FOURIER - SATO TRANSFORM,

BRAID GROUP ACTIONS,

AND FACTORIZABLE SHEAVES

Vadim Schechtman

Talk at

Kavli Institute for the Physics and Mathematics of the Universe,

Kashiwa-no-ha, October 2014

This is a report on a joint work with Michael Finkelberg, cf. [FS].

Plan

1. Serre and Tits symmetries, and their quantization : Lusztig's symmetries.

2. Quantum groups, factorizable sheaves, and Fourier - Sato transform.

3. Balance and BV.

§1. Braid group actions

Let M be a finite dimensional representation of a complex semisimple Lie group G. Then two objects act on G :

a) the Lie algebra $\mathfrak{g} = Lie(G)$; (b) (an extension of) the Weyl group W.

These two actions can be q-deformed.

1.1. Motivation : non-deformed case. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. The set of weights $Poids(L)$ of a finite dimensional $\mathfrak{g}$-module $L = L(\lambda)$ is a convex W-invariant body.

In fact, W acts on $Poids(L)$ and almost acts on L. For $1 \leq i \leq r = \text{rank}\mathfrak{g}$, define

$$\theta_{i,L} = e^{X_i} e^{-Y_i} e^{X_i} : \; L \xrightarrow{\sim} L, \tag{1.1.1}$$

where $\{X_i, Y_i, H_i\}$ is the corresponding $\mathfrak{sl}(2)$-triple. Then

$$\theta_i(L_\mu) = L_{s_i(\mu)}, \tag{1.1.2}$$

cf. [S], Ch. VII, §4, Remarque 1.

Let G be the simply connected Lie group corresponding to $\mathfrak{g}$; we have $W \overset{\sim}{=} N(T)/T$. The elements θ_i considered as elements of G, generate a subgroup $\tilde{W} \subset N(T)$, an *extended Weyl group* (Tits) included into an extension

$$0 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^r \longrightarrow \tilde{W} \longrightarrow W \longrightarrow 0,$$

cf. [T]. The adjoint action of G on $\mathfrak{g}$ thus induces an action of $\tilde{W}$ on $\mathfrak{g}$. On the other hand, (1.1.1) gives rise to an action of $\tilde{W}$ on L, and we have

$$w(gx) = w(g)w(x), \ w \in \tilde{W}, \tag{1.1.3}$$

$w \in \tilde{W}, g \in \mathfrak{g}, x \in L$.

1.2. A q-deformation : Lusztig's action. When we replace $\mathfrak{g}$ by $U_q\mathfrak{g}$, $\tilde{W}$ will be replaced by the braid group $B = B(W)$.

Geometric definition of B. Let $R \subset \mathfrak{h}^*_{\mathbb{R}}$ be the root system of $\mathfrak{g}$. For each $\alpha \in \mathbb{R}$ consider

$$H_\alpha = \mathrm{Ker}(\alpha_{\mathbb{C}}) = \{x \in \mathfrak{h} | \ \alpha(x) = 0\} \subset \mathfrak{h} := \mathfrak{h}_{\mathbb{C}};$$

let

$$\mathfrak{h}^{\mathrm{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in R} H_\alpha.$$

Then $PB = \pi_1(\mathfrak{h}^{\mathrm{reg}}), \ B = \pi_1(\mathfrak{h}^{\mathrm{reg}}/W)$; we have an exact sequence

$$1 \longrightarrow PB \longrightarrow B \longrightarrow W \longrightarrow 1.$$

There is a distinguished central element $c \in PB$ which corresponds to a loop passing through the opposite Weyl chambers ("the square of the longest element of the Weyl group w_0.")

In the simply laced case B is generated by $T_i, 1 \leq i \leq r$, subject to relations

$$T_i T_j T_i = T_j T_i T_j$$

if $a_{ij} = -1$, and $T_i T_j = T_j T_i$ if $a_{ij} = 0$. $\square$

Theorem, cf. [L], Ch. 39. One can introduce an action of B on $\mathfrak{u} = \mathfrak{u}_q \mathfrak{g}$ and on integrable $\mathfrak{u}$-modules M in such a way that (1.1.3) holds true.

The action of the pure braid group on M respects the homogeneous components M_μ.

Our aim will be to give a geometric interpretation of the PB action on M_μ.

§2. Factorizable sheaves and quantum groups

Quantum groups and their representations are realized in some spaces of (generalized) vanishing cycles.

Cf. [BFS].

2.1. We fix a finite root system $R \subset V$ where $(V, (,))$ is a Euclidean vector space with Cartan matrix $A = (a_{ij})$, a base of simple roots $\{\alpha_1, \dots, \alpha_r\}$,

and $q \in \mathbb{C}^*$. Let $\mathfrak{u}_q$ denote the Lusztig's small quantum group.

$$Q_+ = \oplus_{i=1}^r \mathbb{N}\alpha_i \subset Q = \oplus_{i=1}^r \mathbb{Z}\alpha_i$$

For $\lambda \in \Lambda = Hom_{\mathbb{Z}}(Q, \mathbb{Z})$ (the weight lattice), $L(\lambda)$ will denote the irreducible $\mathfrak{u}_q$-module of highest weight λ.

$$L(\lambda) = \oplus_{\mu \in Q_+} L(\lambda)_{\lambda - \mu}$$

2.2. Configurational spaces, local systems. For

$$\mu = \sum_i n_i \alpha_i \in Q_+, \;\; n = \sum n_i$$

we define the spaces

$$X_\mu = Div_\mu(\mathbb{C}) = \mathbb{C}^n / \prod \Sigma_i = \{(t_j)\}$$

and $X_{0,\mu}$ (one point is fixed at 0).

These spaces are naturally stratified ; we denote by

$$j_\mu : \; X^o_\mu \hookrightarrow X_\mu, \; j_{\lambda;\mu} : \; X^o_{0,\mu} \hookrightarrow X_{0,\mu}$$

the respective open strata.

Brading local system : $\mathcal{L}_\mu$ over X^o_μ has monodromy $-q^{(\alpha(i),\alpha(j))}$ when t_i turns around t_j ;

en plus, $\mathcal{L}_{\lambda;\mu}$ has monodromy $-q^{-(\lambda,\alpha(i))}$ when t_i turns around 0.

2.3. From perverse sheaves to quantum groups and their representations.

Middle extension. We set

$$\mathcal{P}_\mu = j_{\mu!*}\mathcal{L}_\mu \in Perv(X_\mu); \mathcal{P}_{\lambda;\mu} = j_{\lambda;\mu!*}\mathcal{L}_{\lambda;\mu} \in Perv(X_{\lambda;\mu})$$

Consider a function "the sum of coordinates"

$$f : X_{\lambda;\mu} \longrightarrow \mathbb{C} \tag{2.3.1}$$

The complex of vanishing cycles $\Phi_f(\mathcal{P}_{\lambda,\mu})$ is supported at the origin 0; denote

$$\Phi(\mathcal{P}_{\lambda,\mu})) = \Phi_f(\mathcal{P}_{\lambda,\mu})_0. \tag{2.3.2}$$

Theorem, cf. [BFS]. *The complex $\Phi(\mathcal{P}_{\lambda,\mu}))$ may have only one, the zeroth, cohomology. We have natural isomorphisms*

$$\Phi(\mathcal{P}_{\lambda,\mu}) \overset{\sim}{=} L(\lambda)_{\lambda-\mu}.$$

In the same manner, the space $\Phi(\mathcal{P}_\mu)$ of vanishing cycles on the main diagonal of X_μ is identified with the homogeneous component $\mathfrak{u}^-_{q,\mu}$.

This theorem is a part of equivalence of ribbon (= braided balanced) categories

$$\Phi : \ \mathcal{FS} \stackrel{\sim}{\longrightarrow} \mathfrak{u} - \text{mod}$$

Objects of $\mathcal{FS}$ are certain special "factorizable" perverse sheaves on the spaces $X_{0;\mu}$.

2.4. Microlocalization (Fourier - Sato transform) and the braid group action. We may vary a function f (2.3.1) and get a local system of spaces of vanishing cycles

$$\tilde{\Phi}_{\lambda,\mu} = \{\Phi_y(\mathcal{P}_{\lambda,\mu})_0\}$$

where $y = dg$ runs through a complement to a finite collection of hyperplanes in the cotangent space $T^*_0(X_{0;\mu})$.

On the other hand we have natural maps

$$\phi_\mu : \mathfrak{h} \longrightarrow T_0^*(X_{0;\mu}),$$

whence a local system

$$\Phi^\vee_{\lambda,\mu} = \phi_\mu^* \tilde{\Phi}_{\lambda,\mu}$$

over some complement of hyperplanes in $\mathfrak{h}$.

Theorem, [FS]. Let $q = e^v$ where v is a formal parmeter. (a) The local system $\Phi^\vee_{\lambda,\mu}$ is smooth on $\mathfrak{h}^{\mathrm{reg}}$ (i.e. it has no monodromy around the "superfluous" hyperplanes).

This way we get an action of PB on a fiber

$$(\Phi^\vee_{\lambda,\mu})_e = \Phi(\mathcal{P}_{\lambda;\mu})$$

(b) The isomorphism from [BFS]

$$\Phi(\mathcal{P}_{\lambda;\mu}) \xrightarrow{\sim} L(\lambda)_{\lambda-\mu}$$

is PB-equivariant, where on the rhs we consider the Lusztig's action of PB. $\square$

§3. Ribbon, Casimir (Laplacian), and BV

3.1. Let $\mathcal{C}$ be a braided tensor category, i.e. a tensor category equipped with natural isomorphisms

$$R_{X,Y}: \ X \otimes Y \overset{\sim}{\longrightarrow} Y \otimes X, \ \ X, Y \in \mathcal{C}$$

satisfying Yang - Baxter equations.

A **balance**, aka **ribbon structure** on $\mathcal{C}$ is an automorphism of the identity functor $\mathrm{Id}_{\mathcal{C}}$, i.e. a collection of automorphisms

$$\theta_X: \ X \overset{\sim}{\longrightarrow} X$$

such that

$$\theta_{X\otimes Y}(\theta_X \otimes \theta_Y)^{-1} = R_{X,Y}R_{Y,X}. \tag{3.1.1}$$

Thus "the square of R-matrix is a coboundary".

Example. $\mathcal{C} = \mathfrak{u}_q - Mod$. For each $M \in \mathcal{C}$ we have the action of the braid group B on M, and a central element $c \in PB \subset B$. The action of c on M gives rise to a balance θ_M.

In other words, the action of the braid group may be considered as a generalization of a balanced structure.

As was remarked by M.Kapranov, the formula (3.1.1) is analogous to the classical relation bertween the resultant and the discriminants of two polynomials

$$\frac{D(fg)}{D(f)D(g)} = R(f, g)^2,$$

cf. [Kap].

3.2. Explicit formula and Casimir. Cf. [CP]. Consider a $\mathbb{C}[h]]$-Hopf algebra $U = U_h\mathfrak{g}$, with the antipode S and the R-matrix

$$R \in U \otimes U,\ \ R \equiv 1 \otimes 1(h).$$

Set

$$u = \mu(S \otimes \mathrm{Id})R_{21} \in U,$$

Then

$$z = uS(u) = S(u)u \in Z(U).$$

One has a canonical isomorphism

$$Z(U_h\mathfrak{g}) \overset{\sim}{=} Z(U\mathfrak{g}). \tag{3.2.1}$$

The image of z under (3.2.1) is more or less e^c where $c \in U^{\check{}}$ is the usual quadratic Casimir element.

Note that c is geometrically a Laplace operator.

3.3. Batalin - Vilkovisky structures : another appearance of a Laplacian. Recall that a **Gerstenhaber algebra** is a commutative dg algebra $C^{\cdot}$, so that we have a multiplication $xy \in C^{i+j}, x \in C^i,\ y \in C^j$,

equipped with a (shifted) Lie bracket

$$[x, y] \in C^{i+j-1},\ x \in C^i,\ y \in C^j$$

such that two operations $xy, [x, y]$ form a (-1)-shifted $\mathbb{Z}$-graded Poisson algebra.

Example. If X is a variety, $\Lambda^\cdot T_X$, with the Schouten bracket.

A **BV-algebra** is a Gerstenhaber algebra $C^\cdot$ equpped with an operator $\Delta : C^i \longrightarrow C^{i-1}$ such that $\Delta^2 = 0$ and

$$\Delta(xy) - \Delta(x)y - (-1)^i x\Delta(y) = (-1)^i [x, y],\ x \in C^i.$$

Δ is an odd differential operator of the second order with respect to the multiplication, "an odd Laplacian".

Example. $\Lambda^\cdot T_X$ if we have an integrable connection on ω_X, for example if X is Calabi - Yau.

Hopf algebras are Koszul dual to Gerstenhaber algebras, cf. [K].

Under this duality the even Laplacian from 3.1, 3.2 corresponds to the odd Laplacian from 3.3.

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