Donaldson-Thomas for CY^4.

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§ Donaldson-Floer theory review

$$\mathbb{C}^r \longrightarrow E \longrightarrow Y^3 \qquad \text{Hermitian vector bundle.}$$

$$D_A \in \mathcal{A} = \{\text{connections on } E\}$$

$$= d + \Omega^1(Y, \operatorname{ad} E)$$

$$T^*_{D_A}\mathcal{A} = \Omega^1(Y, \operatorname{ad} E)^*$$

$$F_A \qquad = \Omega^2(Y, \operatorname{ad} E) \quad \text{via} \int (\text{1-form}) \wedge (\text{2-form})$$

Thus $D_A \mapsto F_A =: F(D_A)$ defines a 1-form on $\mathcal{A}$

$$F \in \Omega^1(\mathcal{A})$$

It is an exact 1-form, i.e.

$$F = d(CS)$$

$$\exists \quad CS : \mathcal{A} \longrightarrow \mathbb{R}$$

Explicitly, $CS(d+A) = \int_Y \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right)$

$\rightsquigarrow$ Morse theory $\left(\begin{array}{c}\text{modulo symmetry group} \\ \mathcal{G} = \mathrm{Map}(Y, SU(r))\end{array}\right)$

Crit. pt. of $CS \longleftrightarrow F_A = 0$ flat connection

Pick metric $g_Y \rightsquigarrow g_{\mathcal{A}} \rightsquigarrow$ gradient flow eqt.

$$\frac{d}{ds} D_{A(s)} = * F_{A(s)} \qquad s \in \mathbb{R}$$

4 dim interpretation:

$$X^4 = Y \times \mathbb{R}_s$$

w/ connection $\underbrace{\frac{\partial}{\partial s} \cdot ds + d_Y}_{dx} + A(s) =: D_{A_4}$

Gradient flow eqt. for $CS : \mathcal{A} \to \mathbb{R}$ $\iff$ $F_{A_4}^+ = 0$ ASD eqt. on X

Formally, Chern-Simons-Floer homology is

$$HF_{CS}(Y) := H^{\frac{\infty}{2}}_{\mathcal{G}}(\mathcal{A})$$ ← Witten-Morse for CS on $\mathcal{A}$.

- $\chi(\text{———}\;\text{"}\;\text{———})$ Casson invariant (up to 2)

$=$ "#" $\mathrm{Hom}(\pi_1 Y, SU(r)) / \mathrm{Ad}(SU(r))$.

Donaldson invariant for X^4

$$\mathcal{M}_X^{ASD} := \{ F_A^+ = 0 \} / \mathcal{G}$$

Say (expected) $\dim \mathcal{M}_X^{ASD} = 0$, then

$$\mathrm{Don}(X) := \# \overline{\mathcal{M}}_X^{ASD}$$

is a diffeomorphism inv. (say $b^+ > 1$) assuming

(1) transversality ($\Rightarrow$ $\mathcal{M}$ smooth)

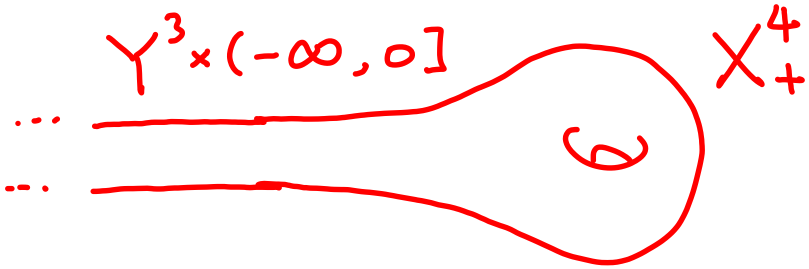
(2) compactness (by including ideal instantons)

(3) orientability ($\Rightarrow$ counting $/\mathbb{Z}$ makes sense).

When $\dim \overline{\mathcal{M}}^{ASD}_X > 0$, imposes constraints (by Donaldson μ-map $H_2(X) \xrightarrow{\mu} H^*(\mathcal{A}/\mathcal{G}) \xrightarrow{\text{restr.}} H^*(\mathcal{M})$) to cut down dim. to zero, then counts, giving Donaldson polynomial invariant:

$$\mathrm{Don} : S^\bullet H_2(X) \longrightarrow \mathbb{R}$$

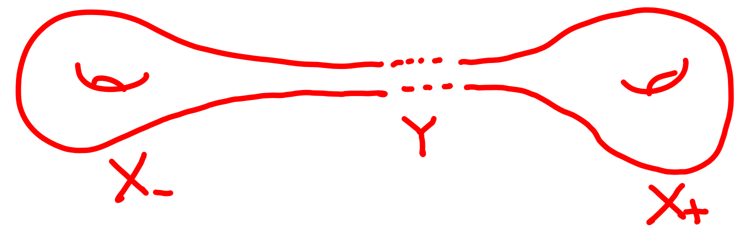
- Relative Donaldson inv.



$$\mathrm{Don}(X_+) \in HF_{cs}(Y)$$

Counting ASD / X_+ w/ flat conn. bdy condition on Y.

- Expect $X = X_- \#_Y X_+$



$$\mathrm{Don}(X) = \left\langle \mathrm{Don}(X_-) \middle| e^S \middle| \mathrm{Don}(X_+) \right\rangle_{HF(Y)}$$

S-matrix (count ASD/$Y \times \mathbb{R}$).

(gluing formula / degeneration formula).

§ Donaldson-Thomas for CY^3

– Complexification of CS theory

$Y^{3\mathbb{C}}$ Calabi-Yau (CY) 3-fold

i.e. compact Kähler-Einstein $Ric = 0$

(for simplicity $\pi_1(Y) = 0$)

(i.e. Holonomy group $Hol(Y, g) \subseteq SU(3)$)

$(Y, g, \underset{\text{Kähler form}}{\omega}, \underset{\text{holom. volume form}}{\Omega} \in \Omega^{3,0}(Y))$ s.t. $\Omega\overline{\Omega} = vol = \omega^3/3!$

Eg. Projective manifolds w/ $c_1 = 0$ (Yau thm)

(eg. smooth quintic 3-folds in $\mathbb{CP}^4$)

$$\mathbb{C}^r \longrightarrow E \longrightarrow Y$$

$$\bar{\partial}_A \in \mathcal{A}^{0,1} := \bar{\partial} + \Omega^{0,1}(Y, \operatorname{End} E)$$

$$T^*_{\bar{\partial}_A} \mathcal{A}^{0,1} = \Omega^{0,1}(Y, \operatorname{End} E)^*$$

$$\underbrace{F_A^{0,2}}_{(\bar{\partial}_A)^2} \qquad = \Omega^{0,2}(Y, \operatorname{End} E) \quad \text{via} \int_Y \varphi^{0,1} \wedge \eta^{0,2} \wedge \Omega$$

$\rightsquigarrow$ $F^{0,2}$: (Complex) 1-form on $\mathcal{A}^{0,1}$

$$\begin{matrix} F^{0,2} \\ \| \\ d\, CS_{\mathbb{C}} \end{matrix} \qquad \exists \quad CS_{\mathbb{C}} : \mathcal{A}^{0,1} \longrightarrow \mathbb{C}$$

Explicitly, $CS_{\mathbb{C}}(\bar{\partial} + A^{0,1}) := \int \operatorname{Tr}\left(A^{0,1} \bar{\partial} A^{0,1} + \frac{2}{3}(A^{0,1})^3\right) \wedge \Omega$

Critical point : $F_A^{0,2} = 0$, i.e. holomorphic VB str. on E/Y

- (Casson inv.)$\otimes\mathbb{C}$ $DT_3(Y) := \#\{F^{0,2}=0\}/\mathcal{G}^{\mathbb{C}} \in \mathbb{Z}$

(i.e. count # holo. VB / Y^3)

- orientability is automatic ($\because$ / $\mathbb{C}$)
- transversality (virtual technique (in alg. geom.))
- compactness : Via alg. geom.

Advantage : Can count 'singular' bundles. i.e. coherent sheaves.

eg. $\mathcal{I}_C$ w/ curve $C \subseteq Y$ ($\sim$ GW inv. of Y)

- Complex Morse Theory (of vanishing cycles)

Yes (Joyce et al., Kiem-Li) $DT_3(Y)$ homology.

(Categorification).

§ DT for CY^4 $(X^{4\mathbb{C}}, g, \omega, \Omega)$

$\Omega \Leftarrow \Omega^{4,0}$ holo. vol. form.

- $\underset{\circlearrowleft}{\Omega^{0,2}} = \Omega^{02}_{+} \oplus \Omega^{02}_{-}$

via $*_4$ defined by $\alpha \wedge *_4 \alpha = |\alpha|^2 \bar{\Omega}$

$(*_4)^2 = 1$

Remark: Unlike real case, $\Omega^{02}_{\pm}$ are related to each other.
namely, $\Omega^{02}_{-} = i\, \Omega^{02}_{+}$.

$$\begin{aligned} so(4) &= so(3) \oplus so(3) \\ \Omega^2 &= \Omega^2_{+} \oplus \Omega^2_{-} \end{aligned}$$

in $4\mathbb{R}$ dim.

$$\begin{aligned} su(4) &\curvearrowright \Omega^{0,2} \\ \cong \quad & \quad\ \cong \\ so(6) &\curvearrowright \mathbb{R}^6 \otimes \mathbb{C} \end{aligned}$$

DT4 $\quad \mathcal{M}^{DT4} := \{ F^{0,2}_{+} = 0 \} / \mathcal{G}^{\mathbb{C}}$

or $\quad \{ F^{0,2}_{+} = 0 = \Lambda F \} / \mathcal{G}$

i.e. use moment map eqt. ($\equiv$ HYM eqt) to perform symplectic quotient.

- $F^{02} = 0 \quad \Longrightarrow \quad F^{02}_{+} = 0 \ \ \& \ \int_X c_2(E) \wedge \Omega = 0$

(i.e. holo. VB) $\qquad$ (DT4) $\qquad \left(\because \begin{array}{l} c_2(E) \in H^{2,2} \\ \Omega \in \Omega^{4,0} \end{array} \right)$

(Lewis) — " — $\Longleftrightarrow$ — " — & — " —

From now on, assume $c_2(E) \in H^{2,2}(X)$.

Remark : Real analog for $X^{4\mathbb{R}}$

$$F = 0 \iff F_{+} = 0 \ \ \& \ \int_X c_2(E) = 0$$

Qu: $\# \mathcal{M}^{DT_4}$?

(1) Transversality (Virtual technique, eg. Li-Tian)

(2) Orientability (Donaldson)

Theorem (Cao-L.) $\forall$ CY^4 $\pi_1 = 0 = H_3 = H^3 \Rightarrow$ ✓

(i.e. trivialization of a suitable det line bdl.)

(3) Compactness. (assuming $c_2 \in H^{2,2}$)

$$\mathcal{M}^{DT_4} \underset{\text{as set}}{=\!=\!=} \{\text{ stable holo. VB}\}$$

Use alg. geom : Gieseker semi-stable sheaves to compactify.

$$\{\text{stable holo. VB}\} \underset{\text{as set}}{=\!=} \mathcal{M}^{DT_4}$$

but different obstruction theory,

$\rightsquigarrow$ different real analytic structure

- Kuranishi theory for holo. VB:

$$\kappa : H^1(X, \mathrm{End}\,E) \longrightarrow H^2(X, \mathrm{End}\,E)$$

$\kappa^{-1}(0)$: local moduli.

- Kuranishi for DT4: (Perfect obstruction theory.)

$$\kappa_+ : H^1(X, \mathrm{End}\,E) \longrightarrow H^2_+(X, \mathrm{End}\,E)$$

$$\kappa_+^{-1}(0) \overset{\text{loc.}}{=\!=} \mathcal{M}^{DT_4}$$

Theorem: As real analytic spaces

$\exists$ closed embedding $\{\text{stable holo. VB}\} \hookrightarrow \mathcal{M}^{DT_4}$

Note: $\kappa_+ : \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \longrightarrow \mathrm{Ext}^2_+(\mathcal{E},\mathcal{E})$

can be defined for ANY coherent sheaf $\mathcal{E}$

(Roughly speaking, $\Omega^{02} \longrightarrow \Omega^{02}_+$ has nothing to do w/ bdl)

$$\{\text{stable holo. VB}\}$$

$$\cap\!\!\mid$$

$$\{\overset{\text{Gieseker}}{\text{semi-stable}} \text{ coh. shf.}\} \leftarrow \text{compactificat}^{\underline{n}}.$$

Locally, can replace κ by κ_+ to obtain $\overline{\mathcal{M}^{DT_4}}$.

Key issue: Can these local models be glued together ?

Good cases:

- s.s. coh. shf. $\equiv$ stable VB $\Rightarrow$ $\overline{\mathcal{M}}^{DT_4} = \mathcal{M}^{DT_4}$ ✓
- $Ext^2(\mathcal{E},\mathcal{E}) = 0 \quad \forall\, \mathcal{E}$ $\Rightarrow$ $\overline{\mathcal{M}}^{DT_4} = \overline{\mathcal{M}}^{Gieseker}$ ✓
- $X^4 = K_{Y^3}$ $\Rightarrow$ $\overline{\mathcal{M}}^{DT_4}_X = \overline{\mathcal{M}}^{DT_3}_Y$

  Y : Fano 3-fold (for $\mathrm{Supp}\,\mathcal{E} \subseteq Y$)

$\left(\because\ (Ext^2_X(\mathcal{E},\mathcal{E}), Q) = (T^*V, Q_{std}) \quad V = Ext^1_Y(\mathcal{E}_3, \mathcal{E}_3)\right)$.

$\rightsquigarrow \quad \exists \ [\overline{\mathcal{M}}^{DT_4}]^{vir} \in H_r(\overline{\mathcal{M}}^{Gieseker})$

- deformation inv.
- vanish if X hyperkähler

  $\exists$ reduced inv.

Example : Genus 0 curves.

$C \subset X \qquad \beta \in [C] \in H_2(X, \mathbb{Z})$

Assume all C's are smooth, $g = 0$.

$$\Rightarrow \quad \overline{\mathcal{M}}^{DT_4}_X = \overline{\mathcal{M}}^{GW}_{0,0}(X, \beta) \quad \text{(as virtual cycles)}$$

Example: $X \subset \mathbb{P}^1 \times \mathbb{P}^4$ bidegree $(2,5)$ $\Rightarrow CY^4$

w/ polarization $\mathcal{O}_X(1, \underset{\sim\sim}{n})$

With suitable choice of $c(E)$

$$\text{(Li-Qin)} \quad \begin{array}{c} \overline{\mathcal{M}}^{\text{Gieseken}} \\ \text{\rotatebox{90}{$\cong$}} \\ \overline{\mathcal{M}}^{DT_4} \end{array} = \begin{cases} \mathbb{P}^5 & n \geq 2 \\ \phi & n = 1 \end{cases} \quad \text{wall-crossing.}$$

$$[\overline{\mathcal{M}}^{DT_4}]^{vir} = [\mathbb{P}^5] \text{ or } 0$$

Example: Count $\underbrace{1}_{\text{ideal shf. of}}$ point. $\overline{\mathcal{M}}^{DT_4} \simeq X$

- Strict CY^4 $\Rightarrow$ $[\overline{\mathcal{M}}^{DT_4}]^{vir} = \pm c_3(X) \in H_2(X)$
- Hyperkähler $\Rightarrow$ $[\text{---}\,''\,\text{---}]^{vir} = 0 \in H_1(X)$

$[\text{---}\,''\,\text{---}]^{vir}_{red} = 0 \in H_2(X)$.

Remark: Borisov-Joyce use (-2)-sympl. geom.

$\rightsquigarrow$ generalized DT_4-moduli spaces

$\rightsquigarrow$ virtual fund. class.

Remark: Complex Morse theory $\rightsquigarrow$

$SU(3) \subset G_2 \subset Spin7$ DT-theory ......