Moonshine and String Theory

Timm Wrase

IPMU August 7, 2014

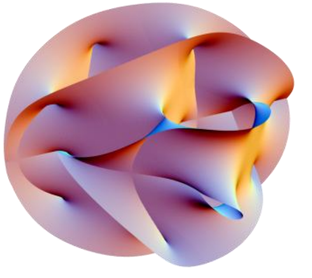
Based on:

N. Paquette, TW to appear
M. Cheng, X. Dong, J. Duncan, S. Harrison, S. Kachru, TW 1406.5502
TW 1402.2973
M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

Outline

- Introduction to moonshine

- Mathieu Moonshine and string compactifications



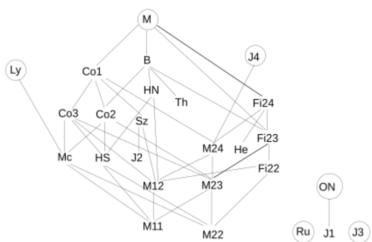
- New moonshine phenomena

Moonshine

Group Theory

Representation Theory

finite (sporadic) groups



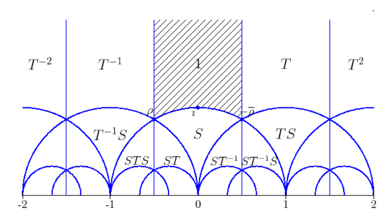
String Theory

Complex Analysis

Number Theory

(mock) modular forms,

Jacobi forms



Finite simple groups

There are 18 infinite families, e.g.

- Alternating group of n elements A_n

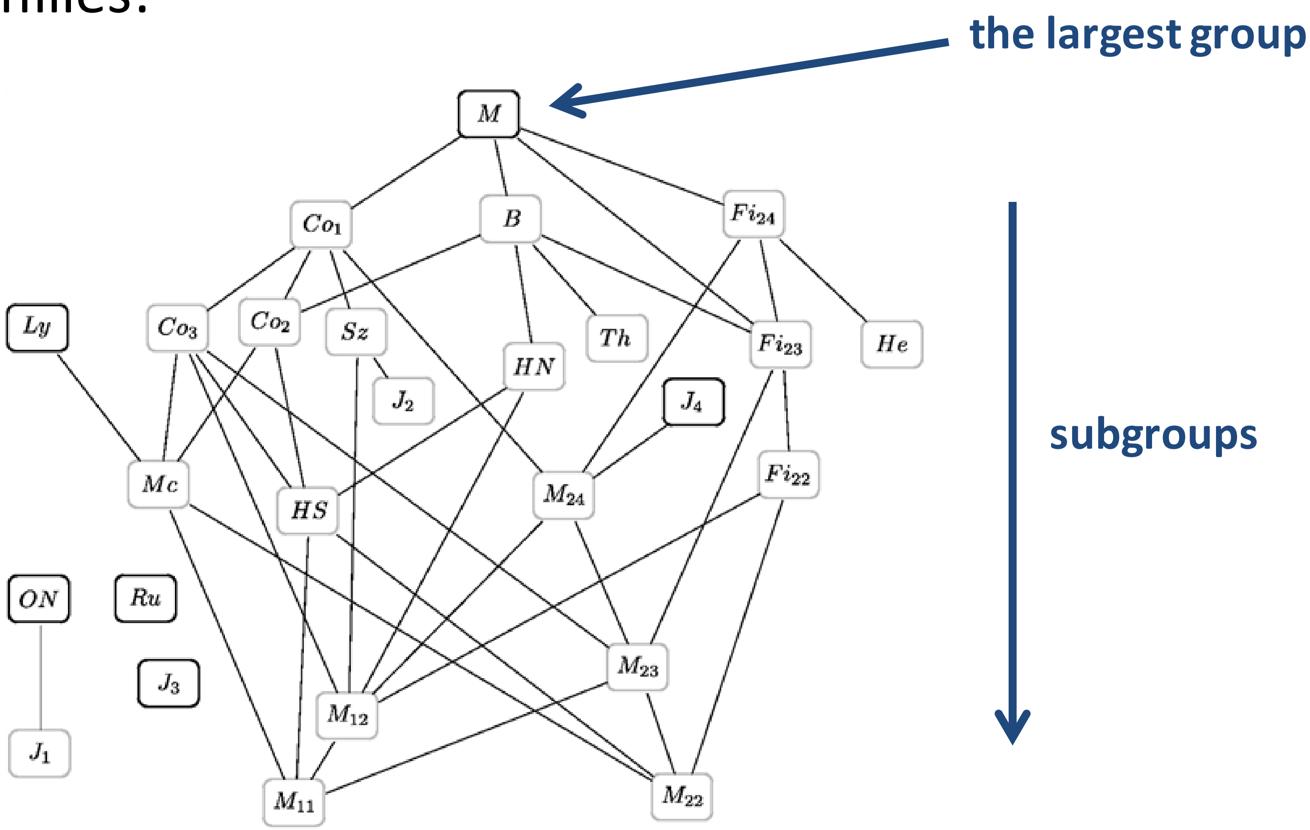
  e.g. A_3: $(123) \leftrightarrow (231) \leftrightarrow (312)$

- Cyclic groups of prime order C_p

  e.g.: $C_p = \mathbb{Z}_p = \langle e^{2\pi i/p} \rangle$

Finite simple groups

There are also 26 so called sporadic groups that do not come in infinite families:



Finite simple groups

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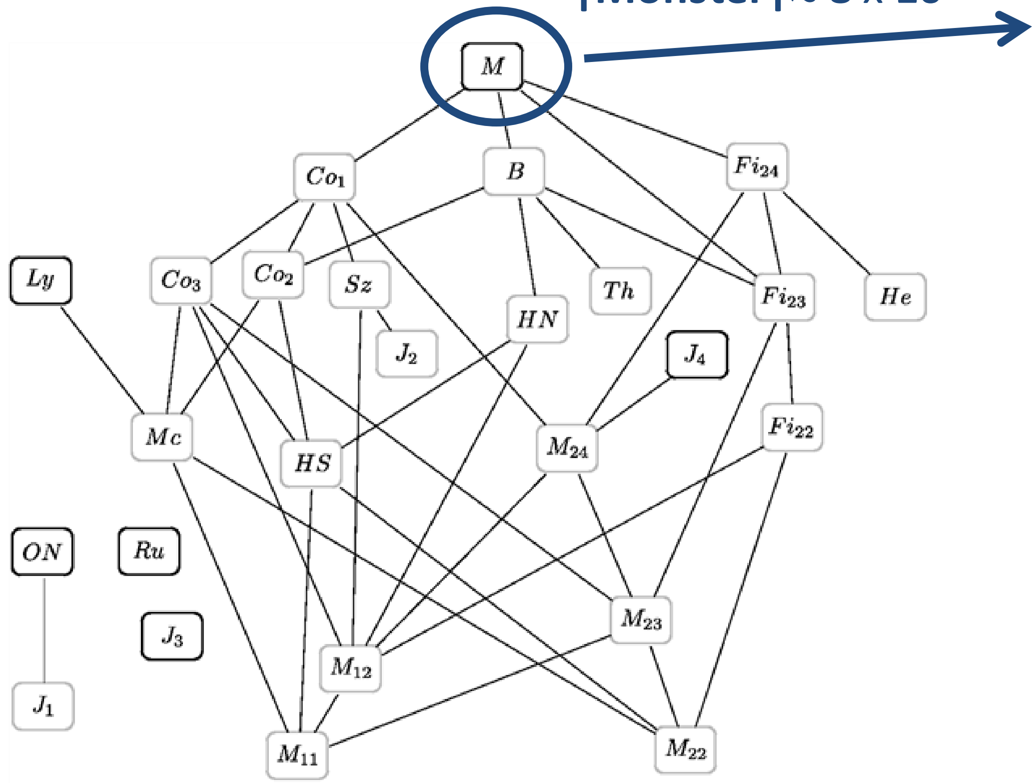
|Monster| ≈ 8 x 10^{53}



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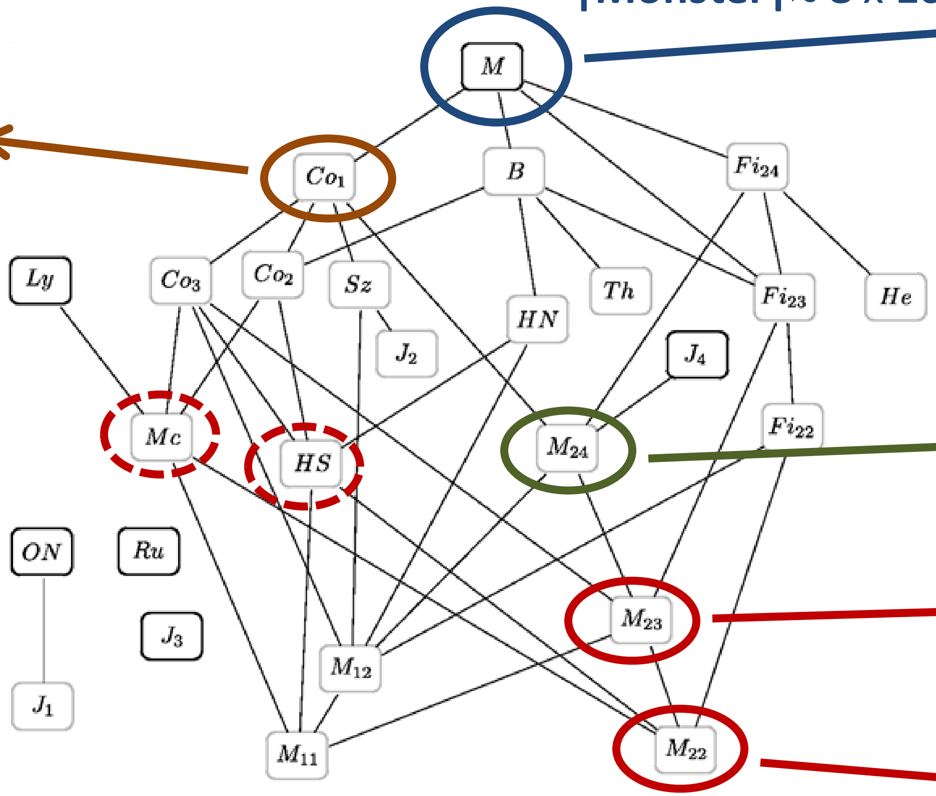

Conway group
≈ 4 x 10^{18}

Largest Mathieu group ≈ 2 x 10^{9}

$|M_{23}|$ = 10,200,960

$|M_{22}|$ = 443, 520

Modular Forms

Modular function of weight *k*

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{k}\, f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z})$$

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Jacobi form of weight *k* and index *m*

$$f\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} f(\tau,z)$$

$$f(\tau, z+\lambda\tau+\mu)=e^{-2\pi i m(\lambda^2\tau+\lambda z)} f(\tau,z), \qquad \lambda,\mu\in\mathbb{Z}$$

Modular Forms

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$$f(\tau,z+\lambda\tau+\mu)=e^{-2\pi i m(\lambda^2\tau+\lambda z)} f(\tau,z), \qquad \lambda,\mu\in\mathbb{Z}$$

Can Fourier expand

$$f(\tau,z)= f(q=\exp[2\pi i\tau], y=\exp[2\pi i z])=\sum_{n\geq 0}\sum_{r^2\leq 4mn} c(n,r)\, q^n y^r$$

Monstrous Moonshine

- The irreducible representations of the Monster group have dimensions 1, 196 883, 21 296 876, …

- The J-function, that appears in many places in string theory, enjoys the expansion

$$J(q) = \frac{1}{q} + 196884\,q + 21493760\,q^2 + \ldots$$

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- as observed by John McKay in the late 70's

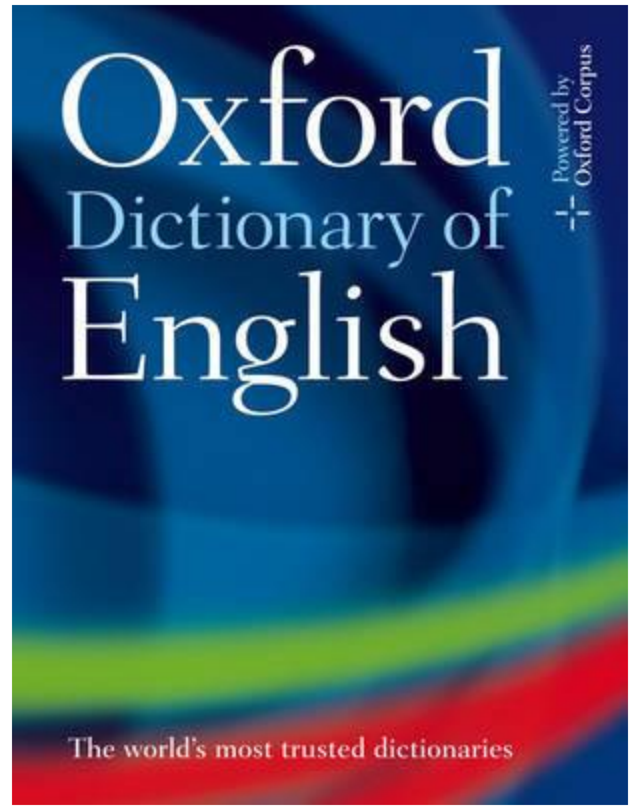
Monstrous Moonshine

moon·shine

ˈmo͞onˌSHīn/

noun informal
noun: **moonshine**

1. foolish talk or ideas.



Monstrous Moonshine

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noun informal
noun: **moonshine**

1. foolish talk or ideas.

2. NORTH AMERICAN
illicitly distilled or smuggled liquor.

Monstrous Moonshine

This surprising connection can be explained by string theory:

Frenkel, Lepowsky, Meurman 1988

- The (left-moving) bosonic string compactified on a $\mathbb{Z}_2$ orbifold of $\mathbb{R}^{24}/\Lambda$ with Λ the Leech lattice has as its 1-loop partition function the *J(q)*-function

$$Z(q) = \mathrm{Tr}_{\mathrm{H}}\, q^{L_0 - \frac{c}{24}} = J(q) = \frac{1}{q} + 196884\, q + 21493760\, q^2 + \ldots$$

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no massless q^0 states

$$Z(q) = \mathrm{Tr}_{\mathrm{H}}\, q^{L_0 - \frac{c}{24}} = J(q) = \frac{1}{q} + \underbrace{196884\, q + 21493760\, q^2 + \ldots}_{\text{supermassive string states}}$$

tachyon of the bosonic string

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- The symmetry group of the compactification space $\mathbb{R}^{24}/\Lambda/\mathbb{Z}_2$ is the Monster group.

Monstrous Moonshine

Since we have a Virasoro algebra we can expand the *J(q)*-function in terms of Virasoro characters (traces of Verma modules)

$$\mathrm{ch}_{h=0}(q) = \frac{q^{-c/24}}{\prod_{n=2}^{\infty}(1-q^n)}\,, \qquad \mathrm{ch}_h(q) = \frac{q^{h-c/24}}{\prod_{n=1}^{\infty}(1-q^n)}$$

$$J(q) = \frac{1}{q} + 196884\,q + 21493760\,q^2 + \ldots$$

$$= \boxed{1}\,\mathrm{ch}_0(q) + \boxed{196883}\,\mathrm{ch}_2(q) + \boxed{21296876}\,\mathrm{ch}_3(q) + \ldots$$

Monstrous Moonshine

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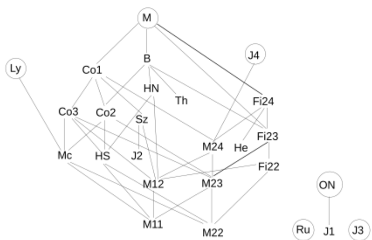

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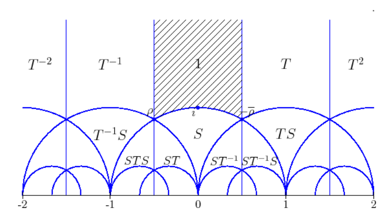

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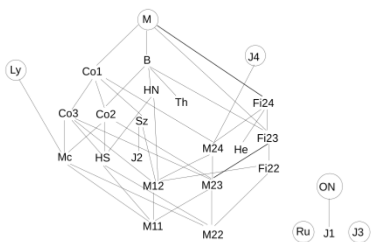

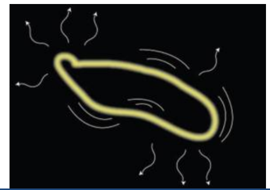

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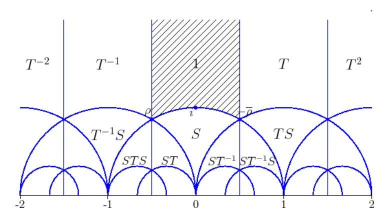

Very interesting for mathematicians!

Monstrous Moonshine

Compactification of the bosonic string:

$\Longrightarrow$ we have a tachyon (instability)

$\Longrightarrow$ spacetime theory has no fermions

Additionally

- Only two spacetime dimensions are non-compact

Monstrous Moonshine

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Additionally

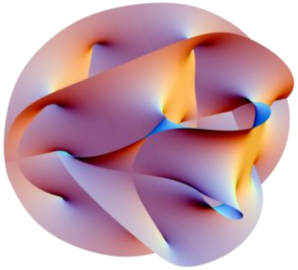
- Only two spacetime dimensions are non-compact

Not so interesting for physicists!

Outline

- Introduction to moonshine

- **Mathieu Moonshine and string compactifications**



- New moonshine phenomena

Mathieu Moonshine

- In 2010 Eguchi, Ooguri and Tachikawa discovered a new moonshine phenomenon that connects K3 to the largest Mathieu group M_{24}

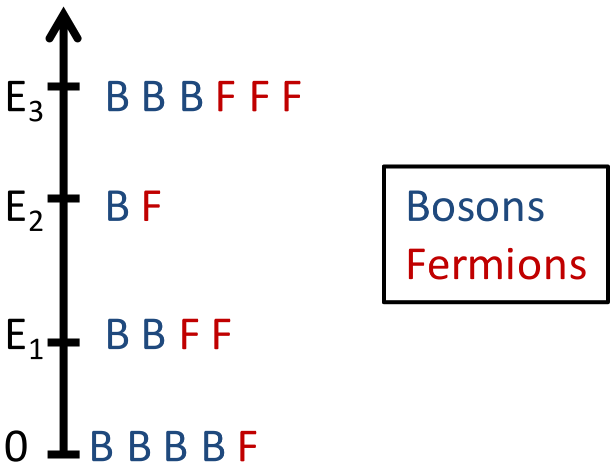
Eguchi, Ooguri, Tachikawa 1004.0956

- They considered a N=(4,4) SCFT with K3 target and calculate an index that is called elliptic genus

Mathieu Moonshine

- The Witten index

$$Z_{\text{Witten}} = \mathrm{Tr}(-1)^F$$
$$= n_B - n_F$$



Mathieu Moonshine

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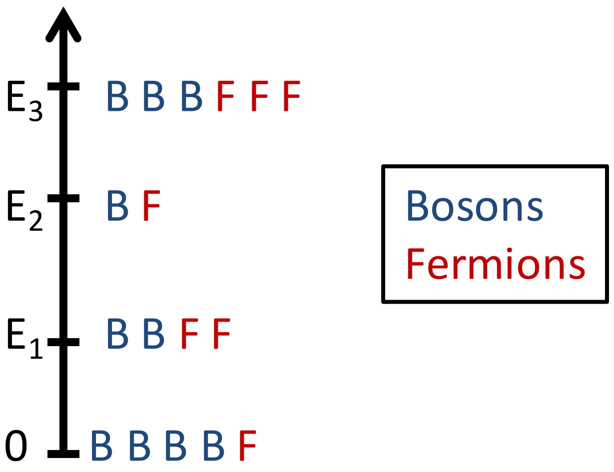

Mathieu Moonshine

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$$
\begin{aligned}
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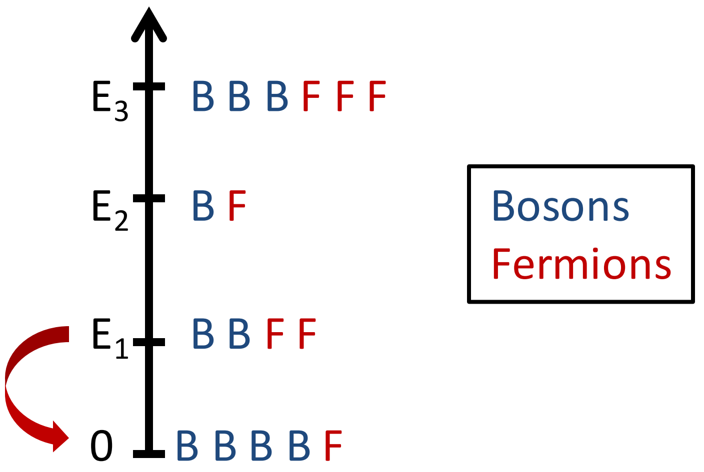
$$



- An index is invariant under deformations of the theory, e.g. masses go to zero

Mathieu Moonshine

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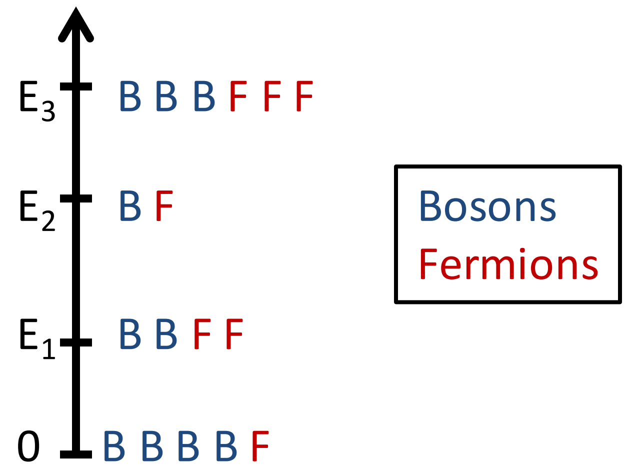

$$Z_{\text{elliptic}}(q,y) = \mathrm{Tr}_{\mathrm{RR}}\left(\overbrace{(-1)^{F_L}\, q^{L_0-\frac{c}{24}}}\, y^{J_0}\, \overbrace{(-1)^{F_R}\, \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}}\right)$$

No dependence on $\bar{q}$!

Mathieu Moonshine

Chemical potential for U(1)
in left-moving N=2 theory

$$Z_{\text{elliptic}}(q,y) = \mathrm{Tr}_{\mathrm{RR}}\left((-1)^{F_L}\, q^{L_0 - \frac{c}{24}} y^{J_0} \underbrace{(-1)^{F_R}\, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}} \right)$$

Witten index:
No dependence on $\bar{q}$

Mathieu Moonshine

$$Z^{\mathrm{K3}}_{\mathrm{elliptic}}(q,y)=8\left(\frac{\theta_2(q,y)^2}{\theta_2(q,1)^2}+\frac{\theta_3(q,y)^2}{\theta_3(q,1)^2}+\frac{\theta_4(q,y)^2}{\theta_4(q,1)^2}\right)$$

T. Eguchi, H. Ooguri, A. Taormina, S.-K. Yang Nucl. Phys. B 315, 193 (1989)

We have N=(4,4) world sheet supersymmetry

$\Longrightarrow$ expand in N=4 Virasoro characters

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N=4 Virasoro characters are defined by the trace over the highest weight state and all its descendants

$$\mathrm{ch}_{\mathrm{h,l}}(q,y)=\mathrm{Tr}\left((-1)^{F_L}\, q^{L_0-\frac{c}{24}} y^{J_0}\right)$$

For the case h=c/24 there are short BPS multiplets

Mathieu Moonshine

$$Z_{\text{elliptic}}^{\text{K3}}(q,y)=8\left(\frac{\theta_2(q,y)^2}{\theta_2(q,1)^2}+\frac{\theta_3(q,y)^2}{\theta_3(q,1)^2}+\frac{\theta_4(q,y)^2}{\theta_4(q,1)^2}\right)$$

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T. Eguchi, K. Hikami 0904.0911

$$Z_{\text{elliptic}}^{\text{K3}}=24\,\mathrm{ch}_{h=\frac{1}{4},l=0}^{\text{short}}-2\,\mathrm{ch}_{h=\frac{1}{4},l=\frac{1}{2}}^{\text{long}}+\sum_{n=1}^{\infty}A_n\,\mathrm{ch}_{h=\frac{1}{4}+n,l=\frac{1}{2}}^{\text{long}}$$

$$A_n=\{90,\ 462,1440,\ldots\}$$

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$$Z^{\mathrm{K3}}_{\mathrm{elliptic}}=\underset{\substack{\downarrow\\ 23+1}}{24}\,\mathrm{ch}^{\mathrm{short}}_{h=\frac{1}{4},l=0}-2\,\mathrm{ch}^{\mathrm{long}}_{h=\frac{1}{4},l=\frac{1}{2}}+\sum_{n=1}^{\infty}A_n\,\mathrm{ch}^{\mathrm{long}}_{h=\frac{1}{4}+n,l=\frac{1}{2}}$$

T. Eguchi, H. Ooguri, Y. Tachikawa 1004.0956

$$A_n=\{45+\overline{45},\ 231+\overline{231},770+\overline{770},\ldots\}$$

Dimensions of Irreps of M_{24}

Mathieu Moonshine

Does this imply a connection between M_{24} and K3?

- The geometric symmetries of K3 are contained in $M_{23} \subset M_{24}$

Mukai, Kondo 1988, 1998

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Gaberdiel, Hohenegger, Volpato 1106.4315

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Gaberdiel, Hohenegger, Volpato 1106.4315

- However, all the A_n are sums of dimensions of irreps of M_{24} with positive coefficients

Gannon 1211.5531

Mathieu Moonshine

- K3 has played a central role in string compactifications and string dualities
- What are implications we can derive from Mathieu moonshine for string compactifications?
- Has the elliptic genus of K3 already appeared in the string theory literature?

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YES!

Heterotic String Theory

- Consider the heterotic $E_8 \times E_8$ string theory compactified on K3 x T^2
- We need to embed 24 instantons into $E_8 \times E_8 \longrightarrow (12+n, 12-n)$ $n = 0,1,\ldots,12$ to satisfy the Bianchi identity for H_3

Heterotic String Theory

- Consider the heterotic E_8 x E_8 string theory compactified on K3 x T^2
- We need to embed 24 instantons into E_8 x E_8 $\longrightarrow$ (12+n,12-n) n = 0,1,…,12 to satisfy the Bianchi identity for H_3
- The resulting four dimensional theories preserves N=2 spacetime supersymmetry
- The 1-loop corrections to the prepotential are related to the new supersymmetric index Z_{new}

Dixon, Kaplunovsky, Louis, de Wit, Lüst, Stieberger, Antoniadis, Narain, Taylor, Gava, Kiritsis, Kounnas, Harvey, Moore, ….

$$h(S,T,U) = h^{\text{tree}} + h^{1-loop} + O(e^{-2\pi i S})$$

Heterotic String Theory

For K3 x T^2 compactifications we have for the standard embedding that preserves N=(4,4)

Affine E$_8$

SO(12) characters

$$Z_{\text{new}}(q;T,U)=\frac{i}{2}\frac{\Theta_{\Gamma_{2,2}}(T,U)}{\eta(q)^4}\frac{\mathrm{E}_4(\mathrm{q})}{\eta(\mathrm{q})^8}\left[\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 Z^{\mathrm{K3}}_{\text{elliptic}}(q,-1)\right.$$
$$\left.+\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}} Z^{\mathrm{K3}}_{\text{elliptic}}(q,-q^{\frac{1}{2}})+\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}} Z^{\mathrm{K3}}_{\text{elliptic}}(q,q^{\frac{1}{2}})\right]$$

SO(12) characters

Harvey, Moore hep-th/9510182

T is the complexified Kähler modulus, *U* the complex structure modulus of the T^2

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SO(12) characters

Harvey, Moore hep-th/9510182

So in particular the [...] part has an "SO(12)xM_{24}"-expansion: exactly the same M_{24} as in Mathieu Moonshine due to *N=(4,4)*

Heterotic String Theory

For K3 x T^2 compactifications we have for the standard embedding that preserves N=(4,4)

Affine E_8

$$\downarrow$$

$$Z_{\text{new}}(q;T,U)=\frac{i}{2}\frac{\Theta_{\Gamma_{2,2}}(T,U)}{\eta(q)^4}\frac{\mathrm{E}_4(\mathrm{q})}{\eta(\mathrm{q})^8}\frac{\mathrm{E}_6(\mathrm{q})}{\eta(\mathrm{q})^{12}}=\frac{i}{2}\frac{\Theta_{\Gamma_{2,2}}(T,U)\mathrm{E}_4(\mathrm{q})\mathrm{E}_6(\mathrm{q})}{\eta(\mathrm{q})^{24}}$$

Harvey, Moore hep-th/9510182

So in particular the E_6(q) has an “SO(12)xM_{24}”-expansion

Heterotic String Theory

For K3 x T^2 compactifications we have for the standard embedding that preserves N=(4,4)

Take away message:

Z_{new} depends on *T* and *U* and is connected to $Z_{elliptic}$ and therefore to M_{24}

T is the complexified Kähler modulus, *U* the complex structure modulus of the T^2

Heterotic String Theory

The 1-loop correction to the prepotential is roughly determined by

$$\Delta(T,U) = \int \frac{d^2\tau}{\tau_2} Z_{\text{new}}(q = e^{2\pi i\tau}; T, U)\,(Q^2 - \tfrac{1}{8\pi\tau_2})$$

and knows about M_{24} since Z_{new} does

M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

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M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

The modular invariance of $\tau_2 Z_{\rm new}(Q^2 - \tfrac{1}{8\pi\tau_2})$ actually tells us that there is a unique solution. So for all instanton embeddings (12+n,12-n) the answer is the same.

Kiritsis, Kounnas, Petropoulos, Rizos hep-th/9608034

Henningson, Moore hep-th/9608145

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and knows about M_{24} since Z_{new} does

M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

We have to solve the following second order differential equation

Harvey, Moore hep-th/9510182

$$-\mathrm{Re}\left(\partial_T\partial_U h^{1-loop} + \frac{1}{T_1U_1}(1 - T_1\partial_T - U_1\partial_U)h^{1-loop}\right) - \frac{1}{\pi}\mathrm{Re}(\log[J(iT) - J(iU)])$$

$$= \frac{1}{2\pi}\int\frac{d^2\tau}{\tau_2}\left(-iZ_{new}(q;T,U)\cdot(Q_{E_8}^2 - \tfrac{1}{8\pi\tau_2}) - b(E_8)\right) + \frac{b(E_8)}{2\pi}(\log[2T_1U_1] + 4\,\mathrm{Re}(\log[\eta(iT)\eta(iU)]))$$

Heterotic String Theory

The solution is given by

$$h^{1-loop} = -\tfrac{1}{3}U^3 + C + \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)})$$

where the polylogarithm is given by $Li_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}$

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and the expansion coefficients are the same as in our index (they go along for the ride when integrating)

$$Z_{\text{new}}(q;T,U) = \frac{i}{2}\frac{\Theta_{\Gamma_{2,2}}(T,U)\mathrm{E}_4(\mathrm{q})\mathrm{E}_6(\mathrm{q})}{\eta(\mathrm{q})^{24}} = \frac{i}{2}\Theta_{\Gamma_{2,2}}(T,U)\left(\sum_{m\geq -1} c(m)\, q^m\right)$$

Dimensions of M_{24}

Heterotic String Theory

Dimensions of M_{24}
(appearing in a spacetime quantity)

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Dimensions of M_{24}

Type IIA on CY_3

String duality

Heterotic string on K3 x T^2 with instanton embedding (12+n,12-n)

Type IIA string theory on elliptic fibrations over F_n (Hirzebruch surface)

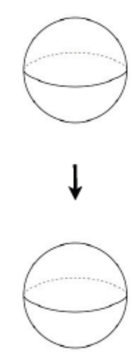
Type IIA on CY_3

String duality

Heterotic string on K3 x T^2 with instanton embedding (12+n,12-n)

Type IIA string theory on elliptic fibrations over F_n (Hirzebruch surface)

F_n



dilaton S $\longleftrightarrow$ Size of base S^2

Type IIA on CY_3

Type IIA string theory on elliptic fibrations over F_{n}:

- Prepotential receives instanton corrections
- These are determined by the Gromow-Witten invariants $\approx$ curve counting (S^2, T^2, …)

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M. Cheng, X. Dong, J. Duncan,
J. Harvey, S. Kachru, TW 1306.4981

$$h(S,T,U) = -STU - \tfrac{1}{3}U^3 + C + \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)}) + O(e^{-2\pi i S})$$

M. Alim, E. Scheidegger 1205.1784
A. Klemm, J. Manschot, T. Wotschke 1205.1795

Dimensions of M_{24}
(appearing in a spacetime quantity)

Type IIB on CY_3

String duality

Type IIA string theory on elliptic fibrations over F_n (Hirzebruch surface)

Type IIB string theory on mirror manifold

Type IIB on CY_3

String duality

Type IIA string theory on elliptic fibrations over F_n (Hirzebruch surface)	Type IIB string theory on mirror manifold

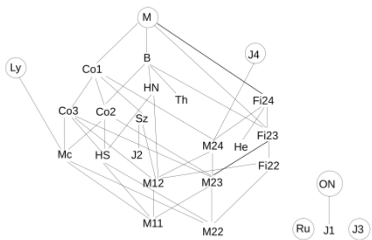
CY_3 manifold X_n ⟷ CY_3 manifold Y_n

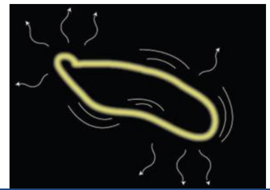
Gromov-Witten invariants ⟷ Periods of the holo-morphic 3-form Ω

New math connections

Group Theory
Representation Theory

finite (sporadic) groups



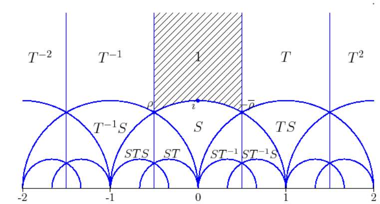


String Theory

NEW

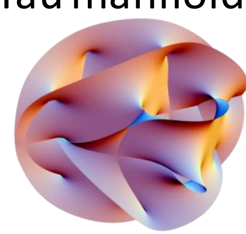
Complex Analysis
Number Theory

(mock) modular forms,
Jacobi forms



(Algebraic) Geometry

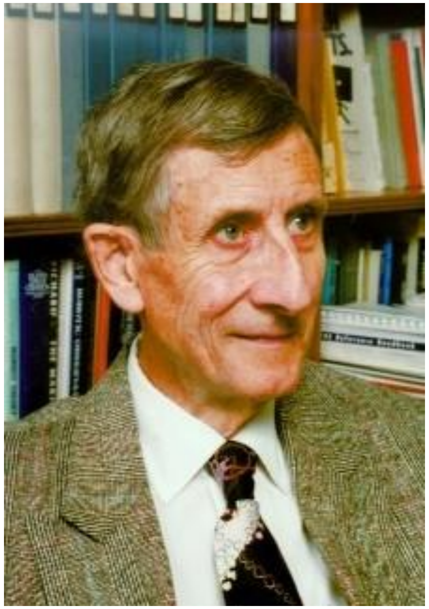
periods of Calabi-Yau manifolds,
Gromov-Witten invariants,
elliptic genus,
…



Moonshine and physics

"I have a sneaking hope, a hope unsupported by any facts or any evidence, that sometime in the twenty-first century physicists will stumble upon the Monster group, built in some unsuspected way into the structure of the Universe."

– Freeman Dyson (1983)



Moonshine and physics

For the $K3 \times T^2$ compactifications, the 1-loop prepotential controls the 1-loop corrections to the gauge couplings in the N=2 spacetime theory

Moonshine and physics

For the K3xT^2 compactifications, the 1-loop prepotential controls the 1-loop corrections to the gauge couplings in the N=2 spacetime theory

For four dimensional N=1 models obtained from orbifold compactifications of the heterotic E_8 x E_8 string theory:

$$f_\alpha(S,T,U) = S + f_\alpha^{1\text{-loop}}(T,U) + O(e^{-2\pi i S})$$

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The (bulk) moduli dependent 1-loop correction to the gauge kinetic function arises only from N=2 subsectors!

Dixon, Louis, Kaplunovsky Nuclear Physics B 355 (1991)

Moonshine and physics

Example $T^6/\mathbb{Z}_{6\text{-}II}$ = $T^2 \times T^2 \times T^2/\mathbb{Z}_{6\text{-}II}$:

$$\mathbb{Z}_{6-II} = \langle g \rangle, \qquad g : (z_1, z_2, z_3) \rightarrow (e^{\pi i/3} z_1, e^{2\pi i/3} z_2, -z_3)$$

Moonshine and physics

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has two N=2 subsector

$$\mathbb{Z}_3 = \{1, g^2, g^4\}, \qquad g^2 : (z_1, z_2, z_3) \to (e^{2\pi i/3} z_1, e^{4\pi i/3} z_2, z_3)$$

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Moonshine and physics

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For which the internal space is $T^4/\mathbb{Z}_3 \times T^2$ or $T^4/\mathbb{Z}_2 \times T^2$ respectively and therefore an orbifold limit of $T^2 \times K3$.

Moonshine and physics

N=2 sectors lead to 1-loop corrections

N=2 prepotential

$$f_{\alpha}^{1-\text{loop}}(T,U) = \sum_{i=1,2,3} \frac{|G_i'|}{|G|} \Bigg[-\frac{1}{2} \partial_{T_i} \partial_{U_i} h_i^{1-loop}(T_i,U_i)$$

N=1 gauge
kinetic coupling

$$- \frac{1}{8\pi^2} \log[J(iT_i) - J(iU_i)] - \frac{b_{\alpha,i}^{N=2}}{4\pi^2} (\log[\eta(iT_i)\eta(iU_i)]) \Bigg]$$

Moonshine and physics

N=2 sectors lead to 1-loop corrections

N=2 prepotential

$$f_\alpha^{1-\mathrm{loop}}(T,U) = \sum_{i=1,2,3} \frac{|G_i'|}{|G|}\left[-\frac{1}{2}\partial_{T_i}\partial_{U_i} h_i^{1-loop}(T_i,U_i) \right.$$
$$\left. -\frac{1}{8\pi^2}\log[J(iT_i)-J(iU_i)-\frac{b_{\alpha,i}^{N=2}}{4\pi^2}(\log[\eta(iT_i)\eta(iU_i)]) \right]$$

N=1 gauge kinetic coupling

where the prepotential was calculated above

$$h^{1-loop}(T,U) = -\tfrac{1}{3}U^3 + C + \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)})$$

Dimensions of M_{24}

Moonshine and physics

Group $\mathbb{Z}_N$	Generator $\frac{1}{N}(\varphi_1, \varphi_2, \varphi_3)$	$\mathcal{N}=2$ moduli
$\mathbb{Z}_3$	$\frac{1}{3}(1,1,1)$	-
$\mathbb{Z}_4$	$\frac{1}{4}(1,1,2)$	T_3, U_3
$\mathbb{Z}_{6-I}$	$\frac{1}{6}(1,1,4)$	T_3
$\mathbb{Z}_{6-II}$	$\frac{1}{6}(1,2,3)$	T_2, T_3, U_3
$\mathbb{Z}_7$	$\frac{1}{7}(1,2,4)$	-
$\mathbb{Z}_{8-I}$	$\frac{1}{8}(1,2,5)$	T_2
$\mathbb{Z}_{8-II}$	$\frac{1}{8}(1,3,4)$	T_3, U_3
$\mathbb{Z}_{12-I}$	$\frac{1}{12}(1,4,7)$	T_2
$\mathbb{Z}_{12-II}$	$\frac{1}{12}(1,5,6)$	T_3, U_3

No N=2 sectors

Moonshine and physics

$\mathbb{Z}_N \times \mathbb{Z}_M$	1^{st} generator $\frac{1}{N}(\varphi_1, \varphi_2, \varphi_3)$	2^{nd} generator $\frac{1}{M}(\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3)$	$\mathcal{N} = 2$ moduli
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{1}{2}(1,0,1)$	$\frac{1}{2}(0,1,1)$	$T_1, U_1, T_2, U_2, T_3, U_3$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\frac{1}{2}(1,0,1)$	$\frac{1}{4}(0,1,3)$	T_1, U_1, T_2, T_3
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\frac{1}{2}(1,0,1)$	$\frac{1}{6}(0,1,5)$	T_1, U_1, T_2, T_3
$\mathbb{Z}_2 \times \mathbb{Z}'_6$	$\frac{1}{2}(1,0,1)$	$\frac{1}{6}(1,1,4)$	T_1, T_2, T_3
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\frac{1}{3}(1,0,2)$	$\frac{1}{3}(0,1,2)$	T_1, T_2, T_3
$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\frac{1}{3}(1,0,2)$	$\frac{1}{6}(0,1,5)$	T_1, T_2, T_3
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\frac{1}{4}(1,0,3)$	$\frac{1}{4}(0,1,3)$	T_1, T_2, T_3
$\mathbb{Z}_6 \times \mathbb{Z}_6$	$\frac{1}{6}(1,0,5)$	$\frac{1}{6}(0,1,5)$	T_1, T_2, T_3

Moonshine and physics

Four dimensional N=1 models obtained from orbifold compactifications of the heterotic E_8 x E_8 string theory receive universal 1-loop corrections to their gauge kinetic functions that are related to M_{24}

TW 1402.2973

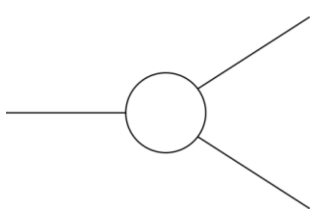
Moonshine and physics

Four dimensional N=1 models obtained from orbifold compactifications of the heterotic E_8 x E_8 string theory receive universal 1-loop corrections to their gauge kinetic functions that are related to M_{24}

TW 1402.2973

For all $T^6/\mathbb{Z}_N$, $N \neq 3,7$, and all $T^6/\mathbb{Z}_N x \mathbb{Z}_M$

$$f(S,T,U) \approx S + \partial_T \partial_U \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)}) + \ldots + O(e^{-2\pi i S})$$



Dimensions of M_{24}

Moonshine and physics

N. Paquette, TW work in progress:

- The holomorphic 3-form Ω plays a role in flux compactifications

Gukov, Vafa, Witten hep-th/9906070
Giddings, Kachru, Polchinski hep-th/0105097

$$W = \int H \wedge \Omega + \ldots$$

Moonshine and physics

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$$W = \int H \wedge \Omega + \ldots$$

- The Yukawa couplings in heterotic models are given by the third derivative of Ω with respect to the moduli

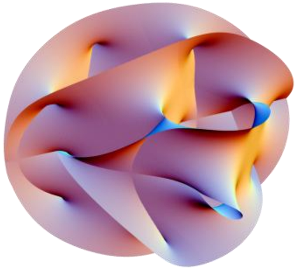
Hosono, Klemm, Theisen, Yau hep-th/9308122

$$Y_{IJK} \approx \partial_I \partial_J \partial_K h(S,T,U)$$

Outline

- Introduction to moonshine

- Mathieu Moonshine and string compactifications



- **New moonshine phenomena**

New Moonshine phenomena

Consider eight (left-moving) bosons and fermions compactified on the orbifold $T^8/\mathbb{Z}_2 = \mathbb{R}^8/\Lambda_{E8} / \mathbb{Z}_2$

$$\mathbb{Z}_2\text{:}\ (X^I, \psi^I) \to -(X^I, \psi^I)$$

The partition function in the NS sector is

Frenkel, Lepowsky, Meurman 1985

$$Z(q) = \frac{1}{\sqrt{q}} + 276\sqrt{q} + 2048\,q + 11202\,q^{\frac{3}{2}} + \ldots$$

Dimensions of representations of Conway group

New Moonshine phenomena

- The Conway symmetry is not manifest in this description

- The theory is equivalent to a theory of 24 chiral fermions orbifolded by a $\mathbb{Z}_2$ symmetry: $\psi^I \to -\psi^I$

J. Duncan math/0502267

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J. Duncan math/0502267

- One can construct an *N=1* superalgebra that breaks the *Spin(24)* symmetry to Co_0
- This explains this Conway moonshine

New Moonshine phenomena

- The toroidal orbifold $T^8/\mathbb{Z}_2 = \mathbb{R}^8/\Lambda_{E8} / \mathbb{Z}_2$ preserves *N=4* worldsheet supersymmetry
- Calculate the partition function and expand in *N=4* characters:

$$Z = \mathrm{Tr}\left((-1)^F q^{L_0 - c/24} y^{J_0}\right) = 21\,\mathrm{ch}^{\mathrm{short}}_{h=\frac{1}{2},l=0} + \mathrm{ch}^{\mathrm{short}}_{h=\frac{1}{2},l=1}$$

$$+ 560\,\mathrm{ch}^{\mathrm{long}}_{h=\frac{3}{2},l=\frac{1}{2}} + 8470\,\mathrm{ch}^{\mathrm{long}}_{h=\frac{5}{2},l=\frac{1}{2}} + \ldots$$

$$+ 210\,\mathrm{ch}^{\mathrm{long}}_{h=\frac{3}{2},l=1} + 4444\,\mathrm{ch}^{\mathrm{long}}_{h=\frac{5}{2},l=1} + \ldots$$

- Two infinite series, coefficients unrelated to Conway

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All coefficients are dimensions of the Mathieu group M_{22}!

New Moonshine phenomena

- The toroidal orbifold $T^8/\mathbb{Z}_2 = \mathbb{R}^8/\Lambda_{E8} / \mathbb{Z}_2$ preserves *N=4* worldsheet supersymmetry
- Let us generalize the above idea and choose chiral super-Virasoro algebras with *N > 1* for 24 fermions

M. Cheng, X. Dong, J. Duncan, S. Harrison, S. Kachru, TW 1406.5502

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M. Cheng, X. Dong, J. Duncan, S. Harrison, S. Kachru, TW 1406.5502

- We find that *N=2* (*N=4*) super-Virasoro algebras break the symmetry group to subgroups of Co_0 that fix a 2-plane (3-plane) in the 24 dimensional representation of Co_0

New Moonshine phenomena

- There are a variety of groups that do not act on a 2- (or 3-) plane in the 24 dimensional representation of Co_0
- For example:

Stabilizers of 2-plane	Stabilizers of 3-plane
M_{23}	M_{22}
Higman-Sims	$U_4(3)$
McLaughlin	
$U_6(2)$	

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$\Longrightarrow$ Coefficient in expansion in *N=2* (*N=4*) characters can be decomposed in irreps of these groups

New Moonshine phenomena

- The toroidal orbifold $T^8/\mathbb{Z}_2 = \mathbb{R}^8/\Lambda_{E8} / \mathbb{Z}_2$ preserves *N=2* worldsheet supersymmetry

- Calculate the partition function and expand in *N=2* characters:

$$Z = \mathrm{Tr}\left((-1)^F q^{L_0 - c/24} y^{J_0}\right) = 23\,\mathrm{ch}^{\text{short}}_{h=\frac{1}{2},l=0} + \mathrm{ch}^{\text{short}}_{h=\frac{1}{2},l=2}$$

$$+ 770\,\mathrm{ch}^{\text{long}}_{h=\frac{3}{2},l=1} + 13915\,\mathrm{ch}^{\text{long}}_{h=\frac{5}{2},l=1} + \ldots$$

$$+ 231\,\mathrm{ch}^{\text{long}}_{h=\frac{3}{2},l=2} + 5796\,\mathrm{ch}^{\text{long}}_{h=\frac{5}{2},l=2} + \ldots$$

All coefficients are dimensions of M_{23}, HS, McL and $U_6(2)$!

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All coefficients are dimensions of M_{22} and $U_4(3)$!

New Moonshine phenomena

- The twined partition functions

$$Z_g = \mathrm{Tr}\left(g\,(-1)^F\, q^{L_0 - c/24} y^{J_0}\right)$$

have special properties only for M_{22} and M_{23}:

The Mathieu groups M_{22} and M_{23} satisfy the extra moonshine property that the mock modular forms in all twined partition functions have only poles at the cusp at $\tau = i\,\infty$

(They can most likely be constructed via Rademacher sums.)

New Moonshine phenomena

- Theory of 24 chiral fermions orbifolded by a $\mathbb{Z}_2$ symmetry:

$$\mathbb{Z}_2 : \ \psi^I \to -\psi^I$$

gives rise to variety of moonshine phenomena:

Symmetry group: Spin(24) $\supset$ Co_0 $\supset$ M_{23} $\supset$ M_{22}

Expansion in: N=0 $\longrightarrow$ N=1 $\longrightarrow$ N=2 $\longrightarrow$ N=4

Conclusion

- Mathieu Moonshine involves K3 that has played a crucial role in superstring compactifications and string dualities
- Certain CY_3 manifolds are now also implicated in Mathieu Moonshine

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- Much more to come!

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THANK YOU!