§1. Elliptic orbifold $\mathbb{P}^1$. works joint with. T. Milanov, Y. Ruan, J. Zhou.

orb $\mathbb{P}^1$s $\mathcal{X} =$: $(2\,2\,2\,2)$ $(3\,3\,3)$ $(4\,4\,2)$ $(6\,3\,2)$

$E/\mathbb{Z}_2$ $E/\mathbb{Z}_3$ $E/\mathbb{Z}_4$ $E/\mathbb{Z}_6$

$\mathcal{X}_r = E/\mathbb{Z}_r$: $r=2,3,4,6$. E: some elliptic curve.

GW correlation function: (Ancestors).

$$\langle\langle \alpha_1\psi_1^{k_1},\cdots,\alpha_n\psi_n^{k_n}\rangle\rangle_{g,n}^{\mathcal{X}_r}(q) = \sum_{d\geq 0}\langle \alpha_1\psi_1^{k_1}\cdots\alpha_n\psi_n^{k_n}\rangle_{g,n,\beta}^{\mathcal{X}_r}\, q^d .$$

- $\alpha_i \in H^*_{CR}(\mathcal{X}_r)$. ψ_i: i-th psi-class over $\overline{\mathcal{M}}_{g,n}$.
- $q = e^t$, t parameter for $P \in H^2_{CR}(\mathcal{X}_N,\mathbb{Z})$. $\beta = d\cdot P$.

Thm: $\langle\langle\cdots\rangle\rangle_{g,n}^{\mathcal{X}_r}$ are quasi-modular forms w.r.t. modular group $\Gamma(r)$.

$$\Gamma(r) := \left\{ \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b\\ c & d\end{pmatrix} \equiv \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \bmod r \right\}.$$

Def: $f:\mathbb{H}\to\mathbb{C}$ is Quasi-modular forms of wt m w.r.t. $\Gamma \hookrightarrow SL(2,\mathbb{Z})$ if $\exists\, 0\leq i\leq K$; $f_i:\mathbb{H}\to\mathbb{C}$ holo on $\mathbb{H}$ & near $\tau = i\infty$. $(f_0 = f)$ s.t.

$$f^*(\tau) = f(\tau) + \sum_{i=1}^{K}\frac{f_i(\tau)}{(\mathrm{Im}\,\tau)^i} \text{ is modular. } f^*(g\tau) = (c\tau+d)^m f(\tau,\bar\tau);\ g = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma.$$

E.g. $G_2(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty}\left(\sum_{d|n} d\right) Q^n$ is NOT modular. $Q = e^{2\pi\sqrt{-1}\tau}$.

$G_2^*(\tau) = G_2(\tau) + \frac{1}{2Y}$ is modular. $Y = 4\,\mathrm{Im}(\tau)$.

Proof 1. (recent) Direct computation & Reconstruction (A-model proof)

Proof 2. Milanov-Ruan (2011) works for $\mathcal{X}_3, \mathcal{X}_4, \mathcal{X}_6$,

- Mirror symmetry. Givental-Teleman formula. (B-model proof).
- Upshot: • in GW cycles (M-R-S)
  • HAE: Holomorphic anomaly eqn. $\Big\}$ elliptic curve.

§2. $\mathcal{F}^0$ GW correlators for $\mathcal{X}_2 := \mathbb{P}^1_{2,2,2,2}$. ($\mathcal{X}_N$: Bouchard's principle).

$$\mathcal{H}^*_{CR}(\mathcal{X}_2) = \mathbb{C}\cdot\mathbb{1} \oplus \bigoplus_{i=1}^{4} \mathbb{C}\,\Delta_i \oplus \mathbb{C}\cdot P. \qquad \Delta_i : \text{twisted sector.}$$

$$\quad\; t_0 \qquad\quad t_i \qquad\; t$$

- Genus-0. with no descendants. (Satake-Takahashi).

$$F_0^{\mathcal{X}_2}(t_i, q) = \text{cubic terms} + X\, t_1 t_2 t_3 t_4 + \frac{Y}{4!}\left(\sum_{i=1}^{4} t_i^4\right) + \frac{Z}{2!\,2!}\left(\sum_{i\neq j} t_i^2 t_j^2\right).$$

with $X := \langle\langle \Delta_1, \Delta_2, \Delta_3, \Delta_4 \rangle\rangle_{0,4}$ pairing $(\mathbb{1}, P) = 1$

$Y := \langle\langle \Delta_i, \Delta_i, \Delta_i, \Delta_i \rangle\rangle_{0,4}$ $\neq 0$, $(\Delta_i, \Delta_i) = \frac{1}{2}$.

$Z := \langle\langle \Delta_i, \Delta_i, \Delta_j, \Delta_j \rangle\rangle_{0,4}$, $i \neq j$.

Let $\theta_q = q\frac{d}{dq}$. use WDVV $[\Delta_1\, \Delta_2 \,|\, \Delta_4\, \Delta_4] = [\Delta_1\, \Delta_4 \,|\, \Delta_2\, \Delta_4] \in H_0(\overline{M}_{0,4})$.

Lifting by $\overline{M}_{0,6} \to \overline{M}_{0,4}$, with Δ_3, Δ_4 new insertions.

$$\Rightarrow \langle\langle \Delta_1, \Delta_2, \Delta_3, \Delta_4, P \rangle\rangle_{0,5} = \theta_q X = 4X(Z - Y)$$

Other choices $\Rightarrow$
$$\theta_q Y = 12Z^2 - 4X^2 - 8YZ;$$
$$\theta_q Z = 4X^2 - 4Z^2.$$

Direct computation $\Rightarrow$ $X = 1 + O(q)$; $Y = -\frac{1}{4} + O(q)$; $Z = 0 + O(q)$;

<u>Fact</u>: <u>WDVV</u> $\Rightarrow$. X, Y, Z is uniquely determined.

§2'. Ramanujan-like identities.

The ring of (quasi) modular forms of $\Gamma_0(4)$ are generated by A, B, C, E.

$$A = \theta_3^2(2\tau);\; B = \theta_2^2(2\tau);\; C = \theta_4^2(2\tau);\; E = \partial_\tau \log(B^4 C^4).$$

satisfying
$$\begin{cases} \theta_q A = \frac{1}{2} A(E + C^2 - B^2) \\ \theta_q B = \frac{1}{2} B(E - A^2) \\ \theta_q C = \frac{1}{2} C(E + A^2) \\ \theta_q E = \frac{1}{2}(E^2 - A^4). \end{cases} \qquad Q = q^2 = e^{2\pi\sqrt{-1}\tau}.$$

Observation: WDVV $\Leftrightarrow$ Ramanujan-like identities.

$$\Rightarrow \begin{cases} X = \frac{1}{4} AC; \\ Y = \frac{1}{8}(-3E + A^2 - 2C^2); \\ Z = \frac{1}{8}(-E + A^2). \end{cases} \Rightarrow X, Y, Z \text{ are (quasi-) modular forms of } \Gamma_0(4)$$

Proposition: they are modular w.r.t. $\Gamma(2)$

§3. higher genus.

- Any genus. (g-reduction). (Ionel, Faber-Pandharipande)

If $M(\psi,\kappa)$ monomial of ψ & κ classes, $\deg M \begin{cases} \geq g & g \geq 1 \\ \geq 1. & g = 0 \end{cases}$

then $M(\psi,\kappa)$ = linear comb of dual graphs on the boundary of $\overline{M}_{g,n}$.

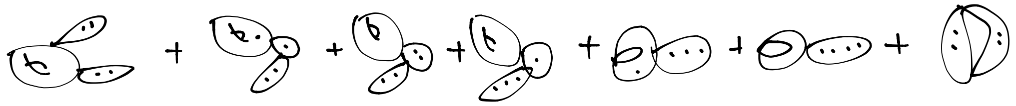
- Dimension axiom: $\langle \alpha_1\psi_1^{k_1} \cdots \alpha_n\psi_n^{k_n} \rangle_{g,n,d} = 0$ unless.

$$\sum \deg \alpha_i + \sum k_i = (3-\dim)(g-1) + c_1(TX_N)\cdot d + n = 2g-2+n.$$

$\Rightarrow \langle\langle \cdots \rangle\rangle_{g,n}^{\mathcal{X}_2}$ is determined by $F_0^{\mathcal{X}_2}, F_1^{\mathcal{X}_2}$.

Explicitly, $\langle\langle \cdots \rangle\rangle_{g,n}^{\mathcal{X}_2}$ = Polynomial of $X, Y, Z, \langle\langle P \rangle\rangle_{1,1}$

• $g=1$ with no ψ: <u>Getzler's relation</u>: in $\overline{M}_{1,4}$. (with some coefficients)



$= 0 \in H_4(\overline{M}_{1,4})$.

<u>Choice of insertions</u>. $\Delta_1, \Delta_2, \Delta_3, \Delta_4$; 4-th term $\sim X \langle\langle P \rangle\rangle_{1,1}$.

$\Rightarrow \langle\langle P \rangle\rangle_{1,1}^{X_2} = \frac{Y}{3} + Z$. $\qquad X$ is divisible on both sides. $\square$

<u>Rmk</u>: This also works for X_3, X_4, X_6. & Elliptic curve.

<u>Cor</u>: All generators of modular forms w.r.t. $\Gamma(N)$ are GW correlators. $N=2,3,4,6$.

· wt 1 modular forms are forms of $\langle\langle \cdots \rangle\rangle_{0,3}$

<u>More</u> <u>Higher genus</u> can be calculated using Pixton's relation on $H^*(\overline{M}_{g,n})$.

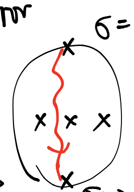
§4. Previous methods. Milanov-Ruan. (works for X_3, X_4, X_6)

$X_3 = \{W=0\} / (G_W / \langle J \rangle)$ $\quad W = X^3 + Y^3 + Z^3$. $\quad G_W = (\mathbb{Z}_3)^3 \ni J$, order $=3$

GW of X_3 $\xleftrightarrow{\text{LG-CY mirror}}$ (sphere) $\Leftarrow$ Saito Givental of $W + \sigma XYZ$ (Global B-model)

$\frac{LG}{CY}$? $\updownarrow$ $\qquad$ (Krawitz-S) $\Uparrow \Downarrow$ $\qquad$ $\sigma = \infty$ $\qquad$ $\sigma = 0$

FJRW of (W, G_W) $\xleftrightarrow[\text{LG-LG mirror}]{}$ (sphere) $\qquad$ Milanov-Ruan.



Thm: $\mathcal{A}^{GW}_{X_3}(q) = \mathcal{A}^{SG}_W(\tau) = \lim_{t \to (\tau,0)} \mathcal{A}^{SG}_W(t)$ $\Leftarrow$ Givental-Teleman formula

· Mirror map: $t = 2\pi i \tau / 3$. $\quad \tau = \frac{\Pi_B(\sigma)}{\Pi_A(\sigma)} \in \mathbb{H}$.

• Π_A, Π_B. Certain periods • $\{A, B\}$ symp basis of $H_1(E_\sigma, \mathbb{Z})$. $\quad E_\sigma = \{W + \sigma XYZ = 0\} \subseteq \mathbb{P}^2$

Antiholomorphic completion for $\mathcal{A}^{SG}_{W}(\tau)$.

$$\mathcal{A}^{SG}_{W}(\tau,\bar\tau) := \widehat{X_{\tau,\bar\tau}(z)}\, \mathcal{A}^{SG}_{W}(\tau) \;.\; X_{\tau,\bar\tau}(z) = \begin{pmatrix} 1 & -\frac{z}{\tau-\bar\tau} \\ 0 & 1 \end{pmatrix} \oplus I_6 \in \mathrm{End}(H)[[z]]$$

Thm: (Milanov-Ruan). Coefficients of $\mathcal{A}^{SG}_{W}(\tau,\bar\tau)$ are quasi-modular. w.r.t. $\Gamma(3)$

Rmk: Modular group for E_6 is $\Gamma_0(3)$.

$\Gamma(3) \hookrightarrow \Gamma_0(3)$ because of twisted sectors.

Rmk. This holds in cohomological field theory level. $\Lambda_{g,n}: H^{\otimes n} \to H^*(\overline{M}_{g,n})$

HAE. $(t-\bar t)\frac{\partial}{\partial \bar t}\Lambda_{g,n}(\mathbb{1},\cdots,\mathbb{1}) = \frac{1}{2}(\text{tree term} + \text{loop term}) + \sum \Lambda_{g,n}(\mathbb{1},\cdots,K\mathbb{1},\cdots\mathbb{1})$

$K(\alpha_i\psi^k) = \begin{cases} \psi \\ 0 \end{cases} :\ \text{if } \alpha_i\psi^k = P$

Conj: HAE holds also for $\mathbb{P}^1_{2,2,2,2}$ & Elliptic curve.

Numerical version of HAE can be verified via g-reduction.