Holography of 3d-3d correspondence at Large N

Dongmin Gang

KIAS -> Kavli-IPMU

Kavli-IPMU

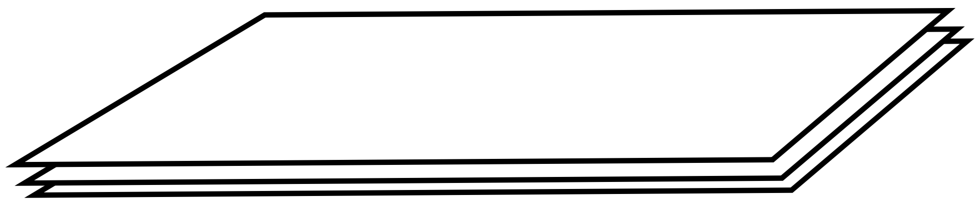
Based on arXiv :1401.3595, 1409.6206

With Nakwoo Kim (Kyunghee U) and Sangmin Lee (SNU)

Introduction

- **6d (2,0) theory**
  - M-theory : M2, M5 branes
  - Low-energy world-volume theory of multiple M2 branes

$N\ M2s$

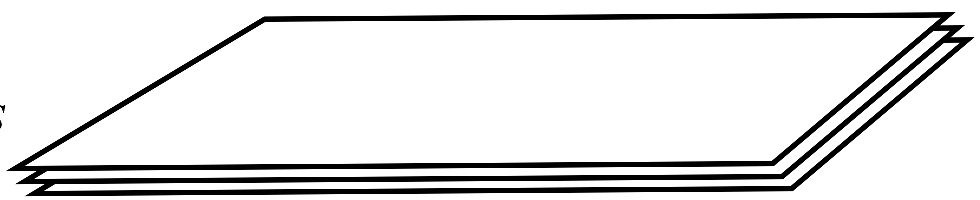


ABJM theory [Aharony,Bergman, Jafferis,Maldacena: `09]

- $3d$ Chern-Simons matter with $G = U(N)\times U(N)$

  - Low-energy world-volume theory of multiple M5 branes

$N\ M5s$



6d $A_{N-1}(2,0)$ theory

- Non-abelian tensor theory (non-Lagrangian)
- Holographically dual to M-theory on $AdS_7 \times S^4$
- $6d$ (2,0) superconformal symmetry ($\supset SO(5)_R$)
- degree of freedom $\sim N^3$
- S^1 – reduction to $5d\ \mathcal{N}=2\ SYM$ ($M5$ to $D4$)

- **Lower-dimensional (d〈6) theory $T_N[M_{6-d}]$**

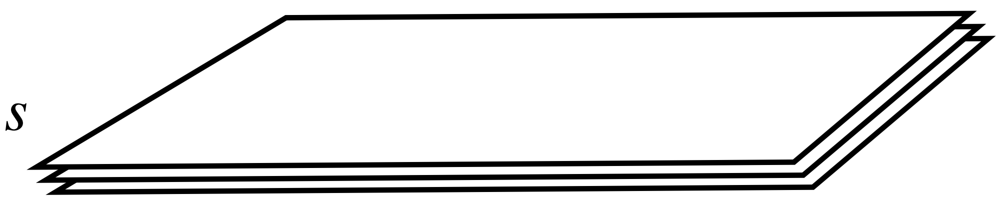
– $6d\ A_{N-1}$ (2,0) theory on M_{6-d} $\xrightarrow{\text{IR limit}}$ d – dimensional SCFT $T_N[M_{6-d}]$

e.g) 4d theories of class S , $T_N[M_2 = \Sigma_{g,h}]$ [Gaiotto-Moore-Neizke: `09]

3d theories of class R , $T_N[M_3]$ [Terashima-Yamazaki: `11] [Dimoft-Gaiotto-Gukov: `11]

Introduction

- 6d (2,0) theory
  - Low-energy world-volume theory of multiple M5 branes



6d A_{N-1}(2,0) theory

- Non-abelian tensor theory (non-Lagrangian)
- $6d$ (2,0) superconformal symmetry ($\supset SO(5)_R$)
- S^1 – reduction to $5d$ $\mathcal{N}=2$ SYM ($M5$ to $D4$)
- Holographically dual to M-theory on $AdS_4 \times S^7$
- degree of freedom ~ N^3

- Lower-dimensional (d〈6) theory $T_N[M_{6-d}]$
  - $6d$ A_{N-1} (2,0) theory on M_{6-d} $\xrightarrow{\text{IR limit}}$ d – dimensional SCFT $T_N[M_{6-d}]$

    e.g) 4d theories of class S , $T_N[M_2=\Sigma_{g,h}]$ [Gaiotto-Moore-Neizke: `09]

    3d theories of class R , $T_N[M_3]$ [Terashima-Yamazaki: `11] [Dimoft-Gaiotto-Gukov: `11]

- Studying $T_N[M]$ is interesting because
  - Dualities between d-dimensional SCFTs [“N=2 dualities”, Gaiotto:`09]
  - AGT-like correspondence: $Z(T_N[M]) = Z(\text{non-SUSY theory on } M)$ [Alday-Gaiotto-Tachikawa:`09]
  - New examples of AdS/CFT [Gaiotto-Maldadena:`09]
  - Window to 6d (2,0) theory : (Superconformal index, N^3) from $5d$ $\mathcal{N}=2$ SYM

    [S.Kim, H.Kim:’12][S.Kim, H.Kim, J.Kim:`12] [S.Kim, H.Kim, S.Kim, K.Lee:`13]

Outline of talk

$3d\ T_N[M]$ theory on 'S_b^3' $M = 3-$manifold

Holography ↔ 3d/3d correspondence

M-theory on 'Pernici-Sezgin AdS4'

$SL(N)$ CS theory on M

$$\exp\left[-\frac{\mathrm{vol}(M)}{12\pi}(b+b^{-1})^2 N^3 + (\text{subleading in } 1/N) + i(....)\right]$$

$$\exp\left[\frac{1}{\hbar}S_0[N;M] + S_1[N;M] + \hbar S_2[N;M] + \ldots\right] \triangleright$$

$$\hbar = 2\pi i b^2$$

- By imposing equality between two calculations

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0] = -\frac{1}{6}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1] = -\frac{1}{6\pi}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2] = \frac{1}{24\pi^2}\mathrm{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_{2j-1}] = \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_{2j}] = 0\ (j\geq 2)\,,$$

- Analytic/Numerical evidences for the conjecture

3d $T_N[M]$ theory

- Compactification of 6d A_{N-1} (2,0) theory on M_3

  - topological twisting along M : $A_{SO(3)_R} = \omega_M$ (spin-connection)

  $1/4$ SUSY : $\mathbf{2}_{SO(3)_R} \otimes \mathbf{2}_{SO(3)_M} = (\mathbf{1} \oplus \mathbf{3})_{SO(3)_{\text{diag}}}$ $\Rightarrow$ 3d $\mathcal{N}=2$ *SCFTs*

  11d : $\mathbb{R}^{1,2} \times T^*M \times \mathbb{R}^2$ (T^*M : Cotangent bundle of M)

  $\cup$

  N $M5s$: $\mathbb{R}^{1,2} \times M$ → 3d $\mathcal{N}=2$ SCFT $T_N[M]$ on $\mathbb{R}^{1,2}$

- The 3d theories enjoy

  - 3d-3d correspondence [Terashima, Yamazaki : `11] [Dimofte, Gukov, Gaiotto : `11]

  3d $T_N[M]$ theory on B $\Leftrightarrow$ $SL(N,\mathbb{C})$ CS theory on M with level (k,σ)

  ($B = S^2 \times_q S^1$, $S^3_b / \mathbb{Z}_k$)

  $$\left(\mathcal{L}_{CS} = \frac{1}{2\hbar} CS[\mathcal{A}] + \frac{1}{2\tilde{\hbar}} CS[\bar{\mathcal{A}}], \quad \frac{4\pi}{\hbar} = k+\sigma, \quad \frac{4\pi}{\tilde{\hbar}} = k-\sigma \text{ with } k \in \mathbb{Z}, \sigma \in \mathbb{R} \text{ or } i\mathbb{R} \right)$$

  $$CS[\mathcal{A}] := \int_M \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}^3 .$$

  - AdS_4/CFT_3 [Pernici,Sezgin : `85] [Gauntlett, Kim, Waldram : `00]

  3d $T_N[M]$ theory on B = M – theory on (Pernici-Sezgin AdS_4 solution)

  $$ds_{11}^2 = \frac{(1+\sin^2\theta)^{1/3}}{g^2}\left[ds^2(AdS_4) + ds^2(M) + \frac{1}{2}\left(d\theta^2 + \frac{\sin^2\theta}{1+\sin^2\theta} d\phi^2\right) + \frac{\cos^2\theta}{1+\sin^2\theta} d\tilde{\Omega}^2 \right]$$

  $$g^3 \sim 1/N, \ \partial(AdS_4) = B$$

3d-3d correspondence

$3d\ T_N[M]$ theory on B $\Leftrightarrow$ $SL(N,\mathbb{C})\ CS$ theory on M with level (k,σ)

$(\ B = S^2 \times_q S^1,\ S_b^3/\mathbb{Z}_k\ )$

$S_b^3 : \{ b^2 |z|^2 + b^{-2} |w|^2 = 1 \} \subset \mathbb{C}^2$

$(\ \mathcal{L}_{CS} = \frac{1}{2\hbar} CS[\mathcal{A}] + \frac{1}{2\tilde{\hbar}} CS[\overline{\mathcal{A}}],\ \frac{4\pi}{\hbar} = k+\sigma,\ \frac{4\pi}{\tilde{\hbar}} = k-\sigma$

with $k \in \mathbb{Z}, \sigma \in \mathbb{R}$ or $i\mathbb{R}$ $)$

- 3d-3d version of AGT relation $(\ Z(T_N[M_2] \text{ on } S^4) = Z(\text{Liouville / Toda on } M_2)\ )$

  $Z[T_N[M] \text{ on } B] = Z[SL(N)\ CS \text{ on } M]$

- Derivation from (2,0) theory

$Z[A_{N-1}\ (2,0) \text{ theory } on\ B \times M]$ [Lee, Yamazaki: `13] [Cordova, Jafferis: `13]

$\swarrow$ $Z[T_N[M] \text{ on } B]$ $\searrow$ $Z[SL(N,\mathbb{C})\ CS \text{ on } M]$

{ 1. The ptn does not depends on relative size betwee
  2. The 6d theory become SL(N) CS theory after co

$B = S^2 \times_q S^1 \quad \Leftrightarrow \quad k = 0,\ 4\pi/\sigma = i \log q$

$B = S_b^3/\mathbb{Z}_k \quad \Leftrightarrow \quad k = k,\ \sigma = k\frac{1-b^2}{1+b^2}$ (when $b \le 1$)

When $k = 1$, $\hbar = 2\pi i(1+b^2)$ and $\tilde{\hbar} = 2\pi i(1+b^{-2})$

But according to [Terashima, Yamazaki :`11]

$k = 1$, $\hbar = 2\pi i b^2$ and $\tilde{\hbar} = 2\pi i b^{-2}$

This puzzle is solved by T.dimofte by showing

$q := e^{\hbar},\ \tilde{q} := e^{\tilde{\hbar}}$ are more relevant parameters

in *quantization* of the *CS* theory

Holography of $T_N[M]$

- $6d$ A_{N-1} (2,0) theory on $\mathbb{R}^{1,2} \times M$

$6d\ A_{N-1}$ (2,0) on $\mathbb{R}^{1,5}$ → $3d\ \mathcal{N}=2$ SCFT $T_N[M]$ on $\mathbb{R}^{1,2}$

UV → IR

- Corresponding Holographic RG [Gauntlett, Kim, Waldram : `00]

$AdS_7 \times S^4$ → Pernici-Sezgin AdS_4 solution

$$ds_{11}^2 = \frac{(1+\sin^2\theta)^{1/3}}{g^2}\left[ds^2(AdS_4) + ds^2(M) + \frac{1}{2}(d\theta^2 + \frac{\sin^2\theta}{1+\sin^2\theta}d\phi^2) + \frac{\cos^2\theta}{1+\sin^2\theta}d\tilde{\Omega}^2\right]$$

analyzed only for closed (no boundary) M

exists for only hyperbolic M ($R_{\mu\nu} = -2g_{\mu\nu}$)

- Holography : $3d\ T_N[M]$ theory = M – theory on Pernici-Sezgin AdS_4 solution

Only for closed (no boundary) hyperbolic 3-manifold

3-sphere ptn of $T_N[M]$

- Squashed 3-sphere partition function

  $Z_{S_b^3}[T_N[M]] = Z[T_N[M] \text{ on } B = S_b^3]$ ($S_b^3 : \{ b^2 |z|^2 + b^{-2} |w|^2 = 1 \} \subset \mathbb{C}^2$)

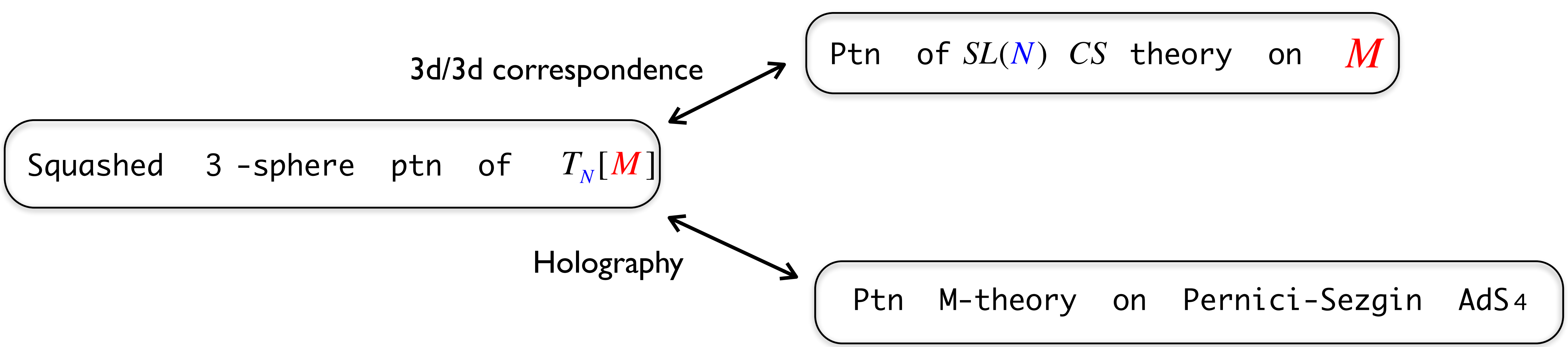
  – Round sphere free-energy $\mathcal{F}_{b=1}[CFT] := -\log \left| Z_{S_{b=1}^3}[CFT] \right|$

  $=$ finte part of entanglement entropy of CFT

  $=$ degree of freedom of CFT

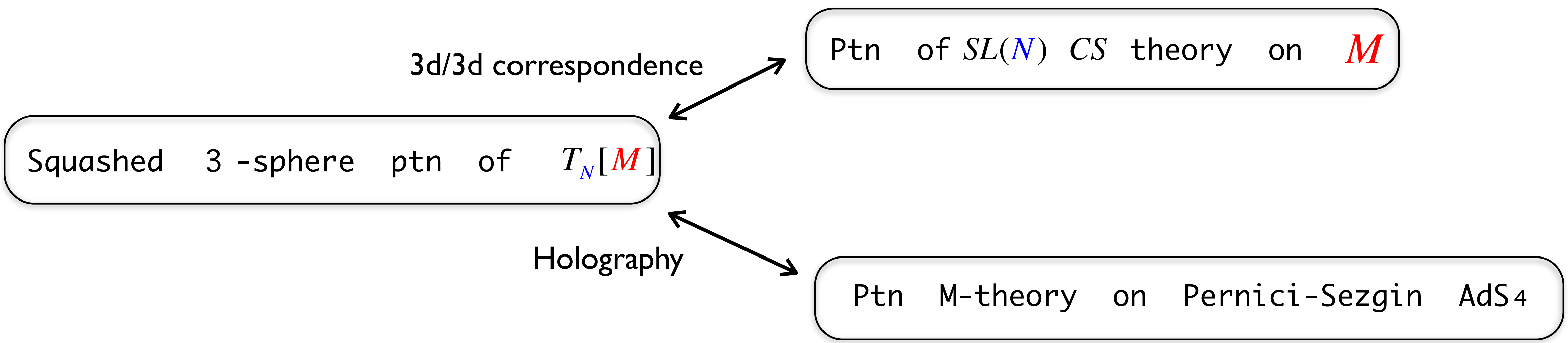
  – We will see $\mathcal{F}_{b=1}[T_N[M]] \sim N^3$ for hyperbolic M

- Two ways of the calculation



3-sphere ptn of $T_N[M]$

- Two ways of the calculation



1) Holography : $3d\ T_N[M]$ theory $=$ M – theory on Pernici-Sezgin AdS_4 solution

– $\mathcal{F}_{b=1}[T_N[M]] = \mathcal{F}^{\text{gravity}}(\text{Pernici-Sezgin } AdS_4)$, $\partial(AdS_4) = S^3$

$$\xrightarrow{\text{Large } N} (\text{Regularized on-shell SUGRA action}) = N^3/(3\pi)\,\text{vol}(M) + \text{subleading in } 1/N$$

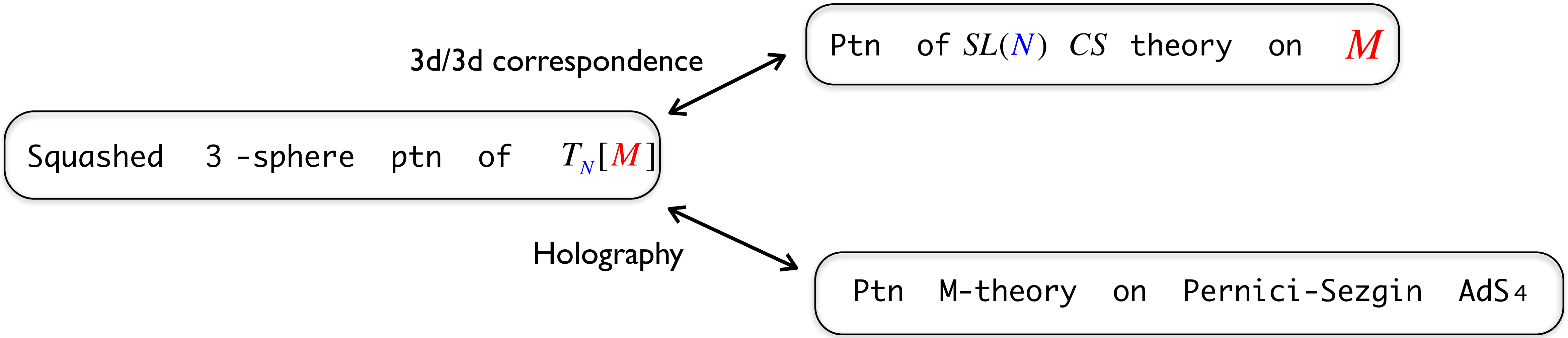
– Squashing parameter can be restored using [Martelli, Passias, Sparks : `11]

$$\mathcal{F}_b = (b+b^{-1})^2/4\ \mathcal{F}_{b=1} + (\text{subleading in } 1/N)$$

$$\Rightarrow \mathcal{F}_b[T_N[M]] = N^3/(12\pi)(b+b^{-1})^2\,\text{vol}(M) + (\text{subleading in } 1/N)$$

3-sphere ptn of $T_N[M]$

- Two ways of the calculation



1) Holography : $3d\ T_N[M]$ theory $=$ $M-$theory on Pernici-Sezgin AdS_4 solution

$$\Rightarrow \mathcal{F}_b[T_N[M]] = N^3/(12\pi)(b+b^{-1})^2 \mathrm{vol}(M) + (\text{subleading in } 1/N)$$

2) 3d-3d correspondence : $Z_{S_b^3}[T_N[M]] = Z_M(\mathrm{SL}(N)\ CS$ with $\hbar = 2\pi i b^2, \tilde{\hbar} = 2\pi i b^{-2})$

$$Z_M^{SL(N)\,CS}(\hbar,\tilde{\hbar}) := \int D\mathcal{A} \exp\left(1/(2\hbar) CS[\mathcal{A}] + 1/(2\tilde{\hbar}) CS[\overline{\mathcal{A}}]\right)$$

– Perturbation when $\hbar \to 0$ ($or\ \tilde{\hbar} \to 0$)

– Non-perturbative symmetry $\hbar \leftrightarrow -4\pi^2/\hbar$ ($b \leftrightarrow b^{-1}$)

– State-integral using ideal triangulation of M : finite-dimensional integral

Perturbative CS theory

- Perturbative expansion of CS theory

$-\ Z_{M}^{SL(N)\,CS}(\hbar) := \int D\mathcal{A} \exp\left(1/(2\hbar) CS[\mathcal{A}] + 1/(2\tilde{\hbar}) CS[\overline{\mathcal{A}}]\right)$

$\xrightarrow{\hbar \to 0} \sum_{\alpha} m_{\alpha}\, Z^{(\alpha)}_{SL(N)\,CS}(\hbar; M)$ ["Analytic continuation of Chern-Simons theory", Witten : `10]

$-$ Saddle point in CS : $d\mathcal{A}^{(\alpha)} + \mathcal{A}^{(\alpha)} \wedge \mathcal{A}^{(\alpha)} = 0$ (Flat connections)

$-\ Z^{(\alpha)}_{SL(N)\,CS}(\hbar; M) = \exp\left(\frac{1}{\hbar} S_0^{(\alpha)}[N; M] + S_1^{(\alpha)}[N; M] + \hbar S_2^{(\alpha)}[N; M] + \ldots\right)$: perturbatvie CS invarians $\{S_n^{(\alpha)}\}$

$S_0^{(\alpha)} = 1/2\, CS[\mathcal{A}^{(\alpha)}]$ (classical part) ,

$S_1^{(\alpha)} = \frac{1}{2} \log \mathrm{Tor}_{\mathrm{adj}}[M, \mathcal{A}^{(\alpha)}]$ (1 − loop) , $S_n^{(\alpha)} =$ (Feynmann diagrams)

($\mathrm{Tor}_{\rho}[M, \mathcal{A}^{(\alpha)}] = \frac{[\det{}' \Delta_0(\rho, \mathcal{A}^{(\alpha)})]^{3/2}}{[\det{}' \Delta_1(\rho, \mathcal{A}^{(\alpha)})]^{1/2}}$, $\Delta_n(\rho, \mathcal{A}^{(\alpha)})$: Laplacian on V_ρ − valued n-form twisted *by* $\mathcal{A}^{(\alpha)}$

Ray-singer torsion in an associated bundle $E_\rho \to M$ twisted by $\mathcal{A}^{(\alpha)}$, $\rho \in \mathrm{Hom}\left(SL(N) \to GL(V_\rho)\right)$)

- We assume only $\mathcal{A}^{(\mathrm{conj})}$ contributes to the perturbative expansion

(*i.e.* $m_\alpha \neq 0$ if and only if $(\alpha) = (\mathrm{conj})$)

$\mathcal{A}_N^{(\mathrm{conj})}$: *a* $SL(N)$ flat connection from the unique hyperbolic sturcture on M

$\mathcal{A}_{N=2}^{(\mathrm{conj})} := \omega + ie = (\text{spin-connection}) + i(\text{dreibein})$

$\mathcal{A}_N^{(\mathrm{conj})} = \rho_N(\mathcal{A}_{N=2}^{(\mathrm{conj})})$, ρ_N : N − dim irreducible representation of $SL(2)$

Conjecture on $S_n^{(\text{conj})}[N;M]$.

Squashed 3-sphere ptn of $T_N[M]$

Holography ↔ Ptn M-theory on Pernici-Sezgin AdS4

3d/3d correspondence ↔ Ptn of $SL(N)$ CS theory on M

$$\log\left|Z_{S_b^3}[T_N[M]]\right| = -\frac{\text{vol}(M)}{12\pi}(b+b^{-1})^2 N^3 + (\text{subleading in } 1/N)$$

$$= -\frac{i}{6\hbar}\text{vol}(M)N^3 - \frac{\text{vol}(M)}{6\pi}N^3 + \hbar\frac{i\,\text{vol}(M)}{24\pi^2}N^3 + (\text{subleading in } 1/N)$$

$(\hbar = 2\pi i b^2)$

$$\log(Z_{S_b^3}[T_N[M]]) = \log Z_M^{SL(N)\,CS}(\hbar) \xrightarrow{\hbar\to 0} \frac{1}{\hbar}S_0^{(\text{conj})}[N;M]$$

$$+ S_1^{(\text{conj})}[N;M] + \hbar S_2^{(\text{conj})}[N;M] + \hbar^2 S_3^{(\text{conj})}[N;M] + \ldots$$

- It leads to a mathematical conjecture

$$\lim_{N\to\infty}\frac{1}{N^3}\text{Im}[S_0^{(\text{conj})}] = -\frac{1}{6}\text{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\text{Re}[S_1^{(\text{conj})}] = -\frac{1}{6\pi}\text{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\text{Im}[S_2^{(\text{conj})}] = \frac{1}{24\pi^2}\text{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\text{Re}[S_{2j-1}^{(\text{conj})}] = \lim_{N\to\infty}\frac{1}{N^3}\text{Im}[S_{2j}^{(\text{conj})}] = 0\ (j\geq 2)\,,$$

For S_0 (classical part), it can be proven from direct computation

$$S_0^{(\text{conj})}[N] = \frac{1}{2}CS[\mathcal{A}_N^{(\text{conj})}] = \frac{1}{2}CS[\rho_N(\omega - ie)]$$

$$= \frac{1}{2}\times\frac{1}{6}N(N^2-1)CS[\omega - ie]$$

$$= \frac{1}{6}N(N^2-1)\left(-i\text{vol}(M) + cs(M)\right)$$

$\text{Tr}(\rho_N[h_1]\rho_N[h_2]) = 1/6N(N^2-1)\text{Tr}(h_1h_2)\,, h_i \in pgl(2)\,.$

$\text{Im}[CS(\omega - ie)] = (\text{Eistein-Hilbert action with } \Lambda = -1)$

Conjecture on $S_n^{(\mathrm{conj})}[N;M]$.

```
Squashed 3-sphere ptn of T_N[M]
```

Holography ↔ Squashed 3-sphere ptn of $T_N[M]$ ↔ 3d/3d correspondence

Ptn M-theory on Pernici-Sezgin AdS4

Ptn of $SL(N)$ CS theory on M

$$\log\left|Z_{S_b^3}[T_N[M]]\right| = -\frac{\mathrm{vol}(M)}{12\pi}(b+b^{-1})^2 N^3 + (\text{subleading in } 1/N)$$

$$= -\frac{i}{6\hbar}\mathrm{vol}(M)N^3 - \frac{\mathrm{vol}(M)}{6\pi}N^3 + \hbar\frac{i\,\mathrm{vol}(M)}{24\pi^2}N^3 + (\text{subleading in } 1/N)$$

$(\hbar = 2\pi i b^2)$

$$\log(Z_{S_b^3}[T_N[M]]) = \log Z_M^{SL(N)\,CS}(\hbar) \xrightarrow{\hbar\to 0} \frac{1}{\hbar}S_0^{(\mathrm{conj})}[N;M]$$

$$+ S_1^{(\mathrm{conj})}[N;M] + \hbar S_2^{(\mathrm{conj})}[N;M] + \hbar^2 S_3^{(\mathrm{conj})}[N;M] + \ldots$$

- It leads to a mathematical conjecture

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0^{(\mathrm{conj})}] = -\frac{1}{6}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1^{(\mathrm{conj})}] = -\frac{1}{6\pi}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2^{(\mathrm{conj})}] = \frac{1}{24\pi^2}\mathrm{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_{2j-1}^{(\mathrm{conj})}] = \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_{2j}^{(\mathrm{conj})}] = 0\ (j\geq 2)\,,$$

For S_0 (classical part), it can be proven from direct computation

For S_1 (1-loop part), it can be proven using a mathematical theorem

$$S_1^{(\mathrm{conj})} = \frac{1}{2}\log \mathrm{Tor}_{\mathrm{adj}}[M, \mathcal{A}_N^{(\mathrm{conj})}]$$

$$= \frac{1}{2}\log\mathrm{Tor}_{\rho_3}[M,\mathcal{A}_{N=2}^{(\mathrm{conj})}] + \frac{1}{2}\log\mathrm{Tor}_{\rho_5}[M,\mathcal{A}_{N=2}^{(\mathrm{conj})}] + \ldots + \frac{1}{2}\log\mathrm{Tor}_{\rho_{2N-1}}[M,\mathcal{A}_{N=2}^{(\mathrm{conj})}]$$

$$= -\frac{1}{4\pi}\mathrm{vol}(M)(3^2+5^2+\ldots(2N-1)^2) + \text{subleding in } 1/N$$

adj representation of $SL(N)$

$$= \rho_3 \oplus \rho_5 \oplus \ldots \oplus \rho_{2N-1}$$

$$\lim_{m\to\infty}\frac{1}{m^2}\log\mathrm{Tor}_{\rho_m}[M,\mathcal{A}_{N=2}^{(\mathrm{geom})}] = -\frac{1}{4\pi}\mathrm{vol}(M)$$

[P. Menal-Ferrer and J. Porti : `11]

Conjecture on $S_n^{(\mathrm{conj})}[N;M]$.

Squashed 3-sphere ptn of $T_N[M]$

Holography

3d/3d correspondence

Ptn M-theory on Pernici-Sezgin AdS4

Ptn of $SL(N)$ CS theory on M

$$\log\left|Z_{S_b^3}[T_N[M]]\right| = -\frac{\mathrm{vol}(M)}{12\pi}(b+b^{-1})^2 N^3 + (\text{subleading in } 1/N)$$

$$= -\frac{i}{6\hbar}\mathrm{vol}(M)N^3 - \frac{\mathrm{vol}(M)}{6\pi}N^3 + \hbar\frac{i\,\mathrm{vol}(M)}{24\pi^2}N^3 + (\text{subleading in } 1/N)$$

$(\hbar = 2\pi i b^2)$

$$\log(Z_{S_b^3}[T_N[M]]) = \log Z_M^{SL(N)\,CS}(\hbar) \xrightarrow{\hbar\to 0} \frac{1}{\hbar}S_0^{(\mathrm{conj})}[N;M]$$

$$+ S_1^{(\mathrm{conj})}[N;M] + \hbar S_2^{(\mathrm{conj})}[N;M] + \hbar^2 S_3^{(\mathrm{conj})}[N;M] + \ldots$$

- It leads to a mathematical conjecture

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0^{(\mathrm{conj})}] = -\frac{1}{6}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1^{(\mathrm{conj})}] = -\frac{1}{6\pi}\mathrm{vol}(M)\,,\ \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2^{(\mathrm{conj})}] = \frac{1}{24\pi^2}\mathrm{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_{2j-1}^{(\mathrm{conj})}] = \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_{2j}^{(\mathrm{conj})}] = 0\ (j\geq 2)\,,$$

For S_0 (classical part), it can be proven from direct computation

For S_1 (1-loop part), it can be proven using a mathematical theorem

For higher S_n , we don't have analytic proof yet.

We will numerically confirm the conjecture
using Dimofte's state-integral for various knot-complements M up to n=3.

When b is finite

- So far, we analyzed the asymptotic limit $b \to 0$

$$\mathcal{F}_{N^3}(M;\hbar = 2\pi i b^2) := -\lim_{N\to\infty} \frac{1}{N^3} \log\left| Z_M^{SL(N)\,CS}(\hbar)\right|$$

$$\mathcal{F}_{N^3}(M;\hbar = 2\pi i b^2) \xrightarrow{b\to 0} \frac{1}{12\pi}\left(\frac{1}{b^2} + 2 + b^2 + 0b^4 + 0b^6 + \ldots + 0b^{2n}\right)\mathrm{vol}(M) \quad \text{(assuming our conjectures } \textit{are} \text{ true)}$$

$$+ \text{ non-perturbative corrections of the form } e^{-\frac{4\pi^2}{\hbar}}$$

- Surprisingly, the strong-weak symmetry $\hbar \leftrightarrow -4\pi^2/\hbar \; (b \leftrightarrow b^{-1})$

exists in the asymptotic expansion (which is is *believed to be* valid when $b \ll 1$)

Only true for $\mathcal{F}_{N^3}$: *no* symmetry in the asymptotic expansion of full $\mathcal{F}(N)$

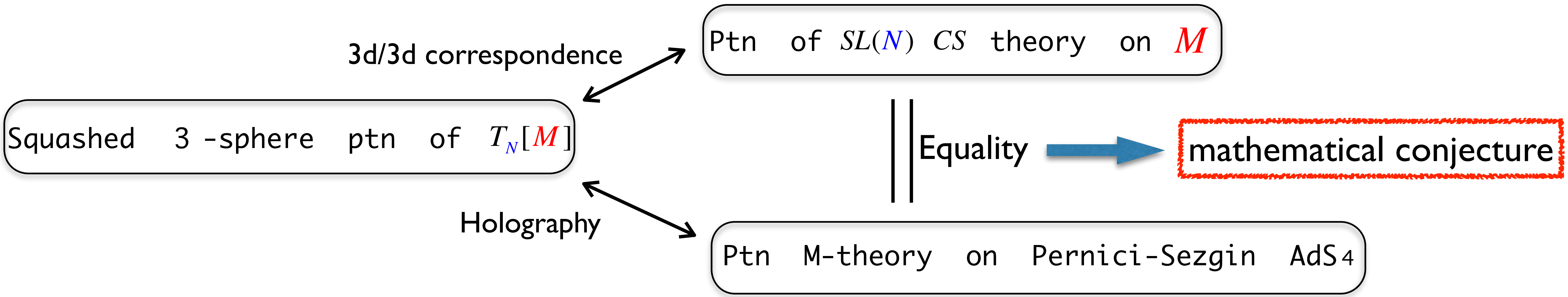
- This strongly suggest that the asymptotic expansion is actually

convergent expansion as far as the leading N^3-term is concerned

- Strong version of our conjecture

$$\mathcal{F}_{N^3}(M;\hbar = 2\pi i b^2) = \frac{1}{12\pi}(b + b^{-1})^2 \mathrm{vol}(M)\,, \text{ for any } b \in \mathbb{R}.$$

(perfect *agreement* with holographic prediction)



We assumed

$$Z_M^{SL(N)\,CS}(\hbar) \xrightarrow{\hbar\to 0} \sum_\alpha m_\alpha\, Z^{(\alpha)}_{SL(N)\,CS}(\hbar; M)\,, \quad m_\alpha \neq 0 \text{ if and only if } (\alpha) = (\text{conj})$$

Our conjecture

$$\lim_{N\to\infty} \frac{1}{N^3} \mathrm{Im}[S_0^{(\text{conj})}] = -\frac{1}{6}\mathrm{vol}(M)\,,\ \lim_{N\to\infty} \frac{1}{N^3} \mathrm{Re}[S_1^{(\text{conj})}] = -\frac{1}{6\pi}\mathrm{vol}(M)\,,\ \lim_{N\to\infty} \frac{1}{N^3} \mathrm{Im}[S_2^{(\text{conj})}] = \frac{1}{24\pi^2}\mathrm{vol}(M)$$

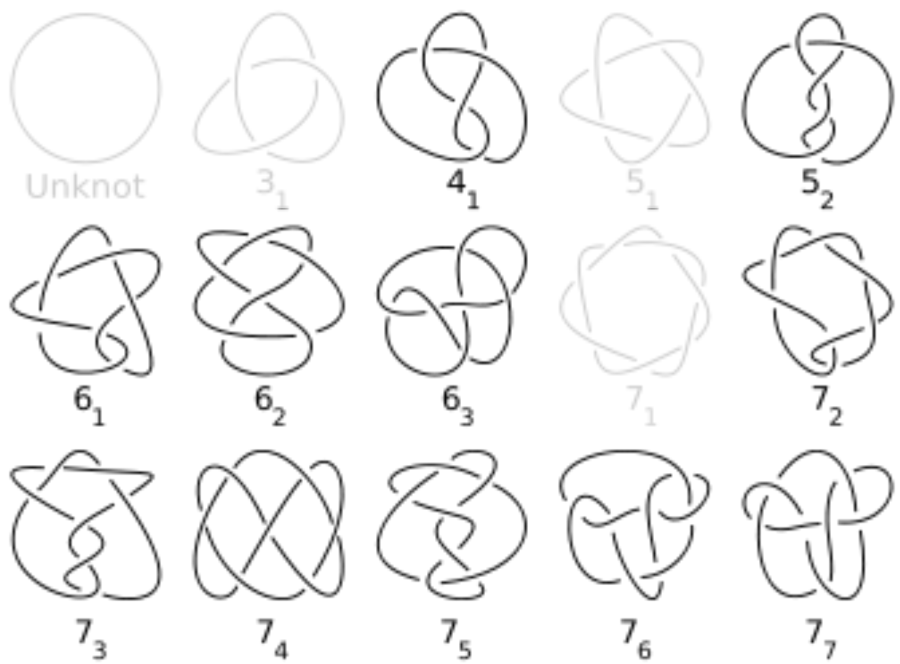
$$\lim_{N\to\infty} \frac{1}{N^3} \mathrm{Re}[S_{2j-1}^{(\text{conj})}] = \lim_{N\to\infty} \frac{1}{N^3} \mathrm{Im}[S_{2j}^{(\text{conj})}] = 0 \ (j \geq 2)\,,$$

In the rest of my talk,
I will give evidences for the assumption and conjecture
for M = (knot-complements) using Dimofte's state-integral

Knot Complements

- Simple hyperbolic 3-manifold w/ finite volume

K =



$M = S^3 \setminus N_K$, $N_{\mathcal{K}}$: Tubular neighborhood of knot $\mathcal{K}$

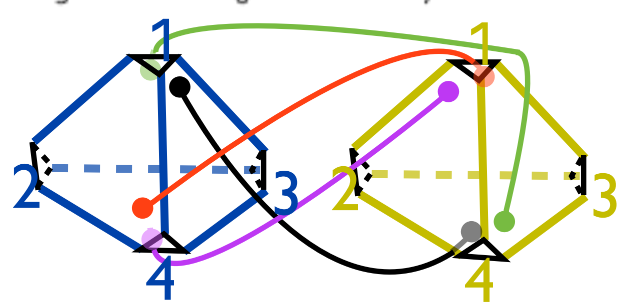
(topologically solid torus)

$cf)\ M_2 = S^2 \setminus (\text{punctures})$

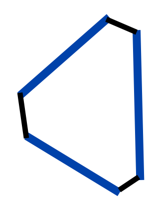
$\partial M = S^1 \times S^1 = \{\text{meridian,longitudinal}\}$

$\mathrm{vol}(M)$, triangulation datum [SnapPy]

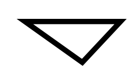
$S^3 \setminus$  =



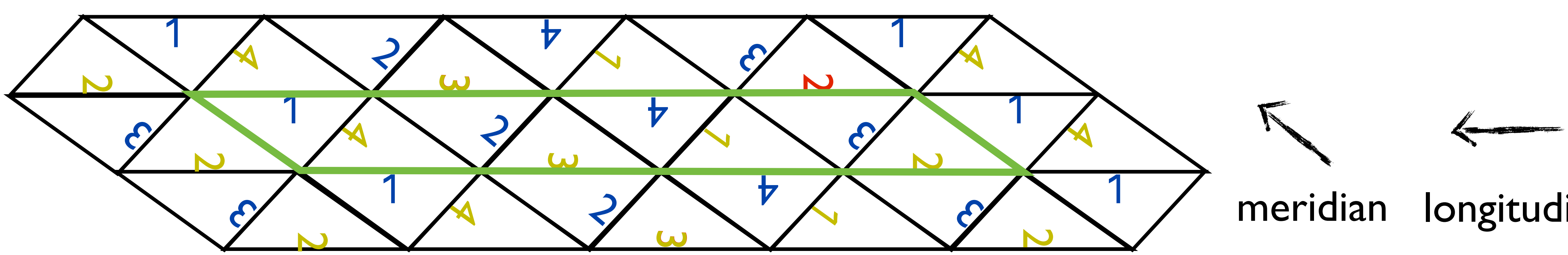
$S^3 \setminus \mathbf{4}_1$



Big boundaries : all glued



Small boundaries : form boundary 2-torus



meridian longitudinal

3d-3d for knot complements

- Knot is realized as intersecting M5-branes

$$\begin{array}{ll} \text{11d} & : \mathbb{R}^{1,2} \times T^{*}S^{3} \times \mathbb{R}^{2} \\ & \qquad\quad \cup \\ N\ M5s & : \mathbb{R}^{1,2} \times S^{3} \\ N\ M5s & : \mathbb{R}^{1,2} \times N^{*}\mathcal{K} \end{array} \longrightarrow \text{3d } \mathcal{N}=2 \text{ SCFT } T_N[S^3 \backslash \mathcal{K}] \text{ on } \mathbb{R}^{1,2}$$

$N^{*}\mathcal{K}$: unit co-normal bundle in $T^{*}S^{3}$

$N^{*}\mathcal{K} \cap S^{3} = \mathcal{K}$

- Basic dictionaries

$3d\ T_N[S^3 \backslash \mathcal{K}]$ theory	$SL(N)\ CS$ theory on $M = S^3 \backslash \mathcal{K}$
Rank of flavor symmetry	1/2 dim $P_N(\partial M)$ ($P_N(\partial M) \coloneqq \{SL(N)$ flat-connections on $\partial M\}$)
$SU(N)$ flavor group	$P_N(\partial M) = \{\mathbf{m} = \text{Hol}_{meridian}(\mathcal{A}), \mathbf{l} = \text{Hol}_{longtude}(\mathcal{A}) : [\mathbf{m},\mathbf{l}] = 0\}$
mass-prameter $\{m_i\}_{i=1,\ldots,N-1}$	boundary meridian holonomy $\mathbf{m} \sim \begin{pmatrix} 1 & \exp(m_1) & 0 & 0 & \cdots \\ 0 & 1 & \exp(m_2) & 0 & \cdots \\ 0 & 0 & 1 & \exp(m_3) & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots \end{pmatrix}$
conformal point, $m_i = 0$	boundary $\mathbf{m}$ *is* parabolic
$Z_B[T_N[M]](m_i)$	$Z_M[SL(N)\ CS](m_i)$

State-integral model for CS ptn

- Finite integral expression for $Z_M^{SL(N)\,CS}$(h)

[Dimofte : `11]
[Dimofte ,Garoufalidis: `12]
[Dimoft, Gabella, Goncharov: `13]

$$Z_M^{SL(N)\,CS}(\hbar) = \int \frac{d^{\sharp_N} X}{(2\pi i \hbar)^{\sharp_N}} \exp\left[-\frac{1}{\hbar}(i\pi + \frac{\hbar}{2}) X \cdot B_N^{-1} \cdot \nu_N + \frac{1}{2\hbar} X \cdot B_N^{-1} A_N X \right] \prod_{i=1}^{\sharp_N} \psi_\hbar(X_i) \quad (\text{ when } m_i = 0\,)$$

Based on ideal triangluation $\mathcal{T}$ of $M = (\bigcup_{i=1}^{k} \Delta_i)/\sim$

X_i : integration variables $i = 1, \ldots, \sharp_N = k/6\; N(N^2-1)$

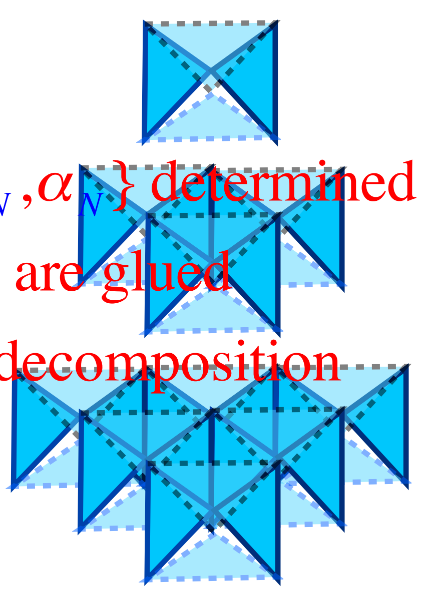
Quantum Dilogarithm $\psi_\hbar(X) := \prod_{r=1}^{\infty} \frac{1 - q^r e^{-X}}{1 - \tilde{q}^{-r+1} e^{-\tilde{X}}}$, when $|q|<1$. Here $q := e^\hbar$, $\tilde{q} := e^{-4\pi^2/\hbar}$, $\tilde{X} := (2\pi i/\hbar)X$.

$\{A_N, B_N, \nu_N\}$: Neunmann-Zagier datum ⟵ N – decomposition of M

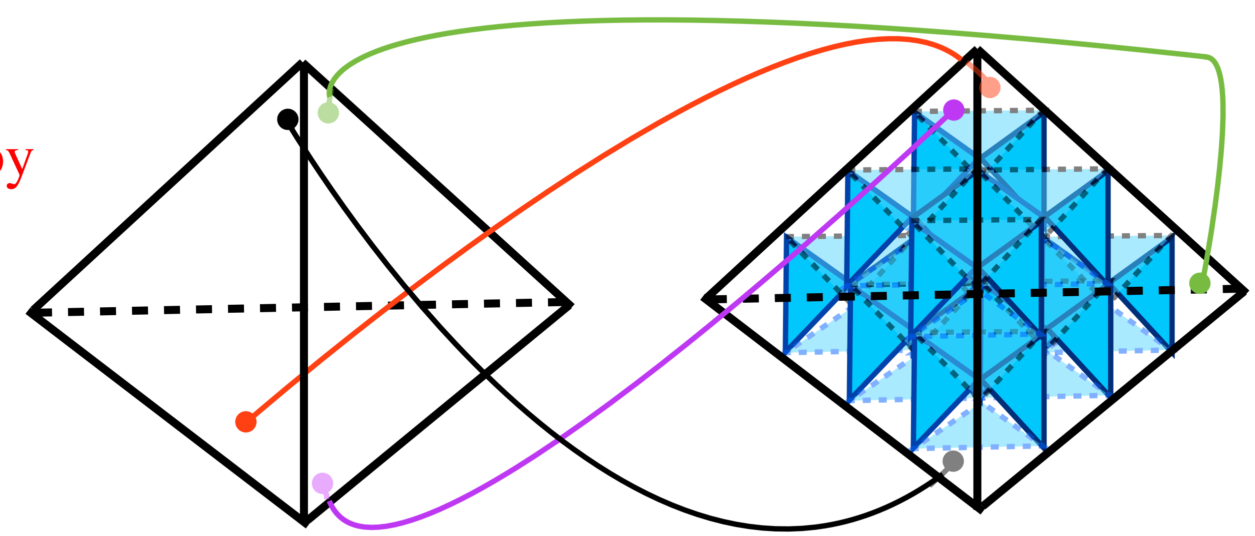
ν_N : $\mathbb{Z}$ – valued vector of size $\sharp_N$

(A_N, B_N) : $\mathbb{Z}$ – valued square matrices of size $\sharp_N$

Decomposition of each Δ into a pyramid of $N(N^2-1)/6$ octahedra .



$\{A_N, B_N, \alpha_N\}$ determined by how $\lozenge s$ are glued in N – decomposition



$N = 4$, $M = S^3 \setminus 4_1$

State-integral model for CS ptn

- Finite integral expression for $Z_M^{SL(N)\,CS}(\hbar)$

[Dimofte : `11]
[Dimofte ,Garoufalidis: `12]
[Dimoft, Gabella, Goncharov: `13]

$$Z_M^{SL(N)\,CS}(\hbar) = \int \frac{d^{\sharp_N} X}{(2\pi i \hbar)^{\sharp_N}} \exp\left[-\frac{1}{\hbar}(i\pi + \frac{\hbar}{2}) X \cdot B_N^{-1} \cdot \nu_N + \frac{1}{2\hbar} X \cdot B_N^{-1} A_N X \right] \prod_{i=1}^{\sharp_N} \psi_\hbar(X_i) \quad (\text{ when } m_i = 0 \text{)}$$

↑

$\{A_N, B_N, \nu_N\}$ ⟵ N – decomposition of M ⟵ Ideal triangulation $\mathcal{T}$ of $M = (\bigcup_{i=1}^{k} \Delta_i)/\sim$

- Examples

$$Z_{M=S^3\backslash \mathbf{4}_1}^{SL(N=2)}[\hbar] = \int \frac{dYdZ}{2\pi i \hbar} \exp\left[-\frac{1}{\hbar} XY \right] \psi_\hbar(X) \psi_\hbar(Y)$$

$$Z_{M=S^3\backslash 5_2}^{SL(N=2)}[\hbar]$$

$$= \int \frac{dXdYdZ}{(2\pi i\hbar)^{3/2}} \exp\left[-\frac{1}{\hbar}(i\pi + \frac{\hbar}{2})(X+Y+Z) + \frac{1}{2\hbar}(X^2+Y^2+Z^2+2XY+2YZ) \right] \psi_\hbar(X)\psi_\hbar(Y)\psi_\hbar(Z)$$

Basic idea of state-integral model

- Classical geometric datum of CS theory

$P_N(\partial M)$ = Boundary phse space = moduli space of flat $SL(N)$ connections on ∂M

$\mathcal{L}_N(M)$ = moduli space of flat $SL(N)$ connections on M , a Lagrangian submanifold of $P(\partial M)$

- Quantization

$P_N(\partial M) \xrightarrow{\text{quantization}} \mathcal{H}_N(\partial M)$, *a* Hilbert-space

$\mathcal{L}_N(M) \xrightarrow{\text{quantization}} |M\rangle$, *a* state in the Hilbert-space , $Z_M^{SL(N)\,CS}(\hbar)(m_i) = \langle m_i | M\rangle$

- N-decomposition

A mathematical tool to construct the classical datum $P_N(\partial M), \mathcal{L}_N(M)$

by 'gluing' the classical datum of basic block, octahedron $\lozenge$.

$P_N(\partial M) = \prod_{i=1}^{\sharp_N} P(\partial \lozenge_i) // (C_I = 0)$, Sympletic reduction using internal vertex variables $\{C_I\}$ as momnetum map

$P(\partial\lozenge) = \{Z, Z', Z'' \in \mathbb{C}^3 : Z + Z' + Z'' = i\pi\} \simeq \mathbb{C}^2$, with symplectic form $\{Z, Z''\}_{P.B} = \hbar$

$\mathcal{L}(\partial\lozenge) = \{e^Z + e^{-Z''} - 1 = 0\} \subset P(\partial\lozenge)$

Quantize $P(\partial\lozenge), \mathcal{L}(\partial\lozenge) \xrightarrow{\text{"quantum symplectic reduction"}}$ Quantize $P_N(\partial M), \mathcal{L}_N(M)$

$\mathcal{H}(\partial\lozenge) = L^2(\{Z\})$, $\hat{Z}|Z\rangle = |Z\rangle$, $\hat{Z}''|Z\rangle = -\hbar\partial_Z$

$(e^{\hat{Z}} - e^{-\hat{Z}''} - 1)|\lozenge\rangle = 0 \Rightarrow \langle Z|\lozenge\rangle = \psi_\hbar(Z)$

$\langle \mu_i = 0 | M\rangle = \int d\mathbf{Z}$ (Gaussian factor (A_N, B_N, ν_N)) $\prod_{i=1}^{\sharp_N} \psi_\hbar(Z_i)$

Perturbation in state-Integral

- The state integral is schematically,

$$Z_M^{SL(N)\,CS}(b) = \int_{\mathcal{C}_{\mathbb{R}}} d^{\sharp_N}\mathbf{X}\; e^{\frac{1}{2\pi b^2}\mathcal{I}(\mathbf{X};b)}\;, \quad \mathcal{C}_{\mathbb{R}} : \textit{contour along } \mathbb{R}+i\epsilon\,.$$

- We want to asymptotic expansion of the integral when $b \to 0$ using saddle point approximation .

  Saddle point $\mathbf{X}^{(\alpha)}$, $(\partial_{\mathbf{X}}\mathcal{I})|_{\mathbf{X}=\mathbf{X}^{(\alpha)}} = 0 \quad \Rightarrow$ Flat-connections $\mathcal{A}^{(\alpha)}$ on M

  $\exists\, \mathbf{X}^{(\mathrm{conj})}$ which corresponds to $\mathsf{A}^{(\mathrm{conj})}$ (if ideal triangulation T meet certain condition)

  One might think we should consdier all the saddle points in asymptotic expansion

$$Z_N^{CS} \xrightarrow{b\to 0} \sum_{\alpha} Z^{(\alpha)}\;,\quad Z^{(\alpha)} = \text{Saddle point approximation around } \mathbf{X}^{(\alpha)}$$

  But

$$Z_N^{CS} \xrightarrow{b\to 0} \sum_{m_\alpha\in\mathbb{Z}} m_\alpha Z^{(\alpha)}\;,\ (m_\alpha = \text{number (with sign) of upward flows from } \mathbf{X}^{(\alpha)} \text{ to } \mathcal{C}_{\mathbb{R}})$$

$$\frac{d\mathbf{X}}{dt} = \frac{\partial \mathrm{Re}[\mathcal{I}]}{d\bar{\mathbf{X}}}\,,\ \frac{d\bar{\mathbf{X}}}{dt} = \frac{\partial \mathrm{Re}[\mathcal{I}]}{d\mathbf{X}} \qquad : \text{upward flow}$$

- To prove our first assumption (only from $\mathcal{A}^{(\mathrm{conj})}$), it's neccesary and sufficient to show that

  $m_\alpha \neq 0$ if and only if $(\alpha) = (\mathrm{conj})$. (note : $\mathrm{Re}[\mathcal{I}(\mathrm{X}^{(\mathrm{conj})})] < \mathrm{Re}[\mathcal{I}(\mathrm{X}^{(\alpha)})]$ for any $(\alpha) \neq (\mathrm{conj})$)

  We checked it for $M = S^3 \backslash \mathbf{4}_1$ and $S^3 \backslash \mathbf{5}_2$ when $N = 2$.

$S_n^{(\mathrm{conj})}$ from State-Integral

$$Z_N^{CS}[\hbar;M]=\int\frac{d^{\sharp_N}X}{(2\pi i\hbar)^{\sharp_N}}\exp\left[-\frac{1}{\hbar}(i\pi+\frac{\hbar}{2})X\cdot B_N^{-1}\cdot\nu_N+\frac{1}{2\hbar}X\cdot B_N^{-1}A_N X\right]\prod_{i=1}^{\sharp_N}\psi_\hbar(X_i)$$

$\{A_N,B_N,\nu_N\}$ ⟵ N – decomposition of M ⟵ Ideal triangulation T of $M=(\bigcup_{i=1}^{k}\Delta_i)/\sim$

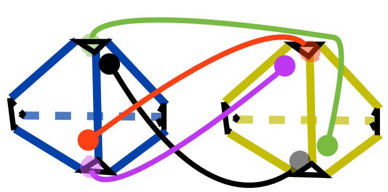
- $Z^{(\mathrm{conj})}=$ (perturbative calcuation of the finite integral aroud a saddle point $\mathbf{X}^{(\mathrm{conj})}$)

$$=\exp\left(\frac{1}{\hbar}S_0^{(\mathrm{conj})}+S_1^{(\mathrm{conj})}+\hbar S_2^{(\mathrm{conj})}+\ldots\right)\quad \text{using}\quad \psi_\hbar(X)\xrightarrow{\hbar\to 0}\exp\left(\sum_{n=1}\frac{B_n}{n!}\hbar^{n-1}\mathrm{Li}_{2-n}(e^{-X})\right)$$

- $S_n^{(\mathrm{conj})}$ can be expressed in terms of $\{A_N,B_N,\nu_N,\mathbf{X}^{(\mathrm{conj})}\}$ using Feynmann-diagram of integral .

  loop : Summation over $i=1,\ldots,\sharp_N$ ⇒ Finite expression

- Test of our conjecture for various knot-complements

$M=S^3\setminus$  = 

$$\mathrm{Im}[S_0^{(\mathrm{conj})}(N)]=1/6N(N^2-1)\,2\mathrm{Im}[\mathrm{Li}_2(e^{-i\pi/3})]=-1/6N(N^2-1)\mathrm{vol}(M)$$

$$\mathrm{Re}[S_1^{(\mathrm{conj})}(N)]=\mathrm{Re}[-\frac{1}{2}\log\det(A_N e^{i\pi/3}+B_N e^{-i\pi/3})]$$

$=\{-0.274653,-1.52226,-4.68107,-10.4071,-19.338,$
$-32.13,-49.4353,-71.902,-100.178,-134.909,$
$-176.745,-226.33,-284.312,-351.337,-428.051660,$
$-515.103361,-613.137064,-722.799596,-844.737248,$
$-979.596285,-1128.022958,\ldots\}$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0^{(\mathrm{conj})}]=-\frac{1}{6}\mathrm{vol}(M)\,,$$

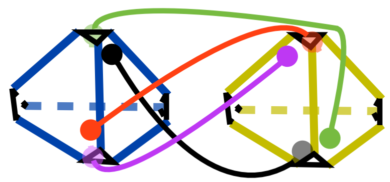
$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1^{(\mathrm{conj})}]=-\frac{1}{6\pi}\mathrm{vol}(M)\,,$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2^{(\mathrm{conj})}]=\frac{1}{24\pi^2}\mathrm{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_{2j-1}^{(\mathrm{conj})}]=\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_{2j}^{(\mathrm{conj})}]=0\ (j\geq 2)\,,$$

$S_n^{(\mathrm{conj})}$ from State-Integral

- Test of our conjecture for various knot-complements

$M = S^3 \setminus$  $=$ 

$\mathrm{Im}[S_0^{(\mathrm{conj})}(N)] = 1/6N(N^2-1)\, 2\mathrm{Im}[\mathrm{Li}_2(e^{-i\pi/3})] = -1/6N(N^2-1)\mathrm{vol}(M)$

$\mathrm{Re}[S_1^{(\mathrm{conj})}(N)] = \mathrm{Re}[-\frac{1}{2}\log\det(A_N e^{i\pi/3} + B_N e^{-i\pi/3})]$

$\simeq -\frac{\mathrm{vol}(M)}{6\pi} N^3$

$= \{-0.274653, -1.52226, -4.68107, -10.4071, -19.338, -32.13, -49.4353, -71.902, -100.178, -134.909, -176.745, -226.33, -284.312, -351.337, -428.051660, -515.103361, -613.137064, -722.799596, -844.737248, -979.596285, -1128.022958, \ldots\}$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0^{(\mathrm{conj})}] = -\frac{1}{6}\mathrm{vol}(M),$$

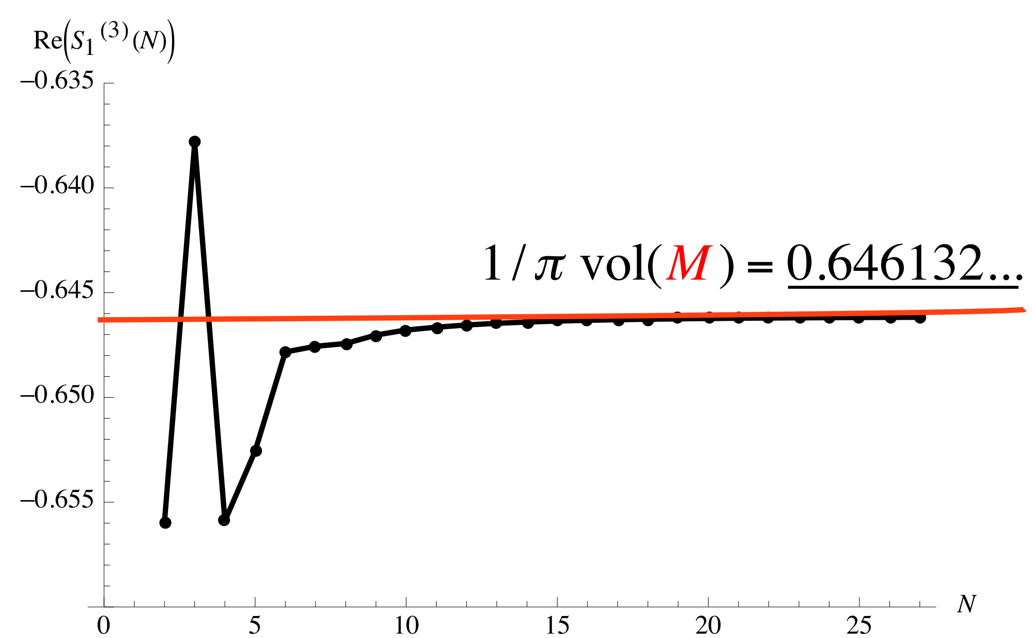
$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1^{(\mathrm{conj})}] = -\frac{1}{6\pi}\mathrm{vol}(M),$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2^{(\mathrm{conj})}] = \frac{1}{24\pi^2}\mathrm{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_{2j-1}^{(\mathrm{conj})}] = \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_{2j}^{(\mathrm{conj})}] = 0\ (j \geq 2),$$

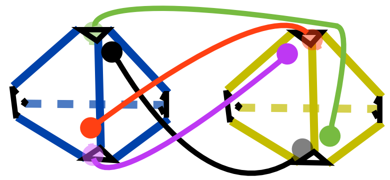
Its third differece sequence

$\mathrm{Re}[S_1{'''}(N)] = \{-0.655958, -0.637856, -0.655893, -0.652562, -0.64783, -0.64756, -0.647428, -0.647022, -0.646783, -0.646649, -0.646543, -0.646462, -0.646402, -0.646356, -0.646319, -0.646291, -0.646267, -\underline{0.646248}\}$



$S_n^{(\mathrm{conj})}$ from State-Integral

- Test of our conjecture for various knot-complements

$M = S^3 \setminus$  $=$ 

$$\mathrm{Im}[S_0^{(\mathrm{conj})}(N)] = 1/6N(N^2-1)\,2\mathrm{Im}[\mathrm{Li}_2(e^{-i\pi/3})] = -1/6N(N^2-1)\mathrm{vol}(M)$$

$$\mathrm{Re}[S_1^{(\mathrm{conj})}(N)] = \mathrm{Re}[-\frac{1}{2}\log\det(A_N e^{i\pi/3} + B_N e^{-i\pi/3})]$$

$$\sim -\frac{\mathrm{vol}(M)}{6\pi} N^3$$

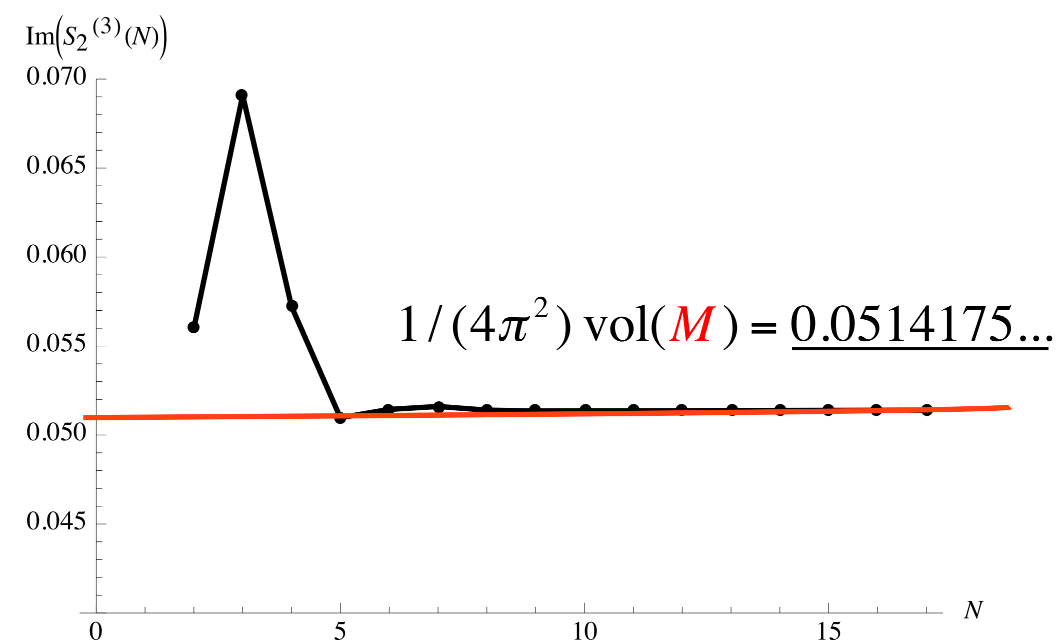
$$\mathrm{Im}[S_2^{(\mathrm{conj})}(N)] = \mathrm{Im}[(7\text{ Feynmann diagrams})]$$

$$= \{0.0882063, 0.289984, 0.618779, 1.13059, 1.89451, \ldots$$

$$\sim \frac{\mathrm{vol}(M)}{24\pi^2} N^3$$

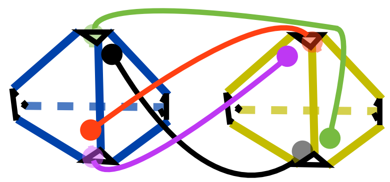
$$\mathrm{Im}[S_2'''(N)] = \{\,0.0560005, 0.0690888, 0.0572193,$$
$$0.0509708, 0.0514399, 0.0516042, \ldots$$

$$\lim_{N\to\infty} \frac{1}{N^3}\mathrm{Im}[S_0^{(\mathrm{conj})}] = -\frac{1}{6}\mathrm{vol}(M),$$
$$\lim_{N\to\infty} \frac{1}{N^3}\mathrm{Re}[S_1^{(\mathrm{conj})}] = -\frac{1}{6\pi}\mathrm{vol}(M),$$
$$\lim_{N\to\infty} \frac{1}{N^3}\mathrm{Im}[S_2^{(\mathrm{conj})}] = \frac{1}{24\pi^2}\mathrm{vol}(M)$$
$$\lim_{N\to\infty} \frac{1}{N^3}\mathrm{Re}[S_{2j-1}^{(\mathrm{conj})}] = \lim_{N\to\infty} \frac{1}{N^3}\mathrm{Im}[S_{2j}^{(\mathrm{conj})}] = 0\ (j \geq 2),$$



$S_n^{(\mathrm{conj})}$ from State-Integral

- Test of our conjecture for various knot-complements

$M = S^3 \setminus$  = 

$$\mathrm{Im}[S_0^{(\mathrm{conj})}(N)] = 1/6N(N^2-1)\, 2\mathrm{Im}[\mathrm{Li}_2(e^{-i\pi/3})] = -1/6N(N^2-1)\mathrm{vol}(M)$$

$$\mathrm{Re}[S_1^{(\mathrm{conj})}(N)] = \mathrm{Re}[-\frac{1}{2}\log\det(A_N e^{i\pi/3} + B_N e^{-i\pi/3})] \sim -\frac{\mathrm{vol}(M)}{6\pi}N^3$$

$$\mathrm{Im}[S_2^{(\mathrm{conj})}(N)] = \mathrm{Im}[(7\text{ Feynmann diagrams})] \sim \frac{\mathrm{vol}(M)}{24\pi^2}N^3$$

$$S_3^{(\mathrm{conj})}(N) = (30\text{ Feynmann Diagrams}) = \{-0.0185185, -0.0362503, -0.0425853, -0.0396434, -0.0348546, -0.0312819, -0.0284423, -0.0260191\} = N^3$$

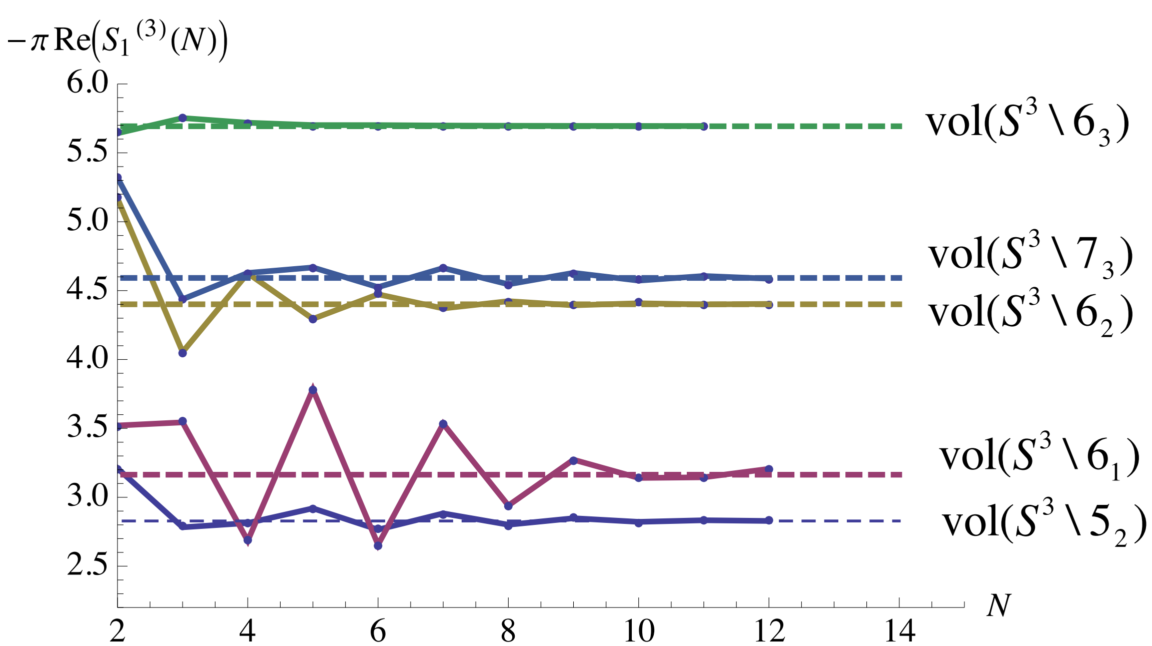
$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0^{(\mathrm{conj})}] = -\frac{1}{6}\mathrm{vol}(M),$$

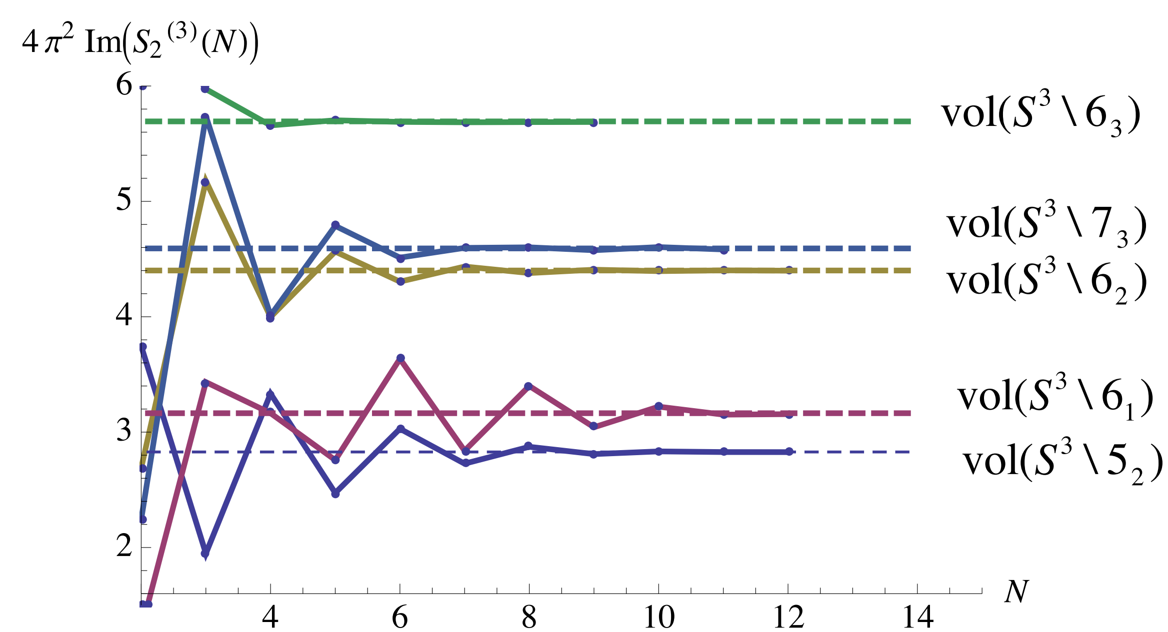
$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1^{(\mathrm{conj})}] = -\frac{1}{6\pi}\mathrm{vol}(M),$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2^{(\mathrm{conj})}] = \frac{1}{24\pi^2}\mathrm{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_{2j-1}^{(\mathrm{conj})}] = \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_{2j}^{(\mathrm{conj})}] = 0\ (j \geq 2),$$

For other knot compelements $M = S^3 \setminus \mathcal{K}$





Discussion

- Large N free energy of 3d $T_N[M]$ on S^3_b .

$$\mathcal{F}_b(T_N[M]) = \frac{N^3}{12\pi}(b+b^{-1})^2 \mathrm{vol}(M) + (1/N \text{ corrections})$$

Holography : Pernici-Sezgin AdS4 solution

3d-3d correspondence

$$Z_M^{SL(N)CS}(\hbar) \xrightarrow{\hbar\to 0} \sum_\alpha m_\alpha Z^{(\alpha)}_{SL(N)CS}(\hbar; M), \quad m_\alpha \neq 0 \text{ if and only if } (\alpha) = (\text{conj})$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_0^{(\text{conj})}] = -\frac{1}{6}\mathrm{vol}(M), \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_1^{(\text{conj})}] = -\frac{1}{6\pi}\mathrm{vol}(M), \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_2^{(\text{conj})}] = \frac{1}{24\pi^2}\mathrm{vol}(M)$$

$$\lim_{N\to\infty}\frac{1}{N^3}\mathrm{Re}[S_{2j-1}^{(\text{conj})}] = \lim_{N\to\infty}\frac{1}{N^3}\mathrm{Im}[S_{2j}^{(\text{conj})}] = 0 \ (j\geq 2),$$

- Future direction

  - Prove the assumption/conjecture (new state-integral model with manifest SU(N) structure)

  - Include defects

$M2 : AdS_2(\subset AdS_4)\times\gamma(\subset M)$	Wilson loop along γ in $SL(N)$ CS theory
$\mathcal{F}_{M2} \propto (1+b^2)N\,\text{Length}(\gamma) + \text{subleading } 1/N$	$\langle W_\gamma\rangle \propto \hbar^0(N\,\text{Length}(\gamma)+\text{subleading } 1/N) + \hbar^1$

  - 3d to 2d : $T_N[M]$ on S^1

$$N\ M5s\ \mathbb{R}^{1,1}\times S^1\times M \subset \mathbb{R}^{1,1}\times S^1\times T^*M\times\mathbb{R}^2 \xrightarrow{S^1\to 0} N\ D4s\ \mathbb{R}^{1,1}\times M \subset \mathbb{R}^{1,1}\times T^*M\times\mathbb{R}^2$$

$$Z_{S^2\times S^1, D^2\times S^1}(T_N[M]) \xrightarrow{S^1\to 0} Z_{S^2, D^2}(T_N[M\times S^1])$$

Thank you !!!

Gravity free energy

- Using the Pernici-Sezgin AdS4 solution, we calculate

$$\mathcal{F}_b^{\text{gravity}} = \frac{(b+b^{-1})^2}{4}\mathcal{F}_{b=1}^{\text{gravity}} = \frac{N^3}{12\pi}(b+b^{-1})^2 \text{vol}(M)$$

- But the SUGRA solution is incomplete when $M = S^3 \backslash \mathcal{K}$

  cf) Gaiotto-Maldacena solution

  $T_N[\Sigma_{g,h}]$: 4d theory of class S associated a Riemann sufrace $\Sigma_{g,h}$

  of genus g with h punctures

- Nevertheless, we believe the calculation is valid at leading order in $1/N$

  Effect of knot $\mathcal{K}$ enter only through vol($S^3 \backslash \mathcal{K}$) factor

  cf) Conformal anomaly coefficients for 4d class S theory

  - $a, c(T_N[\Sigma_{g,h}]) \propto (2-2g-h)N^3 + \text{subleadings} = \chi(\Sigma_{g,h})N^3 + ..$

    when punctures are all maximal

  - Hyperbolic volume vol(M) is anlalogous to Euler characteristic $\chi(\Sigma)$

    Both are topological

    can be written as integration of scalar curvature (of hyperbolic metic)