NANHUA XI
CAS

IPMU 2009.8.

Kazhdan – Lusztig Theory

For what?

For symmetric group S_n of n letters, ~~one~~ or more generally, Weyl groups, affine Weyl groups, and further, Coxeter groups and their Hecke algebras.

Why the groups and their Hecke algebras.

Let $G = GL_n(\mathbb{C})$ T be the subgroup of G consisting of diagonal matrices Let W be the subgroup of G consisting of permutation ~~group~~ matrices, (each row and each column has exactly one non zero entry, which is 1). Then $W \subset N_G(T)$, $W \simeq S_n$ and ~~$N_G(T)/T$~~ $N_G(T) = \cancel{W\times T}\ WT$. $N_G(T)/T \simeq W$.

Now for the structure of G. We have the very useful Bruhat decomposition

$$G = \bigcup_{w\in W} BwB \qquad B = \{\text{upper triangular matrices in } G$$

G/B is the flag variety

$\overline{BwB}/B$ is ~~the~~ an Schubert variety. important objects in geometry, representation theory

W acts on T by conjugation. Consider $\mathrm{Hom}(T, \mathbb{C}^*) = X$
This is an abelian group generated by $\varepsilon_i : \mathrm{diag}(a_1, \cdots, a_n) \to a_i$.
So $X \simeq \mathbb{Z}^n$ and W acts on X: $w: \varepsilon_i \to \varepsilon_{w(i)}$
Then $\varepsilon_i - \varepsilon_j$ $(i \neq j)$ is the root system R of type A_{n-1}
and W is the Weyl group.

You know root systems and Weyl groups are fundamental in Lie theory, ~~both~~ in structure, representats. and applns.

~~why these~~ Hecke

$W \ltimes X$ is an extended affine Weyl group appearing naturally in reductive ~~p~~ adic group theory.

why the ~~Hec~~ Hecke algebras

Example. $S_n = \langle (i, i+1) \mid i = 1, 2, \cdots, n-1 \rangle$.

Denote by s_i the transpose $(i, i+1)$

Let ~~$A = \mathbb{Z}[q, q^{-1}]$ be~~ $A = \mathbb{Z}[v^{1/2}, v^{-1/2}]$ be ~~the~~ the ring of Laurent polynomials in ~~the indeterminant~~ indeterminant $v^{1/2}$ with integer coefficients the Hecke algebra H of S_n is a free A-module with a base T_w, $w \in S_n$ and multiplicats is defined by

$$T_{s_i} T_{s_{i+1}} T_{s_i} = T_{s_{i+1}} T_{s_i} T_{s_{i+1}} \qquad i = 1, 2, \cdots, n-1$$
$$T_{s_i} T_{s_j} = T_{s_j} T_{s_i} \qquad \text{if } |i-j| \geq 2.$$

$$(T_{s_i} - v)(T_{s_i} + 1) = 0.$$

Hecke algebra is ~~import~~ naturally appeared in ~~knot the~~ the study of representats of finite groups of Lie type. braid, knot, quantum group.

$G = GL_n(\mathbb{F}_q)$ $\mathbb{F}_q$ is a finite field of q elements

Let B be the subgroup of G consisting of upper triangular matrices in G. Let 1_B be the trivial complex representats. ~~over~~ G

Consider $M = \operatorname{Ind}_B^G 1_B$ ~~the~~ $= \mathbb{C}[G] \otimes_{\mathbb{C}[B]} 1_B$ ~~the~~ the induced representats to G of 1_B then ~~the~~ ring

$\mathrm{End}_G M = \mathrm{Hom}_{\mathbb{C}[G]}(M, M)$ is isomorphic to $\mathbb{C} \otimes_A H$ where $\mathbb{C}$ is regarded as an A-module by specializing ~~q to~~ v to q.

So the much part of the study of M is reduced to the study of $\mathbb{C} \otimes_A H$ such as composition factors ~~ect~~ etc

What the Kazhdan-Lusztig says

Let W ~~(W, S)~~ be a Coxeter group. i.e. W is generated by some elements in a ~~set~~ subset S of W with relation

$$\cancel{s^2 \in S} \quad s^2 = 1 \qquad \text{if } s \in S$$

$$(st)^{m_{st}} = 1 \qquad \text{for } s \neq t \in S \quad \text{here } m_{st} \in \{2, 3, 4, \cdots, \infty\}$$

Ex. For S_n $\quad S = \{(i, i+1) \mid i = 1, 2, \cdots, n-1\}$

$$(s_i s_{i+1})^3 = 1 \qquad (s_i s_j)^2 = 1 \quad \text{if } |i-j| \geq 2$$

The Hecke algebra of W over $A = \mathbb{Z}[v^{\frac{1}{2}}, v^{-\frac{1}{2}}]$ is a free A-module with a basis T_w, $w \in W$ and ~~fund~~ multiple relations

$$(T_s - v)(T_s + 1) = 0$$

$$\underbrace{T_s T_t T_s \cdots}_{m_{st} \text{ factors}} = \underbrace{T_t T_s T_t \cdots}_{m_{st} \text{ factors}} \qquad \text{if } s, t \in S$$

H has a ring involution:

$$\bar{\ } : H \to H$$

$$v^{1/2} \to v^{-\frac{1}{2}} \qquad T_w \to T_{w^{-1}}^{-1}$$

~~A fundament result in Kazhd~~ The foundation of the Kazhdan-Lusztig theory is the following ~~theorem~~ result.

Theorem (KL, 1979) Let W be a Coxeter group with the set S of generators. For any $w \in W$ there ~~exis~~ is a unique element C_w in H such that

$$\overline{C}_w = C_w$$

$$\cancel{C_w = \sum_{y \leq w} (-1)^{l(y)-l(w)} q^{\frac{l(w)}{2}-l(y)}}$$

$$C_w = q^{-\frac{l(w)}{2}} \sum_{y \leq w} P_{y,w} T_y$$

where $P_{y,w}$ is a polynomial in v, and $P_{w,w} = 1$

$\deg P_{y,w} \leq \frac{1}{2}(l(w) - l(y) - 1)$ if $y < w$.

$\{C_w \mid w \in W\}$ is the celebrated KL basis of H

$P_{y,w}$ are the famous KL polynomials

The proof is purely ~~Comit~~ combinatorical, but the information encoded in C_w, $P_{y,w}$ is very rich, ~~including g~~ involving geometry representati, algebra, combinatorics.

A remarkable ~~exp~~ example that $P_{y,w}$ enters into representati theory is the ~~following~~ Kazhdan-Lusztig Conjecture proved by ~~Kashiwara, Brylinski~~ Brylinski - Kashiwara

~~$\mathfrak{g}$~~: Beilinson - Bernstein.

$\mathfrak{g}$: Semisimple Complex Lie algebra

$\mathfrak{h}$: a Cartan subalgebra

$\rho: \mathfrak{h} \to \mathbb{C}$, ~~$\langle \rho, \alpha^\vee \rangle = 1$~~ $\rho(\alpha^\vee) = 1$ for each simple coroot $\alpha^\vee$

M_w: Verma module of highest weight $-w(\rho) - \rho$, where w in the Weyl group W of $\mathfrak{g}$

L_w: the unique irreducible quotient module of M_w

Conj. (Kazhdan - Lusztig, 1979)

$$\mathrm{ch}\, L_w = \sum_{y \le w} (-1)^{l(w) - l(y)} P_{y,w}(1)\, \mathrm{Ch}\, M_y$$

$$\mathrm{Ch}\, M_w = \sum_{y \le w} P_{w_0 w, w_0 y}(1)\, \mathrm{ch}\, L_y$$

where $P_{y,w}$ are the Kazhdan-Lusztig polynomials and $P_{y,w}(1)$ is the value of $P_{y,w}$ at 1. w_0 is the longest element of W

The Conjecture was proved by ~~B~~ Brylinski-Kashiwara, Beilinson-Bernstein

when $w = w_0$. ~~we get~~ the conjecture simply ~~says~~ the Weyl's character formula.

Using affine Weyl groups. The formulas above were generalized ~~to~~ ~~Kac-~~ Symmetrizable Kac-Moody algebras affine Lie algebras. algebraic groups over p, quantum group at roots of 1

(Kashiwara, Tanisaki, Kazhdan, Lusztig, Andersen, Jantzen, Soergel, Nakajima, Bezrukavnikov, 1990 - 2007)

A key fact is the proof is that $P_{y,w}$ actually compute intersection cohomology of the Schubert varieties.

Recall we defined $\mathcal{B}_w = \overline{BwB/B} \subset G/B$ for $G = GL_n(\mathbb{C})$

For any reductive group we can define Schubert variety.

$\mathcal{L}$ is the local system $\mathbb{C}$ on ~~$\mathcal{B}$~~ BwB/B

$\mathcal{L}^\#$ is the IC intersection cohomology extension

For $y \le w$, set $n_{y,w,i} = \dim \mathcal{H}^i_x \mathcal{L}^\#$ (the dimension of the stalk at x of $\mathcal{H}^i \mathcal{L}^\#$, $x \in \mathcal{B}_y$)

Th (KL, 1980).

$$P_{y,w}(x^2) = \sum_i n_{y,w,i}\, q^i$$

Th (Vogan 1979)

$$P_{y,w} = \sum_{i\geq 0} q^i \dim \mathrm{Ext}^{l(w)-l(y)-2i}(M_y, L_w)$$

Using KL bases, one can define ~~cells~~ KL cells which is ~~very~~ important in studying of representations of finite groups of Lie type (say $SO_n(\mathbb{F}_q)$, ~~G~~ $E_8(\mathbb{F}_q)$, ...) primitive ideals of universal enveloping algebras representations of Coxeter groups and ~~affine~~ ~~Hec~~ their Hecke algebras

Kazhdan-Lusztig theory is still dynamic. there are many interesting questions, say understanding KL polynomials (including leading ~~D~~ coefficients), a-function, based ring of two-sided cells in affine Weyl groups, relations with K-theory and perverse sheaves

A simple question is whether $\mu(y,w) \to \infty$ when $n \to \infty$ for S_n

The work of Kazhdan-Lusztig of 1979 can be viewed as part of the starting point of the powerful ~~subject~~ and dynamic subject Geometric Representation Theory. the other ~~two~~ main articles are ~~Delig~~ Representations of ~~finite~~ reductive groups over finite fields, Ann. of Math. 1976 by Deligne and Lusztig,

~~Trigonometric~~ Trigonometric sums, Green functions of finite groups and representations of Weyl groups Invent. Math. 1976 by T. A. Springer

Now I go into more technical point

1. Kazhdan – Lusztig polynomials

Still (W, S) is a Coxeter group. S the set of simple reflections. For $y \le w$, we have the Kazhdan – Lusztig polynomials $P_{y,w}$. Computation to $P_{y,w}$ is nontrivial, even for the leading coefficient

Assume $sw \le w$. $s \in S$. Then

$$P_{y,w} = q^{1-c} P_{sy,sw} + q^c P_{y,sw}$$

$$- \sum_{\substack{u \\ y \le u < sw \\ su \le u}} \mu(u, sw)\, q^{\frac{1}{2}(l(w) - l(u))} \cancel{P_{y,su}}\ P_{y,u}$$

~~$\mu(y,w)$~~ (For any $y < w \in W$. $\mu(y, w)$ is defined by ~~$P_{y,w}$ to the~~

$$P_{y,w} = \mu(y, w)\, q^{\frac{1}{2}(l(w) - l(y) - 1)} + \text{lower degree terms})$$

So the coefficient $\mu(y, w)$ is very interesting in understanding $P_{y,w}$

Conj. (KL 1979). $P_{y,w} \in \mathbb{N}[q]$

True for Weyl group. ~~1st~~ affine Weyl group (KL, 1980) Crystallographic Coxeter group (L. 1985) (i.e. $m_{st} \in \{2, 3, 4, 6, \infty\}$ for $s, t \in S$)

~~Finite Coxeter group~~ (Kazhdan Lusztig, Lusztig – Alvis)

Coefficient of q. (Dyer 1997). For D_{2m}. ~~H_3, H_4.~~ (trivial) H_4 (D. Alvis, by computer.)

~~The coeffcient μ is related to some cohomology group~~

Assume that W is an extended affine Weyl group. This means it comes from a semisimple Lie algebra $\mathfrak{g}/\mathbb{C}$. W_0 Weyl group, X the weight lattice. $W = W_0 \ltimes X$. Let w_0 be the longest element in W_0. y, x are dominant weights in X. $y \le x$. Then

$$\cancel{P_{y,w}} \quad \cancel{P_{\dots}} \quad P_{yw_0, xw_0}(1) = \dim V(x)_y$$

$V(x)_y$ the y-weight space of the irr. $\mathfrak{g}$-module of highest weight. (L, 1981)

I. Kato (1985) showed that P_{yw_0, xw_0} is actually a q-analogue of Kostant's partition function ~~Conj by~~ (~~Conj~~ Conj. by ~~R~~ Lusztig)

The coeffcient μ is related to some important cohomology groups in Lie theory.

- For type $\tilde{B}_2$ Lusztig computed some (1976). More by L. Wang
- $\exists\, y, w \in S_{13}, S_{16}$ s.t. $\mu(y,w) > 1$.
  (McLarnan & Warrington, 2003). This ~~gi~~ gave a negative answer to $(0,1)$ conjecture. (says for S_n $\mu(y,w) = 0$ or 1)
- If $y, w \in \tilde{S}_n$ or S_n and $a(y) < a(w)$, then $\mu(y,w) < 1$ (Xi;
- For some special cases $\mu(y,w) < 1$ (B.C. Jones, 09, R.M. Green, 09
- ~~When~~ For S_n, ~~$\mu(y,w)$~~ Kazhdan and Lusztig conjecture that $\mu(y,w) \ne 0$ is equivalent to certain geometry properties between X_w and X_y.

Question: Is μ bounded for ~~all S_n~~ $\bigcup_{n \ge 2} S_n$

For $\tilde{S}_n$ $\exists\, y, w$ s.t. $\mu(y,w) = n+1$. So $\mu \to \infty$ if $n \to \infty$ for $\tilde{S}_n$ (~~L~~ Scott & Xi, 06)

Kazhdan - Lusztig cell

Assume that $\mu(y,w)\neq 0$ or $\mu(w,y)\neq 0$. if there exists $s\in S$ such that $sy\leq y$ but $sw\geq w$, we write $y\leq_L w$. then $\leq_L$ generates a preorder $\leq_L$ on W. (i.e. $x\leq_L u \iff \exists$ sequences $x=x_1, x_2, \cdots, x_n = u$ s.t. $\mu(x_i, x_{i+1})\neq 0$ or $\mu(x_{i+1}, x_i)\neq 0$. and $x_i\leq_L x_{i+1}$)

Write $y\leq_R w$ if $y^{-1}\leq_L w^{-1}$ and

$y\leq_{LR} w$ if there exists a sequence $y=y_1, y_2, \cdots, y_n=w$ such that $y_i\leq_L y_{i+1}$ or $y_i\leq_R w_{i+1}$ for $i=1,2,\cdots,n-1$.

The corresponding equivalence relation is denoted by

$\sim_L,\ \sim_R,\ \sim_{LR}$

The equivalence classes are called left cells, right cells, two-sided cells respectively.

Let $h, h'\in H$, $w\in W$ then

$$h C_w = \sum_{u\leq_L w} \xi_u C_u \qquad \xi_u\in A$$

$$C_w h' = \sum_{u\leq_R w} \eta_u C_u \qquad \eta_u\in A$$

$$h C_w h' = \sum_{u\leq_{LR} w} \zeta_u C_u \qquad \zeta_u\in A$$

So each left cell (Right cell, two-sided cell) gives rise to a left H-module (right H-module, ~~two-sided~~ bi H-module)

Th ~~Each left~~ cell For S_n, each left cell module is irreducible

In general it is not true.

Cells in Weyl groups are studied by Lusztig, Vogan, Barbasch etc.

and play a crucial role in Lusztig's work on representations of finite group of Lie type. and ~~classified~~ classify primitive ideals (the primitive ideals of $\mathfrak{g}$ with a fixed regular integral ~~infinit~~ ~~infinitesl~~ infinitesmal character are in one-to-one correspondence with the left cells in ~~Weyl~~ the Weyl group)

The cells in Weyl groups are well understood. For ~~other~~ dihedral group D_{2n}, trivial and H_3, H_4 also clear by ~~comput~~ means of computer (Alvis)

The next category is affine Weyl group. the cells ~~is~~ are ~~describe~~ described for rank cases (Lusztig 1985) ~~type~~ $\tilde{A}_n$ (1985 Shi), $\tilde{C}_3$ (by Bedard, 1986) and some ~~other~~ low rank cases by Shi, and his students or collegues ~~say by~~

It would be very interesting if one could describe the cells in an affine Weyl group of ~~type~~ $\tilde{C}_n$.

A useful tool to study cells ~~is~~ is Lusztig's a function.

For $w, u \in W$ write

$$C_w C_u = \sum h_{w,u,v} C_v, \qquad h_{w,u,v} \in A$$

Set

$$a(v) = \min\{ i \in \mathbb{N} \mid q^{-\frac{i}{2}} h_{w,u,v} \in \mathbb{Z}[q^{-1/2}] \quad \forall\, w, u \in W\}$$

if no such i exists. set $a(v) = \infty$

Conj. (Lusztig, 1997). Assume (W, S) is a Coxeter group. and $\# S < \infty$ is finite, then a is bounded on W

- when ~~(W,S)~~ the Coxeter graph of (W, S) is complete. the conjecture is ~~true~~ (Xi, 2008)

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Extended affine Weyl groups

The ~~theory~~ cell theory ~~is~~ for (extended) affine Weyl groups is interesting and rich

Let R be a root system and Q the ~~weight~~ root lattice, X ~~the~~ the weight lattice, W_0 the Weyl group then $W_0 \ltimes Q$ ~~xx~~ $= W_a$ is an affine Weyl group it is a Coxeter group $W = W_0 \ltimes X$ is an extended affine Weyl group. There exists an abelian subgroup Ω such that $W \simeq \Omega \ltimes W_a$. One can define the Hecke algebra H of W as $\mathbb{Z}\Omega \otimes_{\mathbb{Z}} H(W_a)$ or define the Hecke algebra H of W as follows. Let S be the set of simple reflections in W_a. Define the length function $l: W \to \mathbb{N}$ $l(\omega w) = l(w)$ and $\omega w = w' u \Leftrightarrow \omega = \omega'$, $w = u$. Then H is a free A-module with basis $w \in W$ and multiplication relations

$$(T_s - q)(T_s + 1) = 0$$

$$T_w T_u = T_{wu} \text{ if } l(wu) = l(w) + l(u), \quad w, u \in W$$

Clearly $H(W_a) \subset H$

(1985, 89)

For affine Weyl group ~~Lu~~ Lusztig proved that

- $a(w) \leq a(w_0) = l(w_0)$ w_0 is the longest element in W_0
- a-function is constant on ~~two~~-sided cells of W
- $\{$~~two~~-sided cells of $W\} \Longleftrightarrow \{$ ~~un~~ nilpotent classes of $\mathfrak{g}\}$ where $\mathfrak{g}$ is semisimple Lie algebra$/\mathbb{C}$ with ~~root syste~~ of R.
- c a ~~two~~-sided cell of W. ~~A~~ $\mathcal{C}_c$ ~~a~~ the corresponding nilpotent class. Let $x \in \mathcal{C}_c$ and $\mathcal{B}_x$ the variety of ~~all~~ ~~B~~ all Borel subalgebra of $\mathfrak{g}$ containing x. Then

$$a(c) = \dim \mathcal{B}_x$$

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~~Lusztig~~ - Xi (1988) each ~~two~~-sided cell of W contains a unique canonical left cell.

Based ring of a ~~two~~-sided cell in W

Write $\cancel{C_x C_y} = \sum_{z \in W} h_{x,}$

$$C_w C_u = \sum_{z \in W} h_{w,u,z} C_z \quad h_{w,u,z} \in \mathcal{A}$$

~~Set~~ write $h_{w,u,z} = \gamma_{w,u,z} q^{\frac{a(z)}{2}}$ + Lower degree terms

Let J be a free $\mathbb{Z}$-module, with a basis t_w, $w \in W$.

~~Define~~ $t_w t_u = \sum_{z \in W} \gamma_{w,u,z} t_z$

- J is an associative ring (Lusztig, 1987)

~~(I)~~ Let c be a ~~two~~-sided cell of W. J_c is the $\mathbb{Z}$-span of t_w, $w \in c$. Then (Lusztig, 1987)

- J_c is a ~~two~~-sided ideal of J and

$$J = \bigoplus_c J_c$$

where c runs through the set of ~~two~~-sided cells of W

For a ~~two~~-sided cell c let x be a nilpotent element in ~~the~~ nilpotent class ~~correspondi~~ corresponding to c.

Let G be ~~the~~ simply connected algebraic group $/\mathbb{C}$ ~~of of g~~ with Lie algebra $\mathfrak{g}$ and let F_c be a maximal reductive subgroup of $C_G(x)$.

Conj. (Lusztig, 1989)

$\exists$ a finite F_c - set Y such that

(1). There exists a bijection

$\pi : C \longrightarrow \{$ isomph classes of F_c-equivariant VB vector bundles on $Y \times Y\}$

(2). Let $K_{F_c}(Y \times Y)$ be the Grothendieck group of the category F_c-equivariant vector bundles on $Y \times Y$. Convolution defines an associative algebra structure on $K_{F_c}(Y \times Y)$ then ~~the~~ bijection π induces a ring homomorphism

$$J_c \longrightarrow K_{F_c}(Y \times Y)$$

$$t_w \longrightarrow \pi(w)$$

(3) ~~94, 2002.~~ Let Γ be the canonical left cell in c. then $J_{\Gamma \cap \Gamma^{-1}} \simeq R_{F_c}$

Th (Xi). The conjecture is true for $G = SL_3, Sp_4, G_2$ (94). $G = GL_n, SL_n$ (2002)

Th (Bezrukavnikov, 2004). (3) is true

Also. Bezrukavnikov ~~proves~~ and Ostrik proved a weak form of ~~the~~ conj. Using quantum group. Nakajima proved ~~the~~ conj. for GL_n (2008). general case is still open

Let q be a non zero complex number specialize v to q. and consider $\mathcal{H}_q = \mathbb{C} \otimes_A H$

- (Kazhdan-Lusztig, 1987). When q is not a root of 1, ~~the~~ Deligne-Langlands conjecture is true for H_q.

· Xi 2007 when $\sum_{w \in W_0} q^{l(w)} \neq 0$ the Deligne-Langlands conj is true for H_q

The main tool is the based ring J of W

Using Macdonald's formula and other things we have
· (Xi ??) when $\sum_{w \in W_0} q^{l(w)} = 0$ the DL conj for H_q needs a modification

geometric
A realization of H.

~~G R~~ $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$, R the root system, X the weight lattice W_0 the Weyl group
$W = W_0 \ltimes X$ the extended affine Weyl group. G the simply connected semisimple algebraic group.

$\mathcal{B}$ the variety of all Borel subalgebras of $\mathfrak{g}$.
$\mathcal{N}$ the set of nilpotent elements in $\mathfrak{g}$

Set $Z = \{ (N, b, b') \mid b, b' \in \mathcal{B},\ N \in b \cap b' \}$
$G \times \mathbb{C}^*$ acts on Z as
$(g, x).(N, b, b') \to (x^{-2} g.x,\ g.b,\ g.b')$
(action of G is the adjoint action).

One can define a multiplication on the ~~equivariant~~ equivariant K group $K^{G \times \mathbb{C}^*}(Z)$ by convolution

Th. (KL Ginzburg 1987)
$$K^{G \times \mathbb{C}^*}(Z) \simeq H$$

Assume Y is a $G^{\times \mathbb{C}^*}$-invariant closed subvariety of $\mathcal{N}$. Set
$Z_Y = \{ (N, b, b') \in Z \mid N \in Y \}$
then inclusion $i_Y : Z_Y \to Z$

$(i_Y)_* : K^{G\times\mathbb{C}^*}(Z_Y) \longrightarrow K^{G\times\mathbb{C}^*}$

Assume c is a two-sided cell, $\mathcal{C}_c$ is the corresponding nilpotent class, $Y = \overline{\mathcal{C}_c}$ is the closure of $\mathcal{C}_c$

Conj. (Ginzburg, 1987)
$\mathrm{Im}\,(i_Y)_*$ is a free A-module, spanned by all C_u, $u \underset{LR}{\leq} c$.

Tanisaki - Xi (2006) the conjecture is true for GL_n

Xi (2008). $\mathrm{Im}\,(i_Y)_* \otimes \mathbb{Q}$ is a free ~~$\mathbb{Q}[v^{1/2}, q^{-1/2}]$~~ $\mathbb{Q}[v^{1/2}, v^{-1/2}]$ 模, spanned by all C_u, $u \underset{LR}{\leq} c$. That is the Conj is true ~~for~~ after tensoring with $\mathbb{Q}$.