From non-local spins to large N matrices

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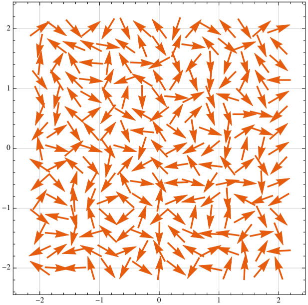


Outline

- Large N random matrices
- Finite Hilbert space
- Non-local spin models
- Outlook

Large N random matrices

Large N random matrices

Systems with $N \times N$ matrix degrees of freedom constitute our best models of 'emergent geometry' in the large N limit...

Some examples: BFSS, AdS/CFT, $c = 1$ matrix models...

Dynamics governed by a single trace Hamiltonian, large N factorization...

More generally, large N matrices appear in many areas of physics (nuclear, quantum chaos, glassiness...)

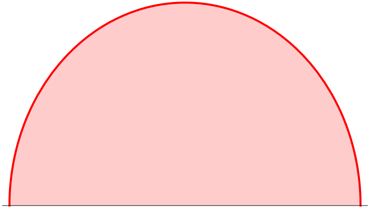
Simplest example: Matrix Integrals (i.e. $(0+0)$-dimensional QFT)

$$Z_N[g_n] = \int dM \; e^{-N \, \mathrm{tr} V(M)} \,, \qquad V(M) = \sum g_n M^n$$

Can diagonalize $M = U \cdot D \cdot U^\dagger$ and solve for eigenvalue distribution ρ.

At large N this is a saddle point problem, eigenvalue repulsion.

e.g. $V(M) = M^2$ we have Wigner semi-circle distribution.



More generally, matrix quantum mechanics:

$$\boxed{S = \mathrm{Tr} \int dt \left(\mathcal{D}_t X \cdot \mathcal{D}_t X^\dagger - m^2 X \cdot X^\dagger + \ldots \right) , \qquad X \in N \times N}$$

Single Hermitean matrix $\rightarrow$ free fermions

More matrices (e.g. BFSS) more complicated (cannot simultaneously diagonalize all matrices)

Add spatial dimensions: large N gauge theory, e.g. AdS/CFT...

Finite Hilbert Space

Most examples describe systems with infinite Hilbert space, such as BFSS and $\text{AdS}_{d+1}/\text{CFT}_d$ with $d > 1$

Situations where the (gravity) system has finite Hilbert space

Geometric examples:

- static patch of dS?
- $\text{AdS}_2 \times S^2 \times CY_3$?
- topological string theory

Region of accessible space by a single de Sitter observer. Surrounded by finite entropy horizon, no arbitrarily large black holes...

$$ds^2 = -dt^2(1-r^2) + \frac{dr^2}{(1-r^2)} + r^2 d\Omega^2$$

Several authors (Fischler,Banks;Susskind;Verlinde...) suggest it is a finite Hilbert space system.

Moreover it is conformally equivalent to $AdS_2 \times S^2$, possibly also suggestive of finite Hilbert space. (Correlators on AdS_2 boundary $\rightarrow$ worldline correlators...)

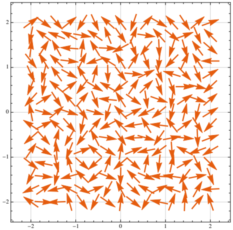
Simple example: Pure Grassman matrices/matrix quantum mechanics (however: no c-number eigenvalues) (Mekeenko,Zarembo)

Simple example: 2d Ising model coupled to random surface has (two) matrix model description (Kazakov)

Chern-Simons gauge theories dual to topological strings (Gopakumar-Vafa)

Non-local spin models

Another possibility: interacting collection of spins.



Simplest case: Ising spins $\sigma_{Ab} \in \{\pm\}$, $A = 1, \ldots, N$; $b = 1, \ldots, M$,

$$\boxed{\dim \mathcal{H} = 2^{N \times M}}$$

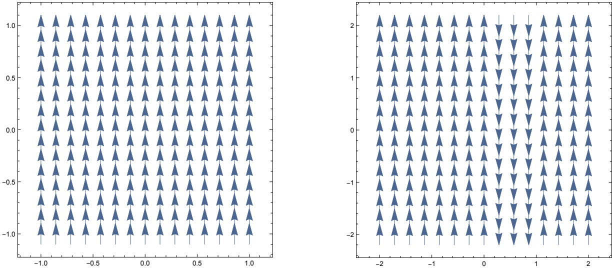
We study the following Hamiltonian for non-locally interacting Ising spins.

$$H = -\frac{1}{16N} \sum_{A,C,b,d} \sigma_{Ab}\sigma_{Cb}\sigma_{Cd}\sigma_{Ad}$$

Like Sherrington-Kirkpatrick model of spin glass, totally non-local interactions, no nearest neighbor... but NO quenched disorder. (Additional index like replica index.)

$\mathbb{Z}_2^N \times \mathbb{Z}_2^M$ symmetry: $\eta_A \sigma_{Ab} \eta_b$ with $\eta \in \{\pm\}$.

2^{N+M} ordered ground states: e.g. $\sigma_{Ab} = \{+\}$ and all those related by symmetries, i.e.



Ground state energy: $E_g = -M^2 N/16$, gap: $\Delta E \sim N$.

Q_{AB} real symmetric $N \times N$ matrix.

$$Z[\beta] = \frac{1}{2^{N/2}} \left(\frac{2N}{\pi\beta} \right)^{N(N+1)/4} \sum_{\sigma_{Ab} \in \{\pm\}^{N \times M}} \int dQ_{AB}\, e^{-\frac{N}{\beta} Q_{AB}^2 + \frac{1}{2} Q_{AB} \sigma_{Ab} \sigma_{Bb}}$$

Repeated b index $\implies$ we end with expression:

$$Z[\beta] = \frac{1}{2^{N/2}} \left(\frac{2N}{\pi\beta} \right)^{N(N+1)/4} \int dQ_{AB}\, e^{-\frac{N}{\beta} Q_{AB}^2 + M \log z(Q)}$$

with:

$$z(Q) = \sum_{\sigma_A \in \{\pm\}^N} e^{Q_{AB} \sigma_A \sigma_B / 2}$$

After another HS transformation we have exact expression:

$$z(Q) = \left(\frac{2}{\pi}\right)^{N/2} \frac{1}{\sqrt{\det Q}} \int dw_A \, e^{-\frac{1}{2} w_A Q^{-1}_{AB} w_B + \sum_A \log\cosh w_A}$$

Small w_A: $\sum_A \log\cosh w_A = \sum_A \frac{w_A^2}{2} - \frac{w_A^4}{12} + \dots$

Would like to expand in small w_A, keep quadratic term in expansion...

w_A propagator: $\boxed{G = Q \cdot (1 - Q)^{-1}}$.

$G_{AB} \sim s\,\delta_{AB} + t/\sqrt{N}$ at large N and small β.

Trace of G is order N and we cannot drop non-planar terms in small w_A expansion.

Define: $\tilde{Q}_{AB} = Q_{AB} - (\gamma - 1)\delta_{AB}$, tune γ s.t. $\mathrm{Tr}\tilde{G} = 0$.

This sets:

$$\boxed{N = \mathrm{Tr}\frac{1}{\gamma - Q}}$$

Now we can expand to quadratic $\tilde{w}_A$ and integrate it out, then at large N:

$$Z[\beta] = \left(\frac{2N}{\pi\beta}\right)^{N^2/4} e^{NM((\gamma-1)/2+\log 2)} \int dQ_{AB}\, e^{-\frac{N}{\beta}\mathrm{tr}Q^2 + \frac{M}{2}\mathrm{tr}\log(\gamma - Q)}$$

All non-planar contributions in Q_{AB} are subleading at large N. System described by a matrix model with single trace potential.

Penner-like matrix model (also appears in study of Grassman matrix models).

We are left with a single trace potential & emergent $SO(N)$ symmetry.

Diagonalize $Q = O \cdot D \cdot O^T$, $D_{ij} = x_i \delta_{ij}$. Introduce $\rho(y) = \frac{1}{N} \sum_i \delta(y - y_i)$.

We then obtain the saddle point equation for $\rho(x)$:

$$\boxed{\beta\, P \int dy\, \frac{\rho(y)}{x - y} = 2x - \frac{1}{2} \frac{M}{N} \frac{\beta}{\gamma - x}}$$

Eigenvalue distribution ρ saddle:

$$\boxed{\rho(x) = \left(1 + \frac{(a+b)}{2(\gamma - x)}\right)\sqrt{(a-x)(b-x)}}, \quad a < x < b < \gamma \; .$$

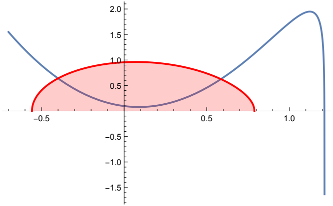
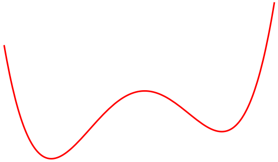


Figure : Plot of $V(x)$ (blue) and $\rho(x)$ (red) for $M = N$, $\beta = 0.4$.

Eigenvalue distribution localized about $x = 0$ only exists for $\beta < \beta_c \sim 1$.

For: $1/N < \beta < \beta_c$ matrix saddle is METASTABLE (exponentially long lived)



For $\beta > \beta_c$ it ceases to exist altogether, crystalline ground state dominates.

Schematic reason: $\boxed{Z[\beta] \sim e^{N} e^{\beta N^3} + e^{N^2} e^{\beta N^2}}$ s.t. $T > N$ entropic term dominates.

Analytic: $$\boxed{\frac{E_{matrix}}{N^2} = \frac{1-\alpha\gamma}{4\beta} - \frac{(b-a)^2}{64\beta^3}\left((3b+a)(b+3a) - 4(\alpha-2)\beta\right)}$$

System can be studied numerically (Metropolis).

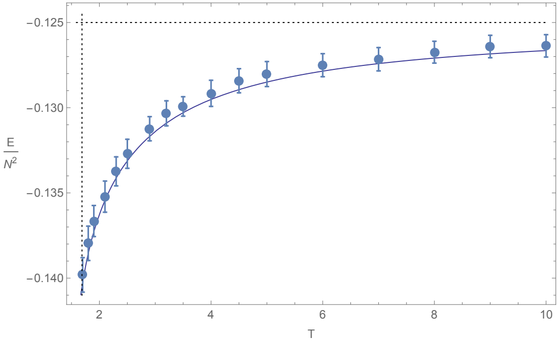


Figure : E vs. T for $N = M = 200$ for $T > T_c$. Solid line is analytic result for E_{matrix}.

$\mathbb{Z}_2^{N+M}$ (and $O(N)$) symmetries broken by low temperature ground states.

Consider: $\boxed{\hat{X}^{1,1}_{Ab} = \sigma^{A,b}\sigma^{A,b+1}\sigma^{A+1,b}\sigma^{A+1,b+1}}$. Ground states: $\hat{X}^{1,1}_{Ab} = +1$

$$\boxed{\langle \hat{X}_{AB} \rangle = \frac{1}{M^2}\sum_{n,b} \hat{X}^{B-A,n-b}_{Ab} = \left(\frac{1}{M}\sum_{b} \sigma_{Ab}\sigma_{Bb}\right)^2}$$

Matrix Phase: $\langle \hat{X}_{Ab} \rangle \to 0$ whereas in ordered phase: $\langle \hat{X}_{Ab} \rangle \to 1$

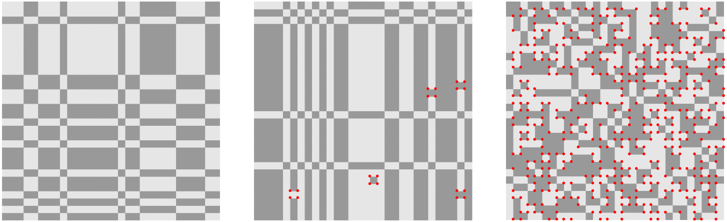


Figure : Defects. Left: Ground state. Center: Low $E \sim -N^3$. Right: High $E \sim -N^2$. $N = M = 30$.

Anti-ferromagnetic case

Anti-ferromagnetic case with $H \to -H$ (Cugliandolo,Kurchan,Parisi,Ritort) has no ordered degenerate ground states.

Exhibits glassy features below a certain T_c. Interesting model of a glass with no quenched disorder.

Outlook

General Class?

Microscopic model is 'tuned'. $\mathbb{Z}_2^N \times \mathbb{Z}_2^M$ symmetries not enough to fix Hamiltonian.

Better characterization of what spin systems have emergent matrix behavior?

Generalize spin system, consider coherent state basis (Haldane). e.g.

$H = \sum_{r,r'} J_{r,r'} \sigma^i_r \sigma^i_{r'}$, $|z\rangle = e^{z_r\,\sigma^-_r}|0\rangle$, $\sigma^\pm_r = \sigma^x_r \pm i\sigma^y_r$, $\sigma^z_r = s|0\rangle$

$$Z[\beta] = \int \mathcal{D}\vec{n}_r\,\delta(|\vec{n}_r(t)|^2 - 1)\,e^{-S[\vec{n}_r]}, \quad S[\vec{n}_r] = \int_0^\beta dt\,\left(i\,s\,\vec{A}(\vec{n}_r)\cdot\dot{\vec{n}}_r + s^2 H(\vec{n}_r)\right)$$

where: $\langle\bar{z}|\vec{\sigma}_r|z\rangle = s\vec{n}_r\langle\bar{z}|z\rangle$, $\vec{A}(\vec{n}_r)$ vector potential of Dirac monopole.

Finite number of states $\Longrightarrow$ Landau levels

What class of non-local spin models (if any) have an MQM representation?

Connections to more interesting geometries?

Both AdS_2 and dS exhibit $SL(2, \mathbb{R})$ symmetries of conformal QM.

Interesting possibility: spin system that exhibits a quantum critical point which is MQM at large N

If $SL(2, \mathbb{R})$ emergent from finite system, pathologies have a cutoff at finite N.

THANK YOU VERY MUCH FOR YOUR TIME!