Modular Operad of Embedded Curves

joint w/ S. Kondo & C. Siegel
(coming for arXiv next week)

Outline

1) Stable curves
2) k-log-canonically embedded curves
3) Gluing embedded curves
4) Modular operads

§1 Stable Curves

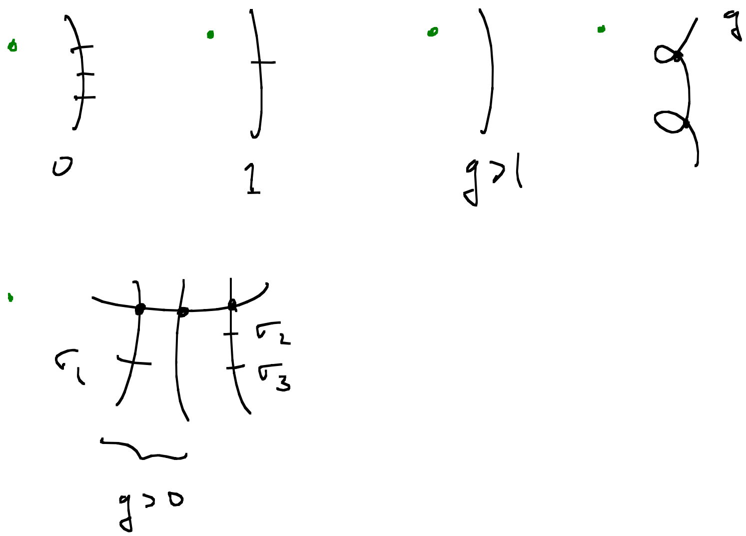
Def: A stable marked curve $(C, \{\sigma_i\}_i^n)$ over a field k is:

1) connected 1-dim'l projective variety C w/ at most nodal singularities
2) n distinct, smooth k-pts $\{\sigma_i\}_i^n$

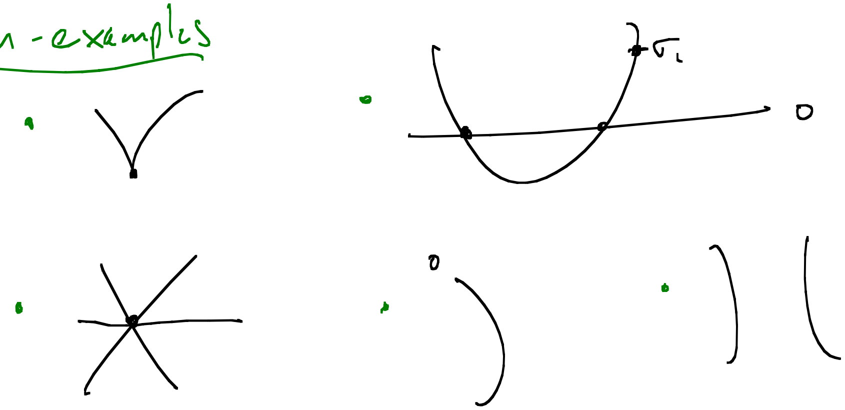
s.t. each irreducible component C_i has at least $3-2g_i$ special pts.
(marked points & preimages of nodes under normalization)

($\Leftrightarrow$ fin. many automorphisms)

Examples:



Non-examples



Def: A family of stable marked curves over S is

1) a projective morphism

$$\mathcal{C} \xrightarrow{\pi} S$$

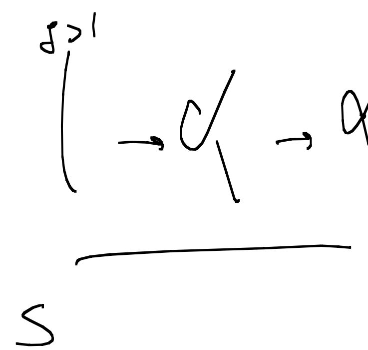
of rel. dim. 1

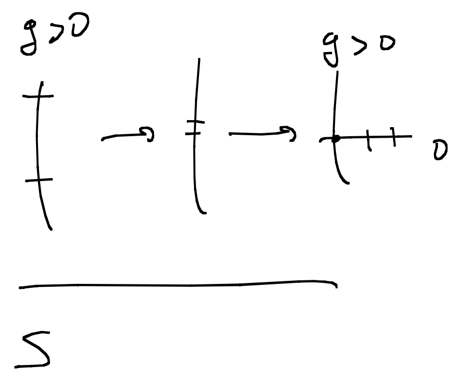
2) n non-intersecting sections

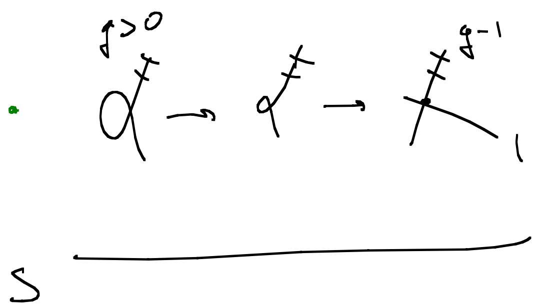
$$\sigma_i : S \to \mathcal{C}$$

s.t. the fiber over any (geometric) point of S is a stable marked curve.

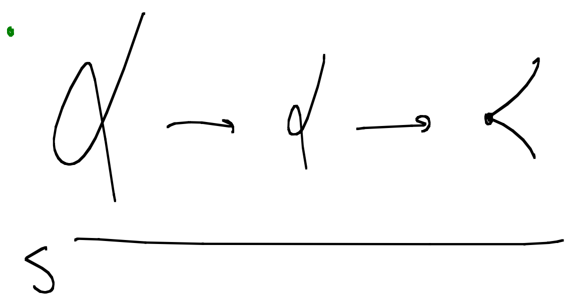
Examples:

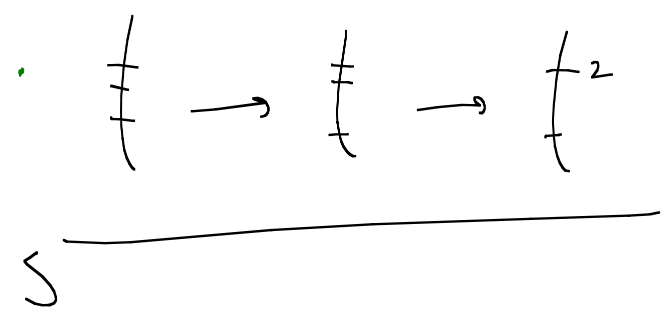






Non-examples





§2. k-Log-Canonically Embedded Curves

$(C, \{\sigma_i\})$ – stable marked curve.

Consider the <u>log-canonical</u> bundle

$$\omega_C(D) := \omega_C\left(\sum_i \sigma_i\right)$$

For smooth C, $U \subseteq C$ open

$$\Gamma_U\big(\omega_C(D)\big) = \left\{ \begin{array}{l} \text{meromorphic 1-forms on } U \\ \text{w/ at most simple poles at each} \\ \sigma_i \text{ \& nowhere else} \end{array} \right\}$$

For singular C w/ nodes $\{p_i\}$

Let $N \to C$ be a normalization.

& let $p_i^{\pm}$ denote the two preimages of each node.

Then

$$\Gamma_U\big(\omega_C(D)\big) = \left\{ \alpha \in \Gamma_U\Big(\omega_N\big(D + \sum_{\pm, i} p_i^{\pm}\big)\Big) \;\middle|\; \forall p_i, \; \operatorname{res}_{p_i^+}\alpha + \operatorname{res}_{p_i^-}\alpha = 0 \right\}$$

For $k \geq 1$, define the <u>k-log-canonical</u> bundle to be

$$\omega_C(D)^{\otimes k}.$$

Line bundles $\mathcal{L} \to X$ w/ nonzero global sections give maps to projective space.

Concretely, have

$$H^0(X, \mathcal{L})$$

If we trivialize a fiber $\mathcal{L}|_x \xrightarrow{\cong} k$

Then $x \in X$ gives a

$$\omega_x : H^0(X, \mathcal{L}) \to k$$
$$\sigma \mapsto \sigma(x)$$

The assignment

$$x \mapsto \omega_x$$

depends on the choice of trivialization, so we only get a map

$$X \to \mathbb{P}(H^0(X, \mathcal{L})^\vee)$$

independent of choices.

<u>Def</u>: A k-<u>log</u>-<u>canonically</u> <u>embedded</u> <u>curve</u> is a stable marked curve (C, D) which is embedded into $\mathbb{P}^n$ by $\omega_C(D)^k$.

Explicitly, it is a stable marked curve (C, D) along w/

1) an embedding $\iota : C \to \mathbb{P}^N$

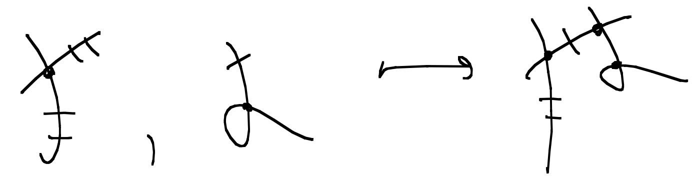
2) an isomorphism $\varphi : \mathbb{P}^N \to \mathbb{P}(H^0(C, \omega_C(D)^{\otimes k})^\vee)$

s.t. $C \xrightarrow{\varphi\iota} \mathbb{P}(H^0(C, \omega_C(D)^{\otimes k})^\vee)$ is $x \mapsto ev_x$

as above.

§ 3. Gluing Embedded Curves

Classically, can glue stable curves.



Key to this talk:

Can glue k-log-canonically embedded curves for $k \geq 5$.

Fix $h \geq 5$, S - scheme

$\mathcal{C} \xrightarrow{\pi} S$ family of curves,

- fibers not necessarily connected, w/at most nodal singularities

$\{\sigma_i : S \to \mathcal{C}\}_{i=1}^{n}$ - smooth non-intersecting sections

$\eta : \mathcal{C} \to \mathbb{P}^N_S$ - embedding by complete linear system

$$\varphi : \mathbb{P}^N_S \xrightarrow{\cong} \mathbb{P}_S\left(\pi_* \omega_{\mathcal{C}/S}\left(\sum_{i=1}^{n} \sigma_i\right)^{\otimes -h}\right)$$

s.t. $\varphi\eta(x) = ev_x \quad \forall x \in \mathcal{C}$.

<u>Thm:</u> (KSW) (Gluing Embedded Curves)

$h \geq 5$, S, $(\mathcal{C}, \{\sigma_i\}_{i=1}^{n}, \eta, \varphi)$ as above.

Let $\ell_{\sigma_1, \sigma_2} = \mathbb{P}^1_S \subseteq \mathbb{P}^N_S$ be the line spanned by σ_1 & σ_2.

Then 1) $\exists$ a natural section

$$p : S \to \ell_{\sigma_1, \sigma_2}$$

2) The projection from p induces an embedding of

$$\eta^\gamma : \mathcal{C}^\gamma := \mathcal{C}/_{\sigma_1 \sim \sigma_2} \longrightarrow \mathbb{P}^{N-1}_S$$

3) The isomorphism φ induces in isom. $\varphi^\gamma : \mathbb{P}^{N-1}_S \xrightarrow{\cong} \mathbb{P}_S\left(\pi_* \omega_{\mathcal{C}^\gamma/S}\left(\sum_{i=3}^{n} \sigma_i\right)^{\otimes -h}\right)$

Sketch Proof:

$\alpha : \mathcal{C} \to \mathcal{C}^\gamma$ quotient map

$\tilde{\sigma} := \alpha\sigma_1 = \alpha_{\sigma_2} : S \to \mathcal{C}^\gamma$

Main tool: $\exists$ SES

$$0 \to \omega_{\mathcal{C}^\gamma/S}\Big(\sum_{i=3}^{n} \sigma_i\Big)^{\otimes h} \to \alpha_* \omega_{\mathcal{C}/S}\Big(\sum_{i=1}^{n} \sigma_i\Big)^{\otimes h} \to \tilde{\sigma}_* \mathcal{O}_S \to 0$$

RR + degree considerations $\Rightarrow$ push-forward to S is exact

$$0 \to \pi^\gamma_* \omega_{\mathcal{C}^\gamma/S}\Big(\sum_{i=3}^{n} \sigma_i\Big)^{\otimes h} \to \pi_* \omega_{\mathcal{C}/S}\Big(\sum_{i=1}^{n} \sigma_i\Big)^{\otimes h} \to \mathcal{O}_S \to 0$$

Now dualize & projectivize $\Rightarrow$

$$\begin{tikzcd} S \arrow[r, "p"] & \mathbb{P}_S\Big(\pi_* \omega_{\mathcal{C}/S}\Big(\sum_{i=1}^{n} \sigma_i\Big)^{\otimes h}\Big)^\vee \arrow[d, dashed, "\text{projection}"] \\ & \mathbb{P}_S\Big(\pi^\gamma_* \omega_{\mathcal{C}^\gamma/S}\Big(\sum_{i=3}^{n} \sigma_i\Big)^{\otimes h}\Big)^\vee \end{tikzcd}$$

1) p lies on line $\ell_{\sigma_1, \sigma_2}$

because projection from it identifies σ_1 & σ_2.

2) Have

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\varphi"] \arrow[d] & \mathbb{P}_S\Big(\pi_* \omega_{\mathcal{C}/S}\Big(\sum_{i=1}^{n} \sigma_i\Big)^{\otimes h}\Big)^\vee \arrow[d, dashed] \\ \mathcal{C}^\gamma \arrow[r, "\varphi^\gamma"] & \mathbb{P}_S\Big(\pi_* \omega_{\mathcal{C}^\gamma/S}\Big(\sum_{i=3}^{n} \sigma_i\Big)^{\otimes h}\Big)^\vee \end{tikzcd}$$

So η^γ & φ^γ are specified by construction.

RTS η^γ is an embedding.

General facts:

(C, D) - stable, Then

$\omega_C(D)$ - embeds all C not in $\mathcal{H}yp_g$ (hyperelliptic locus)

$\omega_C(D)^{\otimes 2}$ - ________ w/ slightly better behavior of deformations

$\omega_C(D)^{\otimes 3}$ - embeds all C

$\omega_C(D)^{\otimes 4}$ - embeds C w/o tri-secants

$\omega_C(D)^{\otimes 5}$ - embeds C w/o quadrisecant planes.

No quadrisecant planes $\Rightarrow$ η^γ is an embedding $\square$

§4. Modular Operads

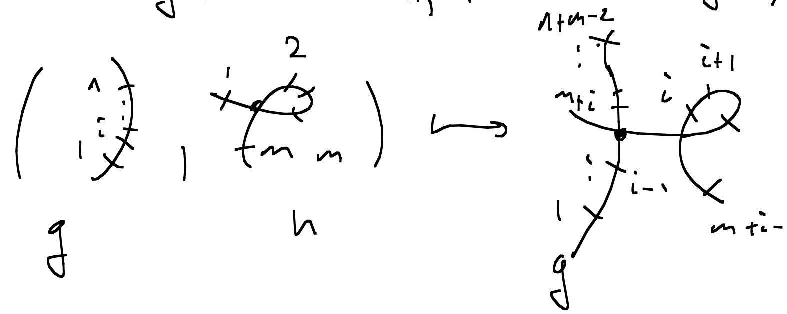
Moduli + operad = Modular operads

$\overline{\mathcal{M}}_{g,n}$ - moduli (stack) of stable genus g curves w/ n marked pts.

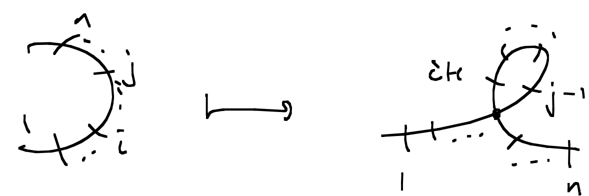
$S_n \circlearrowright \overline{\mathcal{M}}_{g,n}$ by permuting labels

Gluing maps:

$\circ_i : \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{h,m} \longrightarrow \mathcal{M}_{g+h,n+m-2}$

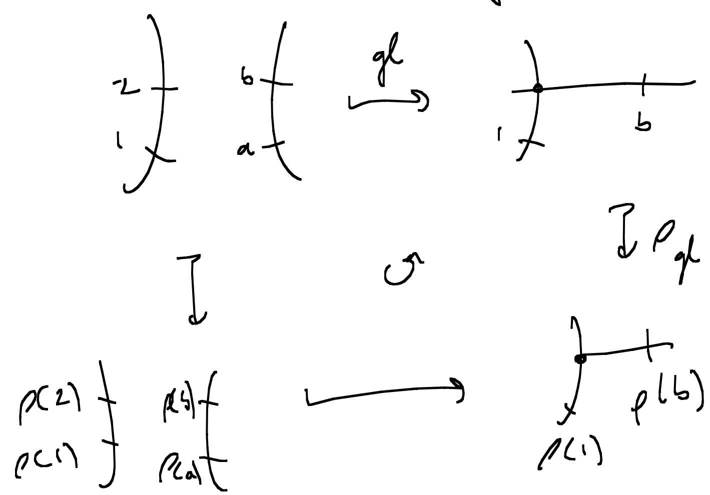


$\xi_{ij} : \overline{\mathcal{M}}_{g,n+2} \longrightarrow \overline{\mathcal{M}}_{g+1,n}$

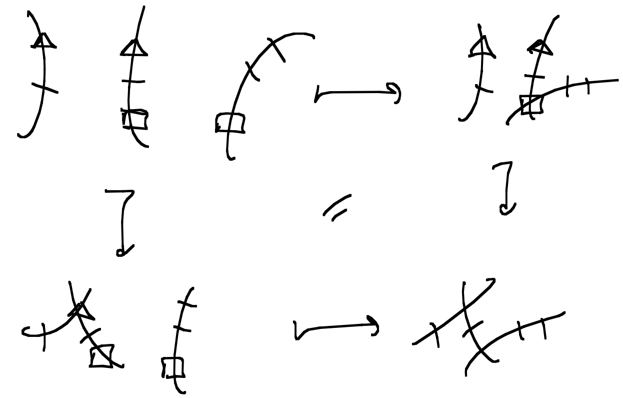


$\overline{\mathcal{M}} := \{\{\overline{\mathcal{M}}_{g,n} \circlearrowleft S_n\}, \circ_i, \xi_{ij}\}$ is defining example of a modular operad.

Key properties: 1) Compatibility btw gluing & S_n-action



2) Associativity of gluing



Fact: $(C, \{\sigma_i\}_1^n)$ stable curve, genus g

$$\Rightarrow h^0\left(\omega_C\left(\sum_1^n \sigma_i\right)^{\otimes k}\right) = 2(k-1)(g-1) + kn =: N(k,g,n)$$

Def: The <u>k-log-canonical Hilbert</u> scheme $\overline{\mathcal{Z}}^k_{g,n}$ is the moduli of k-log-canonically embedded stable genus g curves w/ n marked pts.

Rmk: $\mathcal{X}^k_g \subseteq \mathrm{Hilb}(\mathbb{P}^{N(k,g,n)})$ locus of k-log-canonically embedded stable curves

$\mathcal{C}^k_g \longrightarrow \mathcal{X}^k_g$ - universal curve

$$\overline{\mathcal{Z}}^k_{g,n} = \underbrace{\mathcal{C}^k_g \times_{\mathcal{X}^k_g} \cdots \times_{\mathcal{X}^k_g} \mathcal{C}^k_g}_{n}$$

Rmk: Have forgetful map

$$\overline{\mathcal{E}}^h_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

$$(C,\{\sigma_i\},Z,\varphi) \longmapsto (C,\{\sigma_i\})$$

This makes $\overline{\mathcal{E}}^h_{g,n}$ into a $PGL(N(h,g,n))$ bundle over the stack $\overline{\mathcal{M}}_{g,n}$.

Naturality of gluing embedded curves

$\Longrightarrow$

Cor: $\exists$ maps

$$\xi^e_{ij} : \overline{\mathcal{E}}^h_{g,n+2} \longrightarrow \mathcal{E}^h_{g+1,n}$$

$$o^e_i : \overline{\mathcal{E}}^h_{g,n} \times \overline{\mathcal{E}}^h_{h,m} \longrightarrow \overline{\mathcal{E}}^h_{g+h,n+m-2}$$

and commuting squares

$$\begin{tikzcd}
\overline{\mathcal{E}}^h_{g,n+2} \arrow[r, "\xi^e_{ij}"] \arrow[d] & \overline{\mathcal{E}}^h_{g+1,n} \arrow[d] \\
\overline{\mathcal{M}}_{g,n+2} \arrow[r, "\xi_{ij}"'] & \overline{\mathcal{M}}_{g+1,n}
\end{tikzcd}$$

& similarly for o^e_i & o_i.

<u>Thm:</u> (KSW)

The maps $\circ_i^e$ & ξ_{ij}^e

make $\left(\{ \overline{\Sigma}^h_{g,n} \circlearrowleft S_n \}, \circ_i^e, \xi_{ij}^e \right)$

into a modular operad $\overline{\Sigma}^h$

& the maps $f : \overline{\Sigma}^h_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$

give a map of modular operads

$$\overline{\Sigma}^h \longrightarrow \overline{\mathcal{M}}$$