B_n, C_n, F_4, G_2 Drinfeld-Sokolov Hierarchies and LG-model

YONGBIN Ruan

University of Michigan

(joint with Si-Qi Liu & Youjin Zhang)

arXiv: 1312.7227

Witten Conjecture ('91): (Geometry $\longleftrightarrow$ integrable hierarchy)

Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$

$$\begin{array}{ccc} \mathcal{L}_i & \ni & T^*_{x_i}C \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \ni & \{ C \} \end{array}$$

Intersection theory: $\quad (\psi_i = c_1(\mathcal{L}_i))$

$$\langle \tau_{e_1} \cdots \tau_{e_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{e_1} \cdots \psi_n^{e_n}$$

Generating funct (free energy)

$$\mathcal{F}_{\overline{\mathcal{M}}_{g,n}}(\hbar, t_0, t_1, \cdots) = \sum_{g \geq 0} \hbar^{g-1} \sum_{k \geq 0} \frac{t_1 \cdots t_k}{k!} \langle \tau_{e_1} \cdots \tau_{e_k} \rangle_g$$

$\Uparrow$

encode all information of intersection numbers.

<u>Integrable hierachies</u>: A hierarchy of PDE

$$\frac{\partial u}{\partial t_n} = R_n(u, u_x, u_{xx} \dots)$$

$u = u(x, t_1, \dots t_n \dots)$, R_n - differential polynomials

"classical integrable hierarchies" $\Leftarrow$ representation of affine Kac-Moody algebra

($\Leftarrow$ a long story)

Drinfeld - Sokolov   Kac - Wakimoto

$\Uparrow$ $\Uparrow$

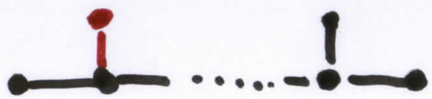
Symplectic reduction of system of PDE $\overset{\text{equivalent}}{\Longleftrightarrow}$ infinite Grassmanian and Hirota Bilinear equ

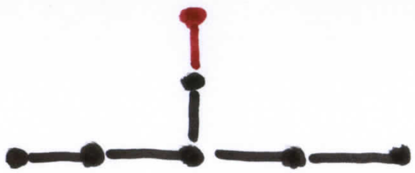
Affine Dykin Diagram: $\Rightarrow$ affine Kac-Moody algebra

$A_n^{(1)}$: 

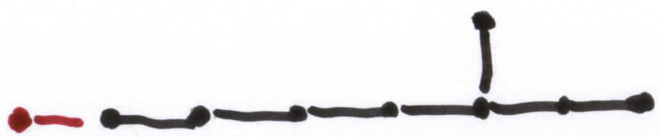
$B_n^{(1)}$: 

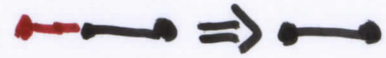
$C_n^{(1)}$: 

$D_n^{(1)}$: 

$E_6^{(1)}$: 

$E_7^{(1)}$: 

$E_8^{(1)}$: 

$F_4^{(1)}$: 

$G_2^{(1)}$ 

Twisted series:

$A_2^{(2)}$, $A_{2\ell}^{(2)}$ $A_{2\ell-1}^{(2)}$ $D_{\ell+1}^{(2)}$

$E_6^{(2)}$ $D_4^{(3)}$

$\mathbb{P}^1/_{ADE}$

$\Uparrow$ (Milanov-Shen-Tseng)

Representation: $\boxed{\text{Principal}}$, homogeneous, other

KP ....

Others: 2 - Toda and its extension $\Rightarrow$ orbifold $\mathbb{P}^1$

Best know examples:

(1) KdV (Kortweg - de - Vries) $\Longleftrightarrow$ $A_1^{(1)}$ with principal rep

Lowest order :

$$\partial_t u = - \partial_x^3 \phi + 6u \partial_x u$$

$$\vdots$$

(2) nKdv (Gelfand - Dickey) $\Longleftrightarrow$ $A_n^{(1)}$ with principal rep

(3) KP $\Leftarrow$ Hurwitz theory

Witten conjecture:

Gromov-Witten theory of point $\overline{\mathcal{M}}_{g,n} \Longleftrightarrow$ KdV

Geometry $\xLeftrightarrow{\text{correspondance}}$ integral hierarchy

More precisely,

$\mathcal{F}_{\overline{\mathcal{M}}_{g,n}}$ satisfies KdV hierarchy

($e^{\mathcal{F}}$ – tau – function of hierarchy)

- Solved by Kontsevich (91) $\rightsquigarrow$ Fields Medal

other examples ?

2 - Toda conjecture :

Gromov-Witten theory of $\mathbb{P}^1$ $\Longleftrightarrow$ 2 - Toda

- Solved by Okounkov-Pandharipande (2000)

  $\Downarrow$

  Fields Medal

- Extension to orbifold $\mathbb{P}^1(a,b)$ by Milanov-Tseng, Johnson

---

- Problem is hard !
- Problem is mysterious ! (case by case)

  If you are lucky, you may run into one !

Witten's ADE conjecture: (91-93)

$\Downarrow$

go to entirely different direction for geometry

Conj: ADE Landau-Ginzburg model $\Leftarrow$ singularity

satisfy

Lie algebra

$\Downarrow$

ADE Drinfeld-Sokolov Hierarchies

- Solved by Fan-Jarvis-— (2009)

what is next ?

Today: B_n, C_n, F_4, G_2

Remark: some new phenomenon!

LG - model

Data: (I) $W : \mathbb{C}^n \to \mathbb{C}$ quasi-homogeneous poly $\quad \lambda \in \mathbb{C}^*$

Super potential

$\exists\ (c_1 \cdots c_n, d)$ s.t $W(\lambda^{c_1} z_1 \cdots \lambda^{c_n} z_n) = \lambda^d W(z_1 \cdots z_n)$

degree $\neq 0$

Nondegeneracy: zero is only critical / singularity pt

i.e: $\partial_1 W = \cdots = \partial_n W = 0 \Rightarrow z = (0, \cdots 0)$

(II) $J \in G \subset \mathrm{Aut}(W) = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \,\middle|\, W(\lambda_1 z_1, \cdots \lambda_n z_n) = W(z_1, \cdots z_n) \right\}$

$$J = \begin{pmatrix} e^{2\pi i \frac{c_1}{d}} & & \\ & \ddots & \\ & & e^{2\pi i \frac{c_n}{d}} \end{pmatrix}$$

Morally, LG-model study singularity with symmetry (W, G)

Classical inv $\quad \mathrm{Jac}_W = \dfrac{\mathbb{C}[z_1, \ldots, z_n]}{\partial_1 W, \cdots \partial_n W} < +\infty$

Quantum inv: quantize Jac_W !

LG – model continued

<u>Witten Equation</u>:

$$\bar{\partial} s_i + \overline{\partial_i W(s_1 \cdots s_n)} = 0$$

$s_i \in \Gamma(L_i)$

$$\begin{array}{c} L_i \\ \downarrow \text{ - orbifold line bundle} \\ \Sigma_g \end{array}$$

orbifold str of $\bigoplus_{i=1}^{n} L_i$ at marked pt $z_j \in \Sigma_g$ is parameterized by $h \in G$.

2009: Fan – Jarvis – , — has worked out a complete moduli theory of Witten equ

invariant = # of sol's of Witten equ

$\updownarrow$

analogous of Gromov – Witten Invariant.

Fan-Jarvis-Ruan Theorem:

W – ADE, $G = G_{max}$ – maximal diagonal symmetry group

Theorem:

$\mathcal{F}_W$ satisfies W^T – Drinfeld Sokolov hierachies

W^T – mirror polynomial in the sense of Berglund - Hubsch - Krawitz

$A_n = x^{n+1}$	$A_n^T = A_n$
$D_n = x^{n-1} + xy^2$	$D_n^T = x^{n-1}y + y^2 \neq D_n$
$E_6 = x^3 + y^4$	$E_{6,7,8}^T = E_{6,7,8}$
$E_7 = x^3 + xy^3$	
$E_8 = x^3 + y^5$	

Proof:

$\boxed{\text{step I}}$ A-model B-model

Theories of Fan-Jarvis-Ruan-Witten for (W, G_{max}) $\overset{\text{Mirror}}{\underset{\text{FJR}}{\Longleftrightarrow}}$ Saito-Givental Theories for W^T

quick route by Dubrovin Zhang

$\boxed{\text{step II}}$ (Frenkel-Givental-Milanov)

Saito-Givental theories for ADE W^T satisfies W^T-Kac-Wakimoto hierarchies

$\boxed{\text{step III}}$ (classical, but nontrivia)

W^T-Kac-Wakimoto hierachies $\overset{\text{equivalent}}{\Longleftrightarrow}$ W^T-Drinfeld-Sokolov hierachies

A logical candiadate for B_n, C_n, F_4, G_2

Saito-Givental theories for simple boundary singularities

$\Uparrow$

index by B_n, C_n, F_4, G_2

- I have been worked on it for several years <u>without</u> success

- A striking recent result by Dubrovin-Liu-Zhang

$$\boxed{\text{Saito-Givental theories of simple boundary singularities are NOT solutions of } BCF_4G_2 \text{ Drinfeld-Sokolov hierarchies}}$$

Behavior of genus one funct $\mathcal{F}_1$:

Key properties: Semi-simplicity

$\exists$ "canonical coordinates" $u_x^1 \cdots u_x^n$ generically s.t

quantum product $\quad u_x^i \cdot u_x^j = \begin{cases} u_x^i & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

All the geometric theories has

$$\mathcal{F}_1 = \sum_{i=1}^{n} \underbrace{\frac{1}{24}}_{\text{central invariant}} \log u_x^i + G(u)$$

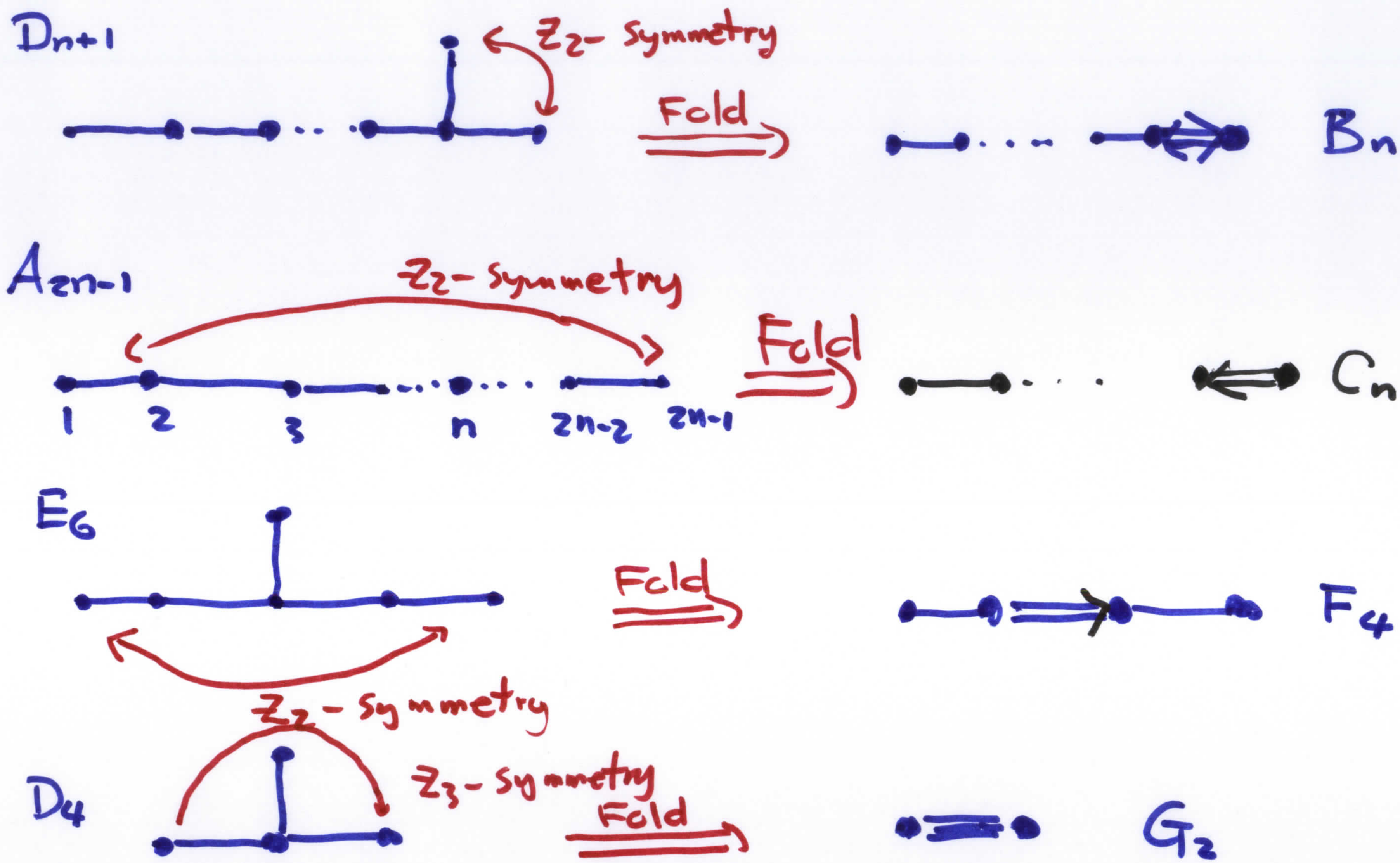
Dubrovin - Liu - Zhang:

For B_n, C_n, F_4, G_2 - Drinfeld Sokolov hierarchies,

its solution has central inv Different from $(\frac{1}{24}, \cdots \frac{1}{24})$

$B_n: (\frac{1}{24}, \cdots \frac{1}{24}, \frac{1}{12})$, $C_n: (\frac{1}{12}, \cdots \frac{1}{12}, \frac{1}{24})$. $F_4: (\frac{1}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{12})$; $G_2: (\frac{1}{8}, \frac{1}{24})$

Next Idea: Symmetry



Classical Fact: These are out automorphism of Lie algebra

Two Questions:

(I) Are these symmetry induces symmetry of Drinfeld Sokolov-hierachies

(II) Can we endow the same symmetry on A-model mirror.

$\boxed{\text{Luckly, Answers to both questions are Yes!}}$

Γ-reduction Theorem:

Let Γ be one of these symmetries. The Γ-invariant flows of an ADE Drinfeld-Sokolov hierarchy define the corresponding B_n, C_n, F_4, G_2 DS-hierachy. Furthermore, the restriction of ADE τ-funct to the Γ-invariant subspace of the big phase space provides a tau-funct of corresponding BCFG-hierachies

Symmetry in A-model:

structure of Gromov-Witten type Invariant:

2D TQFT $\Rightarrow$ $\Lambda_{g,n} : H^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$

H — vector space with pairing, unit $\mathbb{1}$ (state space)

satisfies:

(I) Forgetful morphism: $\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$

$$\pi^* \Lambda_{g,n}(\alpha_1 \cdots \alpha_n) = \Lambda_{g,n+1}(\alpha_1 \cdots \alpha_n, 1)$$

(II) Gluing tree: $\rho_{tree} : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$

$$\rho_{tree}^* \Lambda_{g_1+g_2,n_1+n_2} = \sum_{\sigma,\tau} \Lambda_{g_1,n_1+1}(\cdots e_\sigma)\eta^{\sigma\tau}\Lambda_{g_2,n_2+1}(\cdots e_\tau)$$

pairing

(III) Gluing loop: $\rho_{loop}: \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$

$$\rho_{loop}^{*} \Lambda_{g+1,n} = \sum_{\sigma,\tau} \Lambda_g(\cdots e_\sigma, e_\tau)\, \eta^{\sigma\tau}$$

Symmetry: Finite group G acting on H preserving pairing, unit, all the structure

For example: $\Lambda_{g,n}(g\alpha_1 \cdots g\alpha_n) = \Lambda_{g,n}(\alpha_1 \cdots \alpha_n)$, $g \in G$

<u>Key Properties</u>:

(I) Vanishing: Let $H^G \subset H$ be G- inv subspace
$\alpha_1 \cdots \alpha_{n-1} \in H^G$

Then, $\Lambda_{g,n}(\alpha_1 \cdots \alpha_{n-1}\, \beta) = 0$ if $\beta \notin H^G$

(II) Let $\Lambda^G_{g,n} : (H^G)^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$
be restriction

$\Lambda^G_{g,n}$ satisfies all the structure axioms
except Gluing loop

(III) $\Lambda^G_{0,n}$ satisfies all the structure axioms.
No gluing loop in $g=0$

How to endow symmetry:

(I) Easier case: Gromov-Witten theory

X — Kahler / symplectic mfd

G acts on X preserving the kahler/symph

G induces a symmetry on Gromov-Witten case.

(II) Harder case: LG-model (W, G_{max})

- Theory is independent of deformation of W

2nteresting pheomenon: $W \overset{def}{\rightsquigarrow} W'$ s.t $G_{max,W'} \supsetneq G_{max}$

Then, $G_{max,W'} / G_{max}$ — symmetry of $(W, G_{max,W'})$

Apply to ADE cases:

$$\left.\begin{array}{l} \bullet\ A_{2n-1}, \\ \bullet\ E_6 \end{array}\right\} \exists \text{ an easy } \mathbb{Z}_2\text{-symmetry by degree consideration}$$

- $D^T_{n+1} = x^n y + y^2$, $G_{max, D^T_{n+1}} = \mathbb{Z}/2n\mathbb{Z}$

$$\Big\downarrow \text{deform}$$

$$\widehat{D}^T_{n+1} = x^{2n} + y^2, \quad G_{max, \widehat{D}^T_{n+1}} = \mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

$$\cup\!\!|$$

$(D^T_{n+1}, \mathbb{Z}/2n\mathbb{Z})$ has a symmetry $\mathbb{Z}/2\mathbb{Z}$.

- $(D_4, \langle J \rangle)$, $D_4 = x^3 + xy^2$, $J = \begin{pmatrix} e^{\frac{2\pi i}{3}} & \\ & e^{\frac{2\pi i}{3}} \end{pmatrix}$ ord=3

$G_{max} = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \Rightarrow (D_4, \langle J \rangle)$ has a $\mathbb{Z}_3$-symmetry

Connecting LG to Drinfeld-Sokolov hierarchy

Together with symmetry

I. Older technique (using Frenkel-Givental-Milanov result to Kac-Wakimoto, then to Drinfeld-Sokolov.)

Might work but become ~~i~~ subtle and tedious

II. Short cut by Dubrovin-Zhang theory

providing a much quicker and clearer route

Final Theorem: The restrictions of $\mathcal{F}_{AD_n^TE}$ on G-inv big phase space satisfy corresponding B_n, C_n, F_4, G_2 - Drinfeld-Sokolov Hierarchies

Relation between G-inv $\mathcal{F}_{AD^TE}$ and $\mathcal{F}_{BCFG}$ from Saito - Givental theory

---

(I) Recall $\Lambda^G_{g,n}$ satisfy all axioms except gluing loop (doesn't exist in $g=0$)

$\Downarrow$

$\Lambda^G_{0,n}$ still satisfy all the axioms

(II) In $g=0$. G-inv $\mathcal{F}_{AD^TE}$ and $\mathcal{F}_{BCFG}$ agree

$g>0$ They are Different !

THANK YOU !