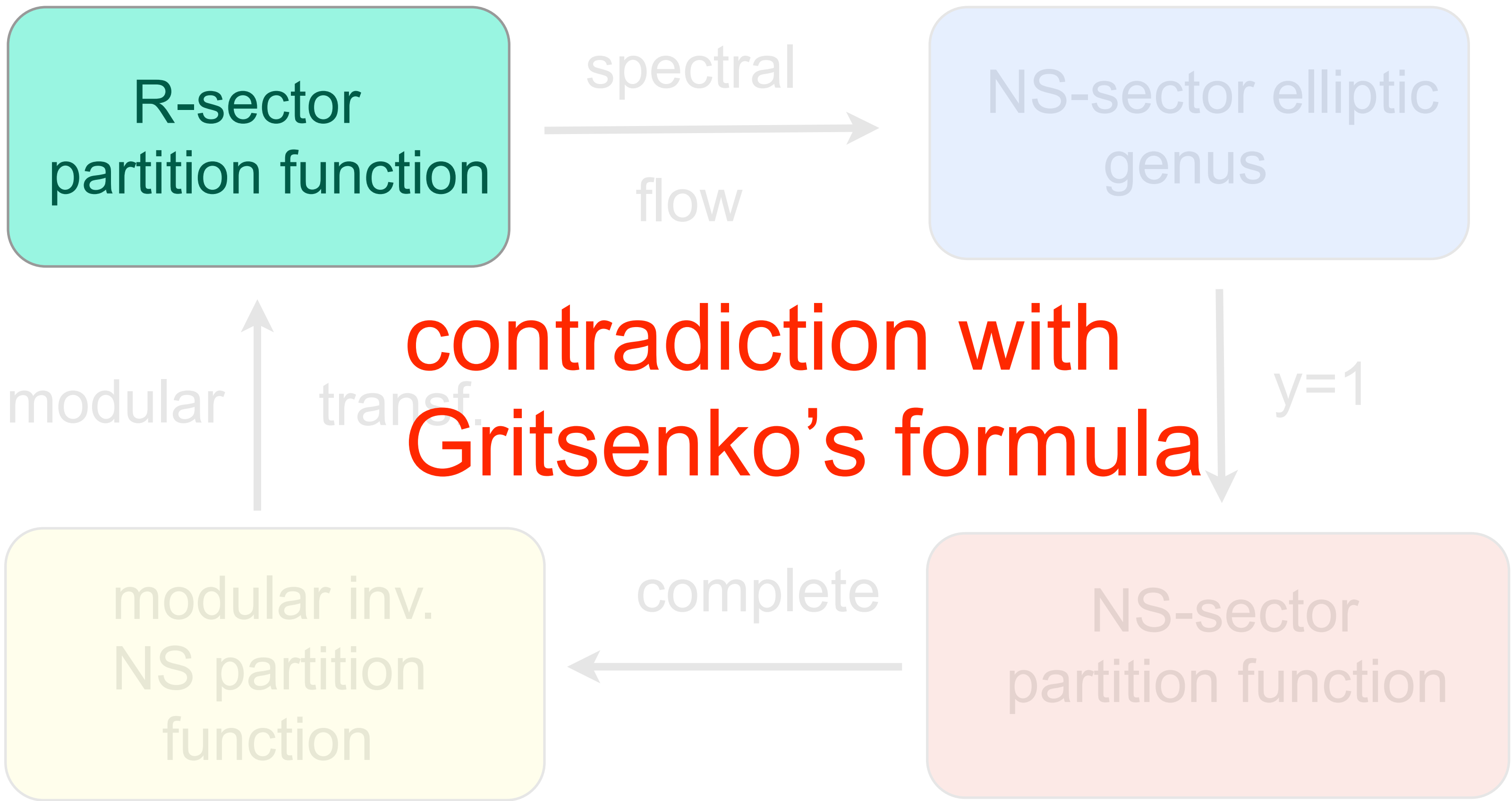

Near extremal N=2 SCFTs

The discrepancy between the number of polar terms and the number of weak Jacobi forms appears at subleading order in m --- on the basis of this argument we cannot therefore exclude the existence of near extremal N=2 SCFTs.

Near extremal: additional states appear only at

$$4mL_0 - J_0^2 \geq -\frac{m}{2} \ .$$

[On the other hand, did not assume holomorphic factorisation: elliptic genus always chiral.]

N=4

We have also studied the similar question for the N=4 analogue of this question (replace N=2 superconformal character by N=4 superconformal character in extremal ansatz).

The result appears to be essentially the same (although we have not yet performed the analytic analysis in detail).

Conclusions & Outlook

Chiral self-dual theories generically satisfy a modular differential equation at order $s \sim \sqrt{c/2}$.

A modular differential equation at order s generically leads to a non-trivial null-vector in the vacuum representation at conformal weight h=2s.

Conclusions & Outlook

Thus any chiral self-dual theory has typically a Virasoro-primary field at conformal weight

$$h \sim \sqrt{2c}\ .$$

This line of reasoning makes it unlikely that extremal CFTs exist for large central charge. This is also in agreement with the N=2 analysis.

Outlook

In order to study the consistency further we could also consider the 3d gravity theory on different quotients of AdS_3 --- this should give access to all genus g vacuum amplitudes of the dual CFT.

[Maloney,Witten,Giombi,Yin,..]

It is believed (Friedan & Shenker) that these determine the CFT uniquely.

[Recent partial result: the genus g amplitudes determine the current symmetries of a CFT uniquely, as well as its representation content.]

[MRG, Volpato]

Outlook

It would then be interesting to understand the consistency conditions these genus g amplitudes have to satisfy in order to define a consistent conformal field theory.

This could give us another handle for analysing the consistency of the dual CFT.

Log gravity

Finally, in the context of chiral gravity there is also another interesting possibility: log gravity.

The chiral gravity theory seems to have logarithmic solutions that suggest that the dual conformal field theory is logarithmic.

[Grumiller, Johansson, ..]

Log gravity

In fact there are many non-trivial c=0 logarithmic conformal field theories (e.g. for systems with quenched disorder, dilute self-avoiding polymers, percolation, ...).

Actually, many of these c=0 LCFTs have fermionic symmetry and have chiral `characters' equal to

[Gurarie, Ludwig, ..]

$$\mathrm{Tr}_{\mathcal{H}}\Big(q^{L_0}(-1)^F\Big) = 1\ .$$

Partition function

Thus from the point of the partition function it may not be possible to distinguish chiral gravity from log gravity.

[cf Maloney, Song, Strominger]

However, the structure of the two theories is very different, since the log theory will generically not be self-dual.

As a consequence the modular differential equation counting argument does not work any longer.