Holographic entanglement entropy beyond AdS/CFT

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Entanglement entropy

“Geometric” or entanglement entropy:

$$S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A), \qquad \rho_A = \mathrm{Tr}_B \rho \,.$$

CFT$_2$ in ground state on plane [Holzhey, Larsen, Wilczek; Cardy, Calabrese]:

$$S_A = \frac{c}{3} \log \frac{L_x}{\epsilon} \,.$$

CFT$_2$ in ground state on cylinder:

$$S_A = \frac{c}{3} \log \frac{\sin(L_\theta/2)}{\epsilon}$$

CFT$_2$ at finite left-moving and right-moving temperatures:

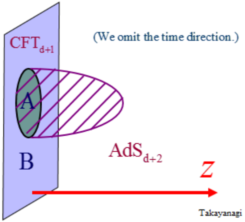
$$S_A = \frac{c}{6} \log \left(\frac{\beta_L \beta_R}{\pi^2 \epsilon^2} \sinh\left(\frac{\pi L_x}{\beta_L}\right) \sinh\left(\frac{\pi L_x}{\beta_R}\right) \right)$$

Holographic entanglement entropy

For time-independent state in AdS/CFT, Ryu-Takayanagi (RT) proposed

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$

for minimal surface γ_A on a given time slice.



Effectively proven by now [Casini, Huerta, Myers; Faulkner; Hartman; Lewkowycz, Maldacena]. Extended to quantum corrections [Barella, Dong, Hartnoll, Martin; Faulkner, Lewkowycz, Maldacena], higher spin theories [de Boer, Jottar; Ammon, Castro, Iqbal], higher curvature theories [Hung, Myers, Smolkin; Dong].

Time-dependent holographic entanglement entropy

Hubeny-Rangamani-Takayanagi (HRT) proposed generalization

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

for *extremal* surface γ_A not restricted to time slice. Formula unproven but satisfies nontrivial checks, e.g. strong subadditivity [Callan, He, Headrick; Wall], and reproduces CFT_2 formulae at finite T_L and T_R.

No extension of HRT proposal to non-AdS UV asymptotics.

Warped AdS$_3$

Spacelike WAdS$_3$ written as fibration over Lorentzian AdS$_2$ base space:

$$ds^2 = \frac{\ell^2}{4}\left(-(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + a^2(du + r\,d\tau)^2\right)$$

in global coordinates and

$$ds^2 = \frac{1}{4}\left(\ell^2\,\frac{-d\psi^2 + dx^2}{x^2} + a^2\left(d\phi + \ell\,\frac{d\psi}{x}\right)^2\right)$$

in Poincaré-like coordinates.

- All coordinates valued in $\mathbb{R}$; $a \in [0,2)$. $R = \frac{2(a^2-4)}{\ell^2}$.
- Solution of Einstein gravity plus matter; exists in string theory.
- Isometry group $SL(2,\mathbb{R}) \times U(1)$, unless $a = 1$.
- No conformal boundary (but there exists anisotropic conformal infinity [Horava, Melby-Thompson]).
- Discrete identification gives warped BTZ black hole.

Warped AdS$_3$

Compactify fiber coordinate ϕ to get 3D part of NHEK geometry:

$$ds^2 = 2J\Omega(\theta)^2 \left(\frac{-d\psi^2 + dx^2}{x^2} + d\theta^2 + a(\theta)^2 \left(d\phi + \frac{d\psi}{x} \right)^2 \right).$$

- Relevant for Kerr/CFT and understanding of astrophysical black holes.

Warped CFT$_2$

Consider theories defined as having $SL(2,\mathbb{R}) \times U(1)$ symmetry and proposed to be holographically dual to warped AdS$_3$.

Symmetry automatically enhanced to infinite-dimensional $Vir \times U(1)$ Kac-Moody [Hofman, Strominger]; this case referred to as warped CFT$_2$.

Cardy-like formula can be derived for density of states [Detournay, Hartman, Hofman].

Trivial warping: AdS$_3$ spacetime

Set $a = 1$ to get AdS$_3$ spacetime in fibered Poincaré-like coordinates:

$$ds^2 = \frac{1}{4}\left(-\ell^2\,\frac{d\psi^2}{x^2} + \ell^2\frac{dx^2}{x^2} + \left(d\phi + \ell\,\frac{d\psi}{x}\right)^2\right).$$

In fibered global coordinates we have

$$ds^2 = \frac{\ell^2}{4}\left(-(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + (du + r\,d\tau)^2\right).$$

HRT proposal can be applied to these spacetimes! We stick to $r = +\infty$. Coordinates on boundary are null.

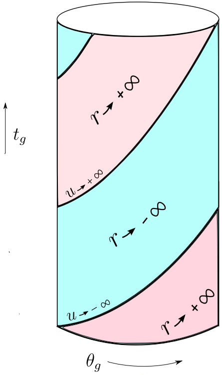


Figure : Adapted from 0905.2612.

HRT proposal for fibered Poincaré-like AdS$_3$ spacetime

Answer in terms of two null distances:

$$S_{EE} = \frac{c}{3} \log\left(\frac{1}{\epsilon}\sqrt{L_\psi \ell \sinh\left(\frac{L_\phi}{2\ell}\right)}\right) = \frac{c}{6}\log\frac{L_\psi}{\epsilon} + \frac{c}{6}\log\left(\frac{\ell}{\epsilon}\sinh\left(\frac{L_\phi}{2\ell}\right)\right)$$

ψ-movers in ground state and ϕ-movers at finite temperature ℓ.
Holographic renormalization shows

$$\langle T_{\psi\psi}\rangle = \langle T_{\psi\phi}\rangle = 0; \qquad \langle T_{\phi\phi}\rangle \neq 0\,.$$

Compactifying fiber coordinate gives near-horizon limit of extremal BTZ, which has $T_L = 0$ and $T_R \neq 0$.

HRT proposal for fibered global AdS$_3$ spacetime

$$S_{EE} = \frac{c}{3}\,\log\left(\frac{1}{\epsilon}\sqrt{\sin\left(\frac{L_\tau}{2}\right)\sinh\left(\frac{L_u}{2}\right)}\right)$$
$$= \frac{c}{6}\log\left(\frac{1}{\epsilon}\,\sin\left(\frac{L_\tau}{2}\right)\right) + \frac{c}{6}\log\left(\frac{1}{\epsilon}\sinh\left(\frac{L_u}{2}\right)\right)$$

τ-movers in ground state on cylinder and u-movers at finite temperature. Coordinates all dimensionless.

WAdS$_3$ Geodesics

Easy to solve for $u(\lambda)$, $\tau(\lambda)$, and $r(\lambda)$ in terms of conserved momenta c_τ, c_u and c_v. For AdS$_3$, $\lim_{\lambda\to\infty} u(\lambda) = k$ for constant k, but for warped AdS$_3$ solution for fiber coordinate has piece linear in λ. We find

$$\text{Length} \sim \lambda_\infty = \frac{\log\left[r_\infty f_1(c_u, c_\tau, a)\right]}{\sqrt{1+(1-1/a^2)c_u^2}}$$

and

$$c_\tau = f_2(c_u, a)\cot\left(\frac{L_\tau}{2}\right)\,,$$

$$2\left(-1+\frac{1}{a^2}\right)c_u\lambda_\infty + \log\left(\frac{c_u+\sqrt{1+c_u^2-\frac{c_u^2}{a^2}}}{c_u-\sqrt{1+c_u^2-\frac{c_u^2}{a^2}}}\right) = L_u\,.$$

Perturbation theory

Troubling equation is

$$2\left(-1+\frac{1}{a^2}\right)c_u\lambda_\infty + \log\left(\frac{c_u+\sqrt{1+c_u^2-\frac{c_u^2}{a^2}}}{c_u-\sqrt{1+c_u^2-\frac{c_u^2}{a^2}}}\right) = L_u\,,$$

transcendental in c_u. Solve perturbatively instead for $a = 1+\delta$:

$$c_u = c_{u,0} + \delta\, c_{u,1} + \delta^2 c_{u,2} + \cdots\,,$$

with

$$|\delta^n c_{u,n}| \ll |\delta^{n-1} c_{u,n-1}|$$

to assure convergence. Guaranteed if

$$L_u \gtrsim 1, \qquad |\lambda_\infty \delta| \ll 1\,.$$

Latter requirement interpreted as remaining in AdS$_3$ part of geometry; this is just AdS/CFT in presence of infinitesimal, irrelevant source! HRT proposal should apply.

Answer

Perturbative answer to all orders:

$$S_{\text{EE}} = \frac{\ell}{4G_N}\left[\left(1+\delta\coth^2\frac{L_u}{2}\right)\log\left(r_\infty \sin\frac{L_\tau}{2}\sinh\frac{L_u}{2}\right)\right]+$$

$$\frac{\ell}{4G_N}\sum_{i=2}^{\infty}\delta^i(-1)^{i+1}\coth^2\frac{L_u}{2}\operatorname{csch}^{2(i-1)}\frac{L_u}{2}\left[\log\left(r_\infty\sin\frac{L_\tau}{2}\sinh\frac{L_u}{2}\right)\right]^i \times\left(\sum_{j=0}^{i-2}c_{ij}\cosh(jL_u)\right).$$

Taking $L_u \gg 1$ and $\delta > -1/2$ lets us sum the entire series to get

$$S_{\text{EE}} = \frac{\ell}{2G_N}(1+\delta)\log\left(\frac{1}{\epsilon}\sqrt{\sin\frac{L_\tau}{2}\exp\left(\frac{L_u}{2}\right)}\right).$$

Reading off the central charge

Answer *to all orders in* δ in the large-L_u limit:

$$S_{\text{EE}} = \frac{\ell}{2G_N}(1+\delta)\log\left(\frac{1}{\epsilon}\sqrt{\sin\frac{L_\tau}{2}\exp\left(\frac{L_u}{2}\right)}\right).$$

We have recovered universal CFT$_2$ answer in large L_u limit, with

$$\boxed{c_L = c_R = \frac{3\ell}{2G_N}(1+\delta).}$$

Peforming same perturbative expansion in Poincaré-like coordinates again gives universal CFT$_2$ answer:

$$S_{EE} = \frac{\ell}{2G_N}(1+\delta)\log\left(\frac{1}{\epsilon}\sqrt{L_\psi \ell \exp\left(\frac{L_\phi}{2\ell}\right)}\right).$$

Warped BTZ black hole

Spacelike warped BTZ black holes are locally spacelike warped AdS$_3$ [Anninos, Li, Padi, Song, Strominger]:

$$\frac{ds^2}{\ell^2} = \frac{3dt^2}{4-a^2} + \frac{dr^2}{4(r-r_+)(r-r_-)} + \frac{6\sqrt{3}}{(4-a^2)^{3/2}}\left(ar - \sqrt{r_+ r_-}\right) dt d\theta$$
$$+ \frac{9r}{(4-a^2)^2}\left((a^2-1)r + r_+ + r_- - 2a\sqrt{r_+ r_-}\right) d\theta^2, \quad \theta \sim \theta + 2\pi$$

Perturbative answer in large fiber-coordinate regime given by

$$S_{\mathrm{EE}} = \frac{\ell a}{G_N} \log\left(\frac{r_+ - r_-}{\epsilon^2} \exp\left(\sqrt{\frac{3}{a^2(4-a^2)}}\Delta t + \frac{\pi \Delta\theta}{\beta_L}\right) \sinh\frac{\pi\Delta\theta}{\beta_R}\right),$$

with dimensionless temperatures

$$\beta_L^{-1} = T_L = \frac{3}{2\pi(4-a^2)}\left(r_+ + r_- - \frac{2}{a}\sqrt{r_+ r_-}\right),$$
$$\beta_R^{-1} = T_R = \frac{3(r_+ - r_-)}{2\pi(4-a^2)}.$$

Nonperturbative (in δ) proposal

Cardy formula is nontrivial check (even for finite δ):

$$S = \frac{\pi^2}{3}(c_L T_L + c_R T_R) = \left(\frac{3\pi\ell}{2G_N(4-a^2)} \left(a r_+ - \sqrt{r_+ r_-} \right) \right) = \frac{A}{4G_N}.$$

Physically relevant range is $a \in [0, 2)$. Our expansion converges for $a \in (1/2, 2)$, i.e. $\delta > -1/2$, so we propose it is valid in that range.

Open questions

- Full nonperturbative application of HEE proposal?
  - Nonlocality in the UV with volume law?
- Independent way to see WCFT$_2$ reproduce CFT$_2$ behavior; correlation functions?

$$\langle \phi_i(x^-, x^+)\phi_j(y^-, y^+)\rangle = \frac{f_{ij}(x^- - y^-)}{(x^+ - y^+)^{\lambda_i+\lambda_j}}$$

- Universal entanglement entropy formulae in WCFT$_2$, without holography [ES, in progress (sort of)].
- Extension to TMG [Castro, Detournay, Iqbal, Perlmutter, in progress].
- Extend perturbative approach to spacetimes continuously connected to AdS$_{d+2}$.
- Produce $c_L = 12J$ in NHEK.

Take-away

- Sometimes useful to compute holographic entanglement entropy perturbatively.
- Warped CFT$_2$ seems CFT$_2$-like at finite temperature, as long as we take an IR limit *and* large fiber coordinate separation.
- Central charge, T_L, and T_R predicted, with Cardy formula satisfied; first quantiatively successful application of (covariant) holographic entanglement entropy to non-asymptotically AdS spacetime!
- Many concrete directions for progress; outlook hopeful! 

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Compactification of common bosonic sector of IIA/B and heterotic SUGRAs (Einstein frame):

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(R_{10} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-\Phi}H_{MNP}H^{MNP}\right).$$

Compactify on $S^3 \times T^3 \times S^1$ and keep KK gauge field from S^1, with background

$$ds_{10}^2 = e^{-3Y/2}\left(e^{X}ds_3^2 + e^{-X}(d\phi + A)^2\right) + e^{Y}L_S^2 ds^2(S^3) + ds^2(T^3)$$
$$H = h_S L_S^3 Vol(S^3) + \hat{H} + \hat{F}\wedge(d\phi + A)$$

for $\hat{H} \equiv d\hat{B} - \hat{F}\wedge A$, L_S the radius of S^3, h_S a constant, and $ds^2(S^3)$ and $\text{Vol}(S^3)$ the metric and volume forms on S^3. We can thus reduce and consistently truncate the resulting 3D action to

$$S_{3D} = \frac{1}{2\kappa_3^2}\int d^3x\sqrt{-g_3}\left(R_3 - \frac{1}{8}e^{3Y-\Phi}F^2 + \mathcal{L}_{kin}(\Phi, Y) - 2h_3^2 e^{2\Phi-6Y}\right)$$
$$+\frac{1}{2\kappa_3^2}\int d^3x\sqrt{-g_3}\left(\frac{12}{L_S^2}e^{-4Y+\Phi/2} - h_S^2 e^{-6Y-\Phi/2}\right) - \frac{h_3}{4\kappa_3^2}\int A\wedge F$$