PARK
Jont Won

No.

Date

① Cylinders in rational surfaces / $\mathbb{C}$ + projective

S = surface

Def: $U \subseteq S$ is a cylinder if $U = \mathbb{A}^1 \times Z$
Z = affine curve

$\Rightarrow S$ is ruled $\to$ non-rational
$\to$ rational.

Affined ruled (Miyanishi)

Lemma: $\forall$ smooth rational surface, $\exists$ cylinder in it.

Proof: easy

Q: singular rational surfaces? Probably NOT always.

H = ample divisor

Def: $U \subseteq S$ is a H-polar cylinder if

1) U is a cylinder ($U = \mathbb{A}^1 \times Z$ for affine Z)

2) $S \setminus Z = \sum_{i=1}^{r} D_i$ and $H \sim_{\mathbb{Q}} \sum a_i D_i$ $\quad a_i \in \mathbb{Q}_{>0}$

Remark: importance of being positive.

Lemma: $\forall$ smooth rat. surface S $\exists H$ s.t.

S contains a H-polar cylinder.

Proof: relatively easy.

Q: For singular? ~~Probably~~ not true.... (Keel+MacKernan)
even with $\pi_0(S^{smooth}) = 0$

(2)

Two ways to proceed:

1. Fix S ⊕ describe H s.t. $\exists / \nexists$ H-polar...

   Ex: Question by Prokhorov: $S_3 \subseteq \mathbb{P}^3$.

2. Find class of surfaces with natural polariz.

   Ex: S = smooth / singular del Pezzo surface.

   $H = -K$.

Theorem (KPZ, CPW)

Let S be smooth del Pezzo surface.

Then S contains $(-K_S)$-polar cylinder

$\Updownarrow$

$K^2 \leq 3$.

Existence: explicit construction (Prokhorov)

Obstructions: Jihun told ~~y~~ 2 days ago

Theorem (CPW) $S = S_3$. Then $\exists$ H-polar cylinder

$\Updownarrow$

$H \neq \lambda[-K_S]$

Theor (CPW) S = del Pezzo with du Val sing.

Then $\exists$ $(-K_S)$ polar cylinder $\iff$
1) $K^2 \geq 4$
2) $K^2 = 3$ + singular
3) $K^2 = 2$ + A_{1_2} or wor
4) $K^2 = 1$ + not A_1, A_2 A_3 D_4

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Obstructions: $H = -K_S$

$$
\begin{tikzcd}
 & & \hat{S} \arrow[lld, bend right, "f"'] \arrow[ddl, bend left, "g"] \\
S \supset U = \mathbb{A}^1 \times Z \arrow[r] \arrow[rd, dashed] & Z \arrow[d, phantom, "\cong"] & \\
 & \mathbb{P}^1 &
\end{tikzcd}
$$

$\mathbb{P}^1$ - general fiber

$D = \sum a_i D_i$, $a_i \in \mathbb{Q}$

$S \setminus \sum D_i = \mathbb{A}^1 \times Z = U$

$-K$

$\hat{S} \setminus \underbrace{\sum D_i + \sum E_i}_{} = U = \mathbb{A}^1 \times Z$

$H^2(S, \mathbb{Q})$

exactly one of them is a sect of g, all others ~~are~~ lie in fibers e.g.

$F =$ fiber

$$\boxed{K_{\hat{S}} + \sum a_i \hat{D} + \sum b_i E_i \equiv f^*(K_S + D)}$$

$\swarrow$ $\exists\, a_k = 2$ (BIG)

$\searrow$ $\exists\, b_k = 2$ (BAD)

$\Rightarrow$ not log canonical

Remark: $\# D_i \geq \operatorname{rk} \operatorname{Pic} S$

Existence

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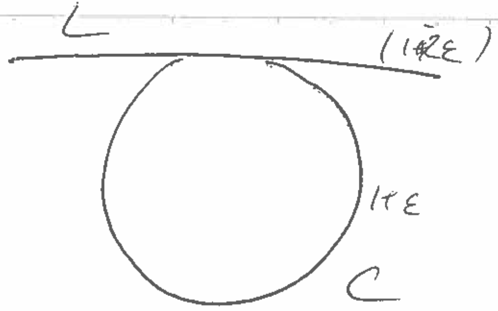
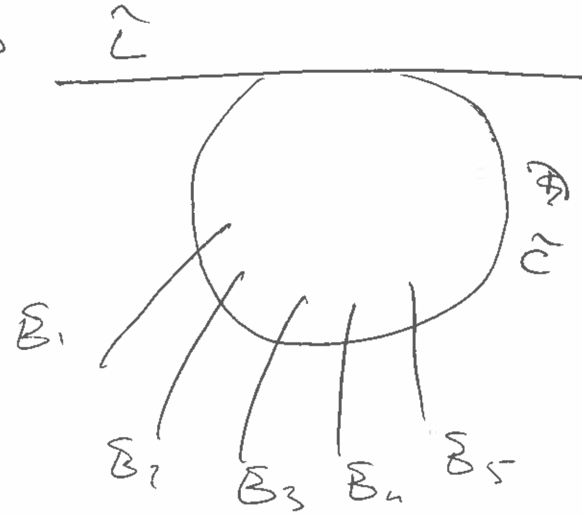


$K_S = -3L$

$D_{\mathbb{P}^2} = (1-2\varepsilon)L + (1+\varepsilon)C \equiv -K_S$

$K_S = 3L$



$f : \hat{S} \to S$

$$\begin{cases} \hat{L} = f^*(L) \\ \hat{C} = f^*(C) - \sum E_i \\ K_{\hat{S}} = f^*(K_S) + \sum E_i \end{cases}$$

$(1-2\varepsilon)\hat{L} + (1+\varepsilon)\hat{C} + \varepsilon \sum E_i \equiv -K_{\hat{S}}$

$$\mathbb{P}^2 \setminus (L \sqcup C) = \mathbb{A}^1 \times \mathbb{A}^* \implies \underbrace{\hat{S} \setminus (\hat{L} \sqcup \hat{C} \sqcup E_1 \ldots)}_{= \mathbb{A}^1 \times \mathbb{A}^*}$$

⑥ Singular del Pezzos.

Obstructions

Du Val → WHAT IS IT?

dP_1 $\quad \underbrace{A_1, A_2, A_3}$ D_4

dP_2 $\quad A_1$

dP_3 smooth

Construction

$dP_1 + A_4$

$dP_1 + A_5$

$dP_2 + A_2$

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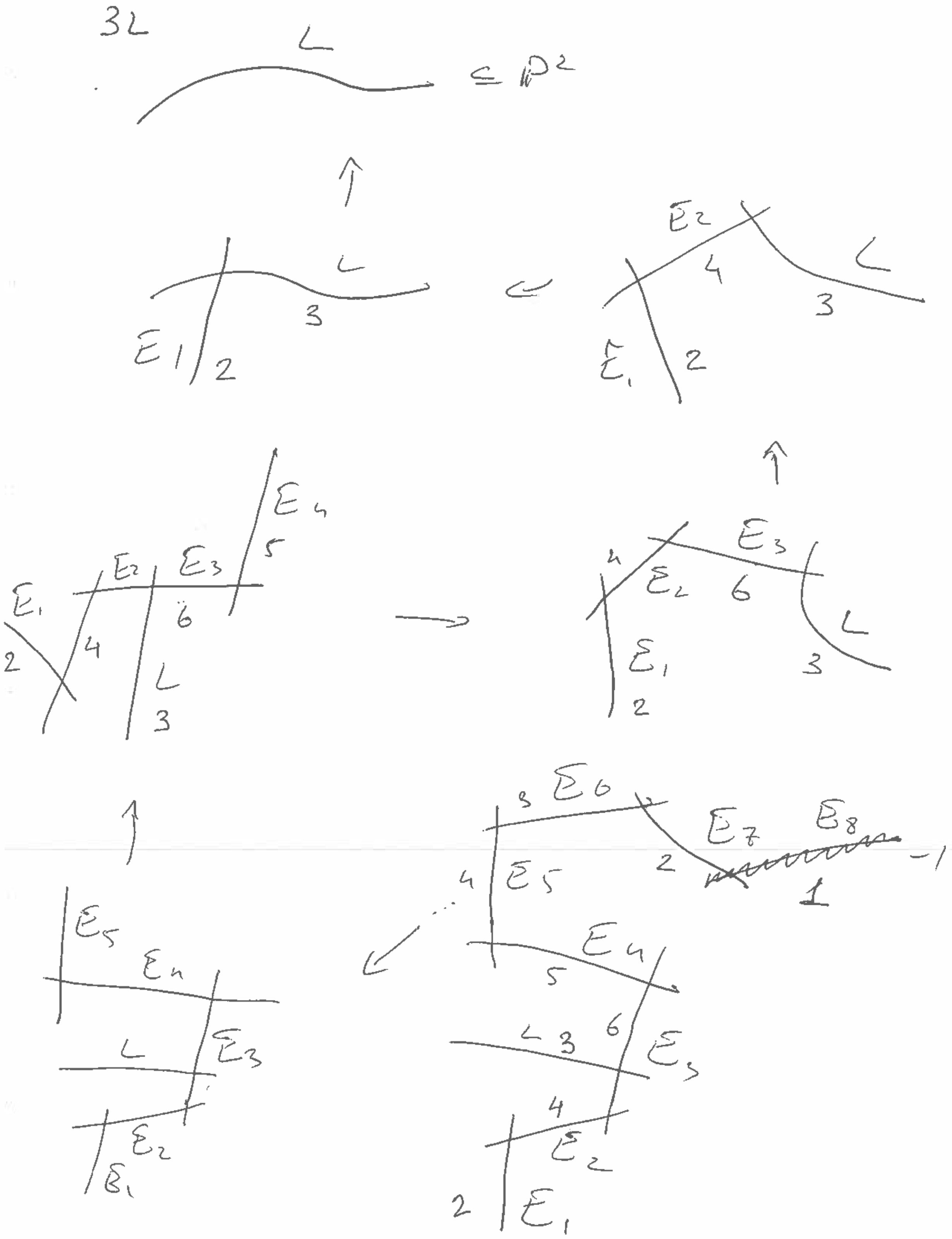
$dP_1 + E_8$


3L
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⊂ P²
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E1
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E1
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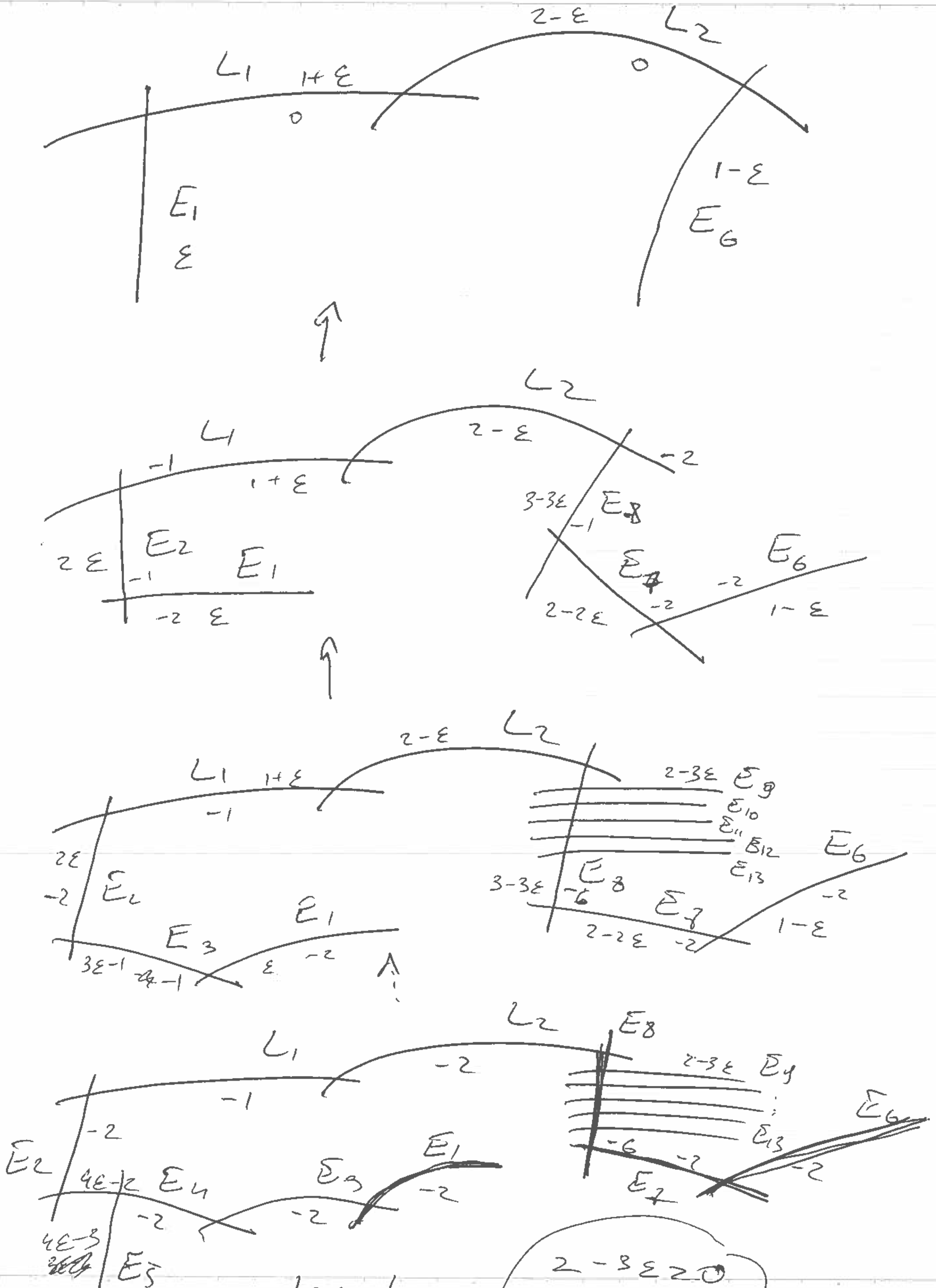
③

$dP_1 + A_4$



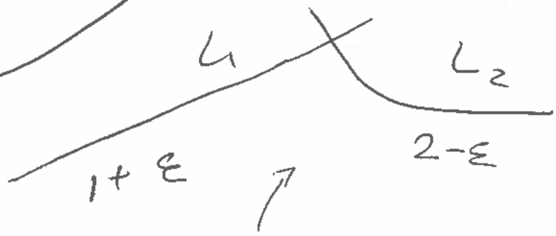
contract
L_1, L_2, E_2, E_4, E_5

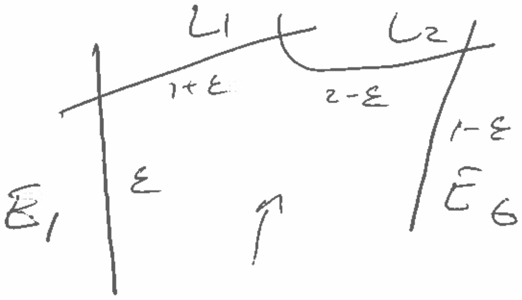
$2-3\varepsilon \geq 0$

$4\varepsilon - 3 > 0$

$A_5 + dP_1$

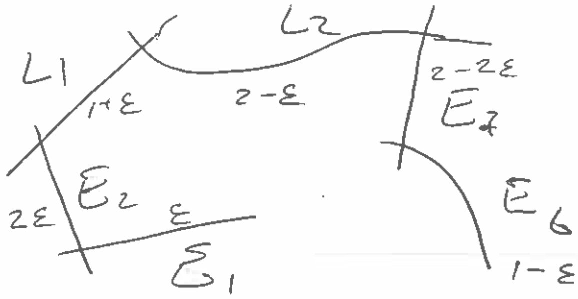
$\varepsilon = 2/5$

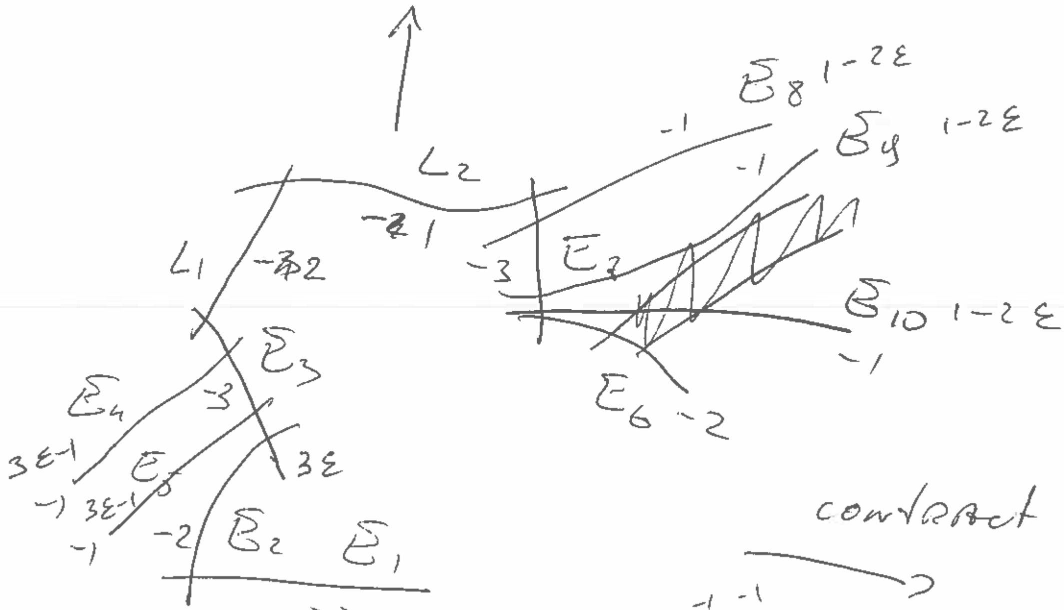




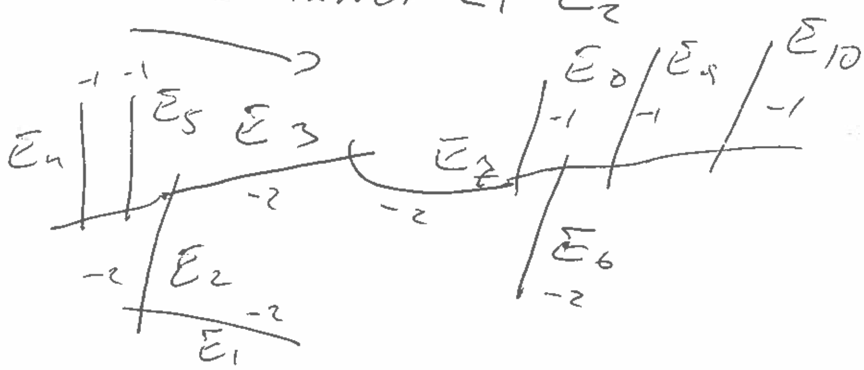
$1-2\varepsilon > 0$

$3\varepsilon - 1 > 0$



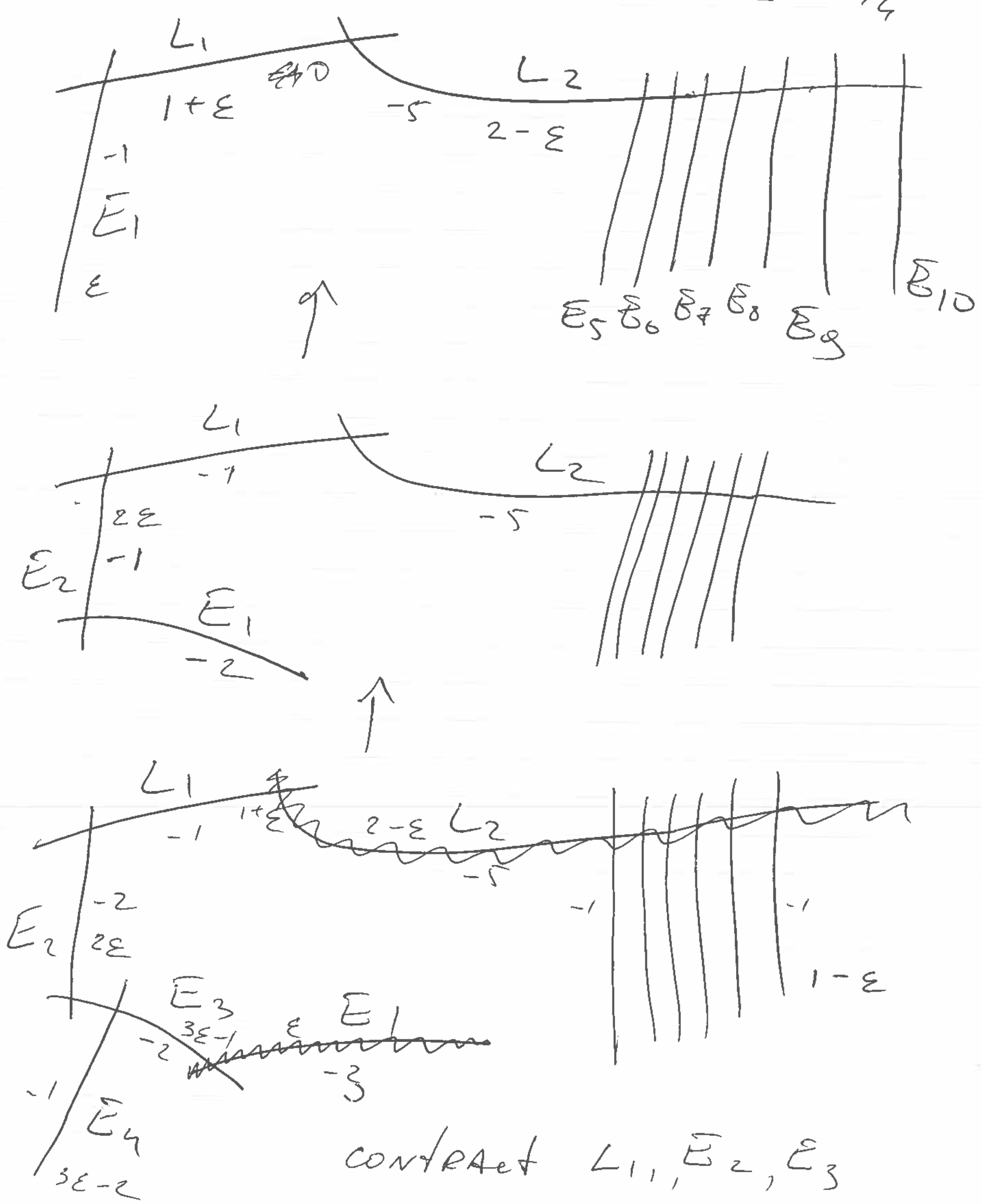


contract $L_1 + L_2$



$dP_2 + A_2$

$\varepsilon = 3/4$

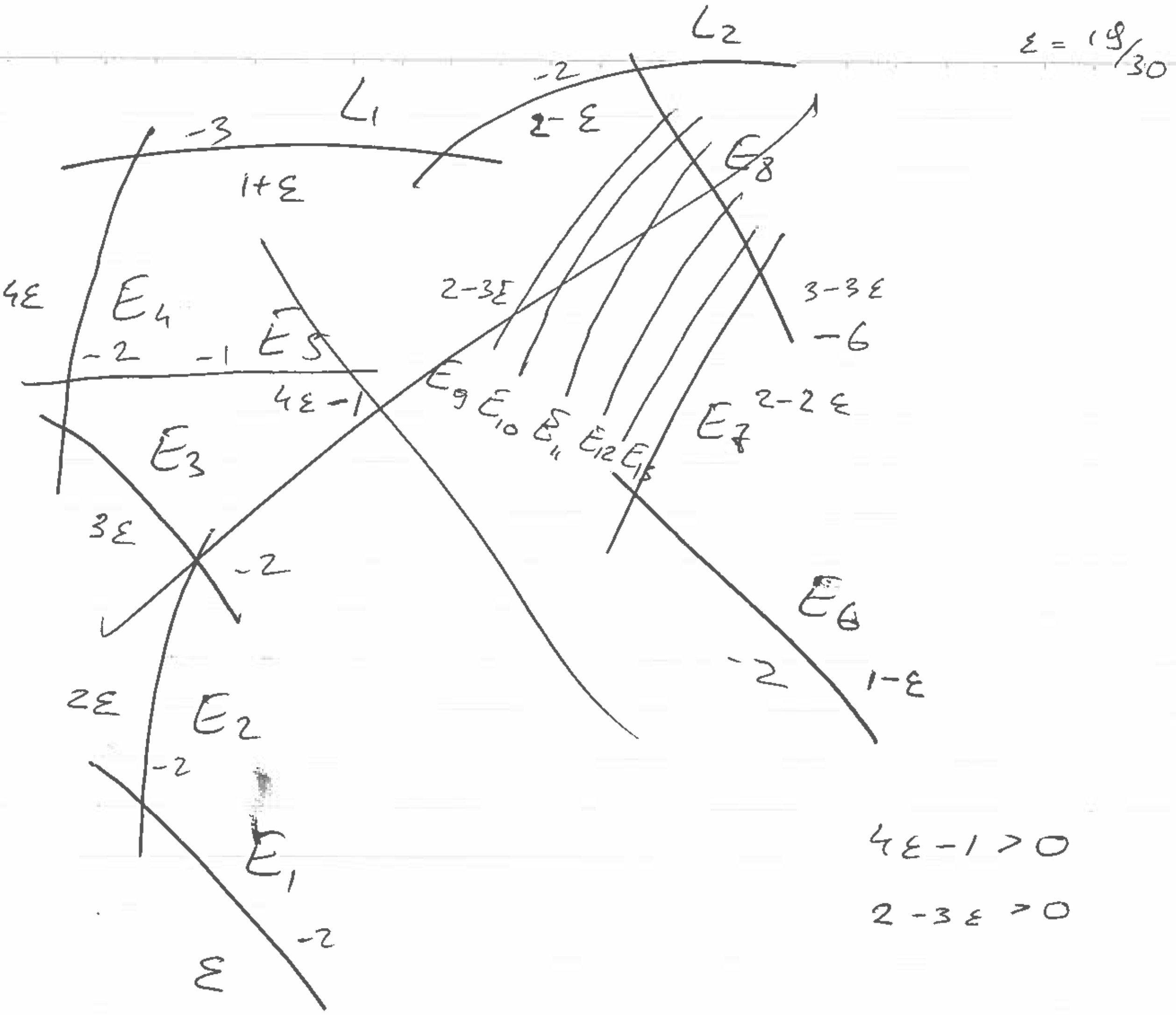


contract L_1, E_2, E_3

$3\varepsilon - 2 > 0$

$1 - \varepsilon > 0$

$dP_1 + A_4$

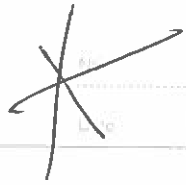


$4\varepsilon - 1 > 0$

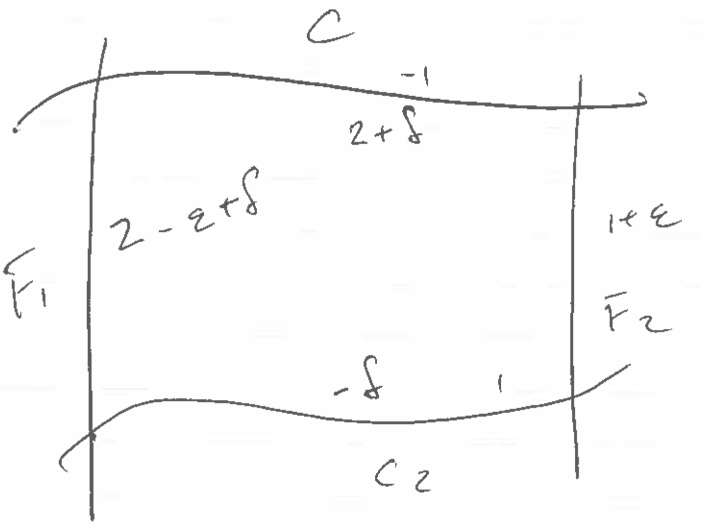
$2 - 3\varepsilon > 0$

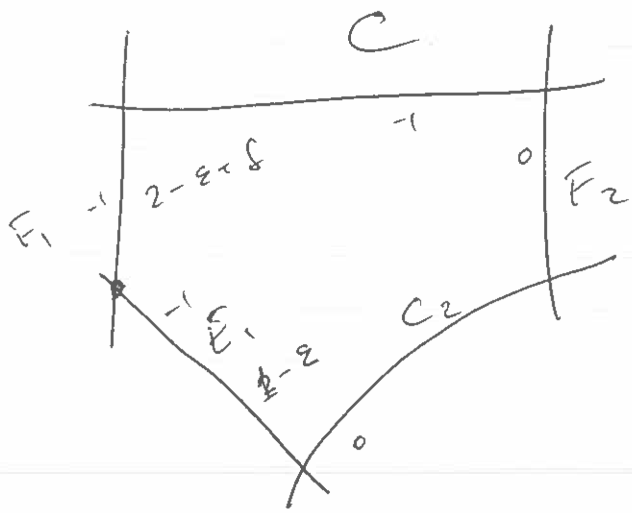
Conic bundle dP_3

(pure)

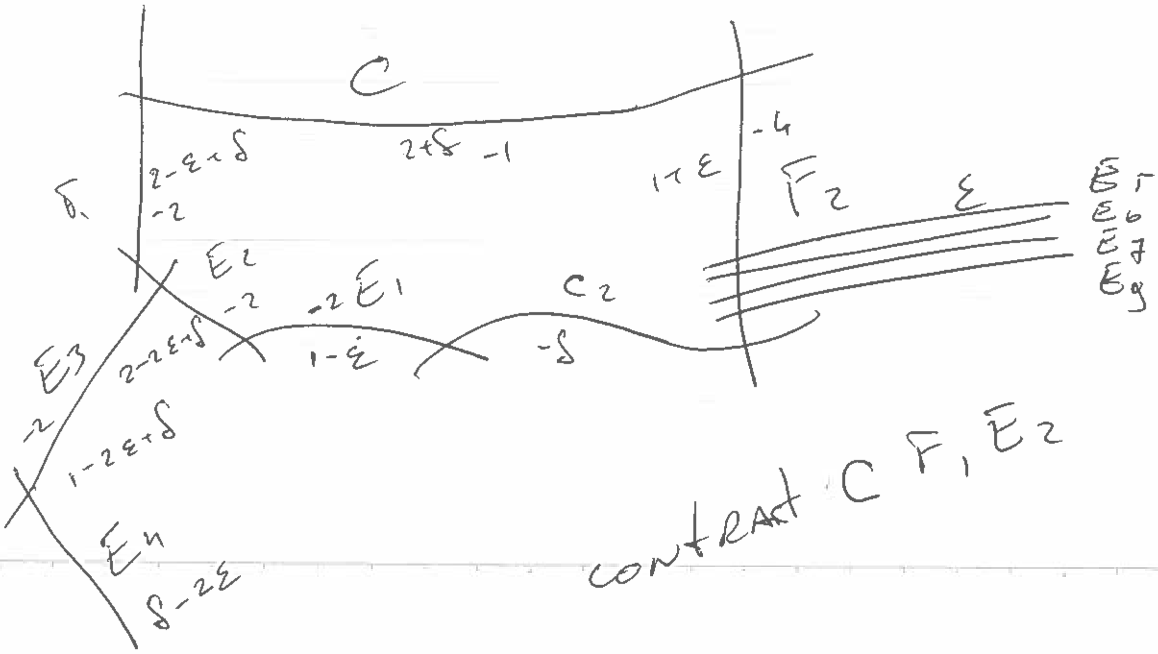


$\mathbb{F}_1$



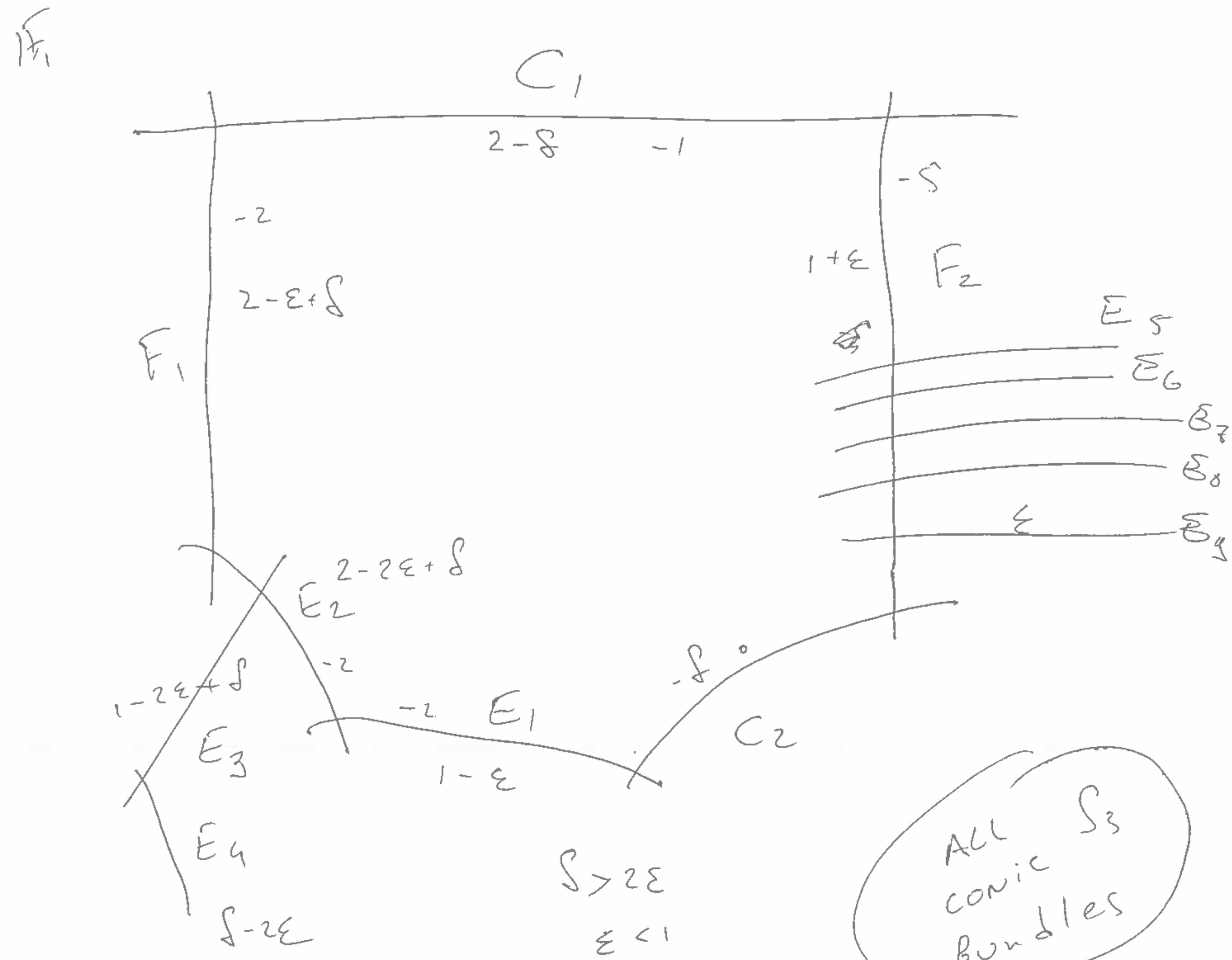


δC_2



contract $C\ F_1\ E_2$

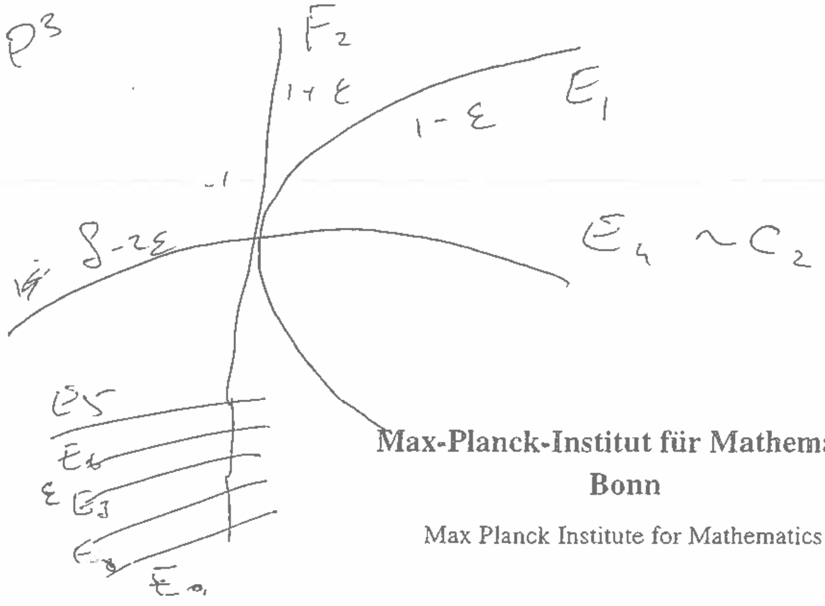
Conic bundle $S_3 \subseteq \mathbb{P}^3$

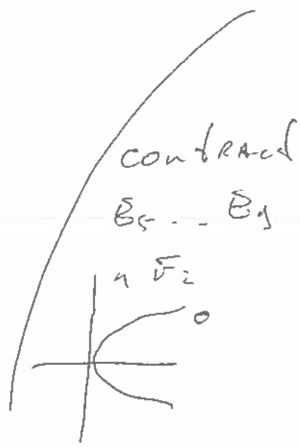


ALL S_3 conic bundles

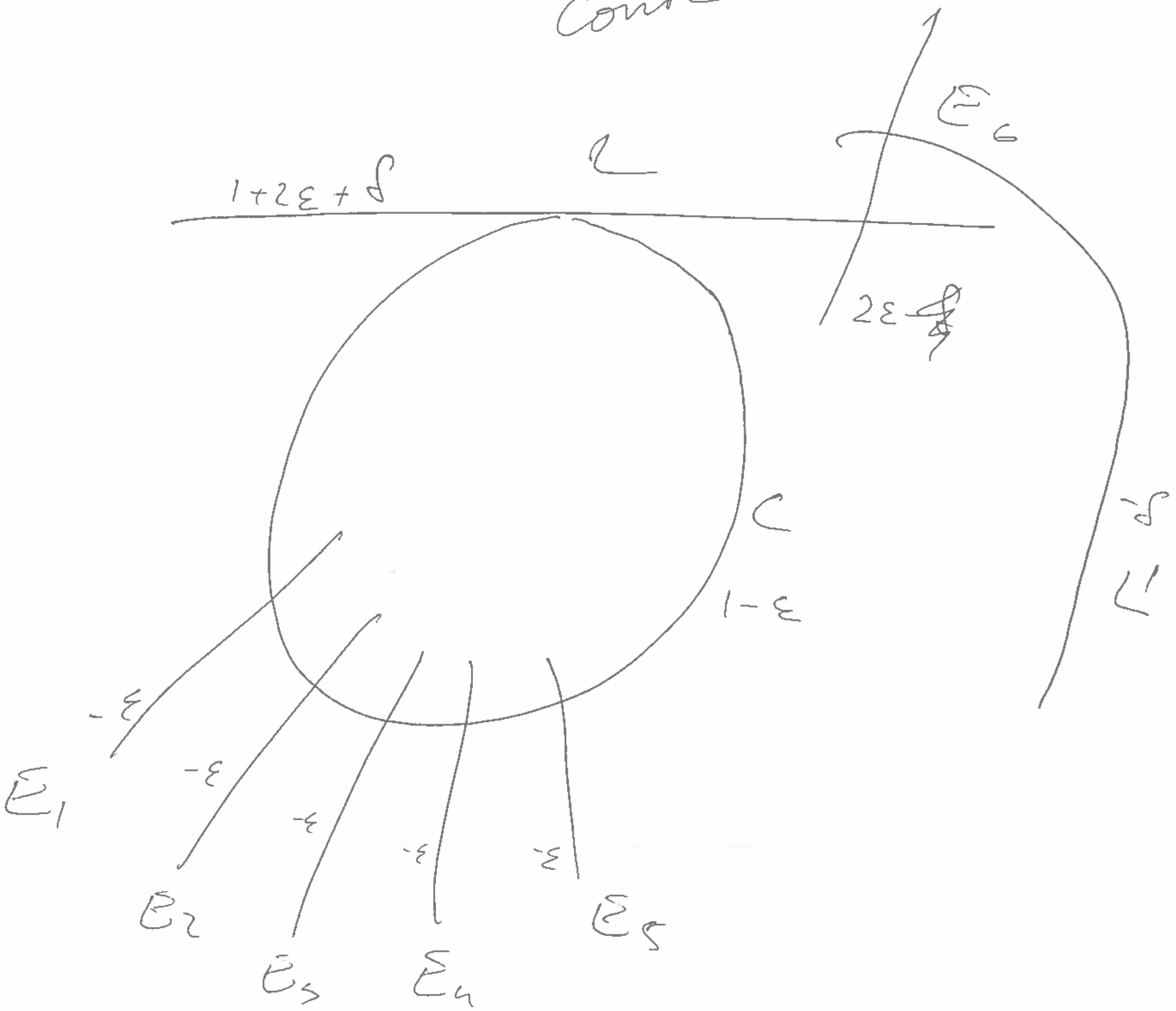
Contract C, F_1, E_2, E_3

$S_3 \subset \mathbb{P}^3$





$\mathbb{P}^2$ Remaining conic bundle



$$-K + \delta L' + \varepsilon_1 E_1 + \quad + \varepsilon_1 E_5 =$$

$$\equiv (1+2\varepsilon)L + (2\varepsilon \cancel{\delta}) E_6 + \sum_{i=1}^{5} (\varepsilon_i - \varepsilon) E_i$$

$$\forall \varepsilon_i > \varepsilon \qquad \forall \varepsilon_i < 1$$

$\cancel{2\varepsilon\delta}$