Exactly solvable models of tilings and Littlewood–Richardson coefficients

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Outline of the talk

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Random tilings

- Random tilings are simple models whose main purpose is to describe quasi-crystals.
- They typically correspond to a high-temperate limit where entropy considerations dominate.
- All (known) random tiling models can be thought of as fluctuating surfaces (i.e. bosonic fields) in a higher-dimensional space.
- Typical configurations may have "forbidden" symmetries. For example, the square/triangle model has 12-fold symmetry!

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Schur functions and Littlewood–Richardson coefficients

- Schur functions are the most important family (basis) of symmetric functions in algebraic combinatorics.
- They are also characters of $GL(N)$.
- They form bases of the cohomology ring of Grassmannians. (related to Schubert varieties)
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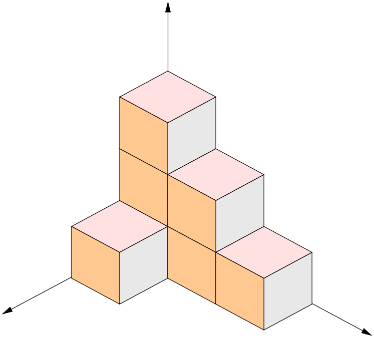
Plane partitions



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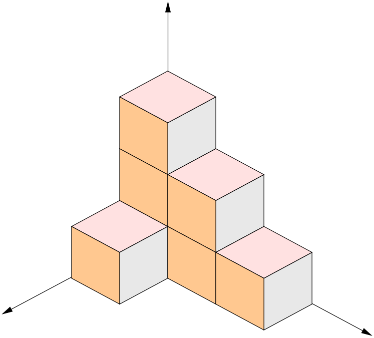
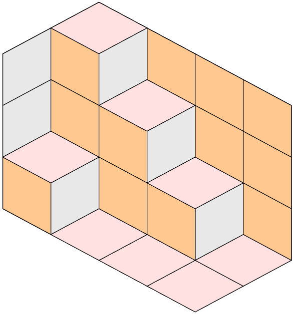
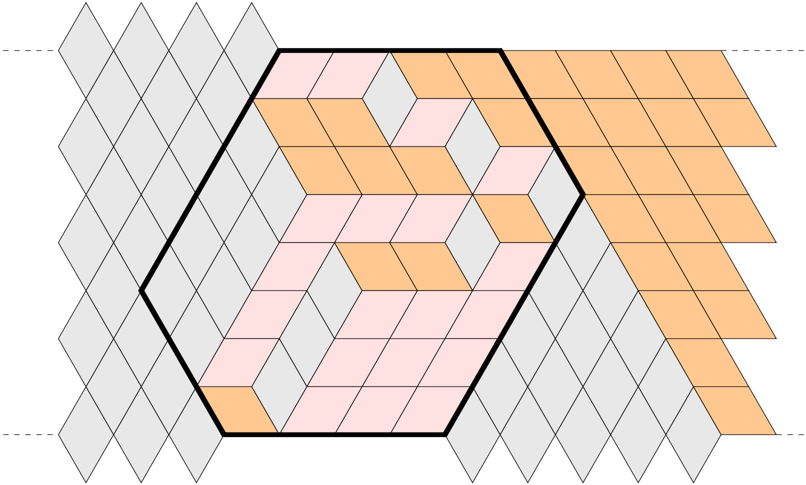
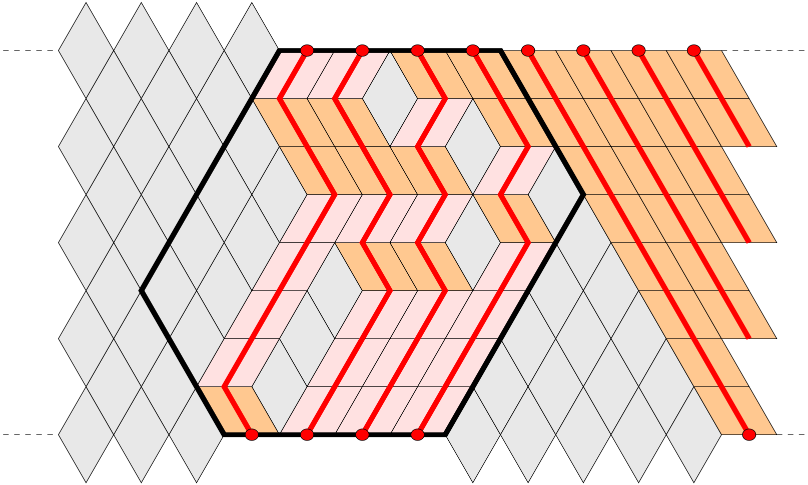




Lozenge tilings



Non-Intersecting Lattice Paths



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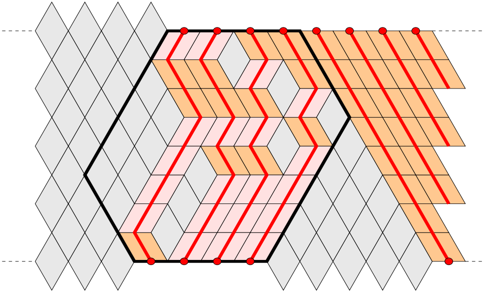
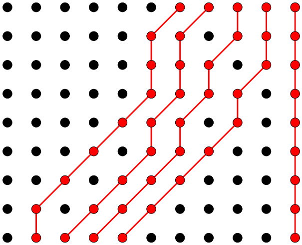
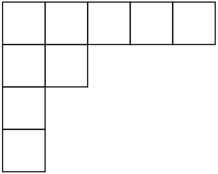


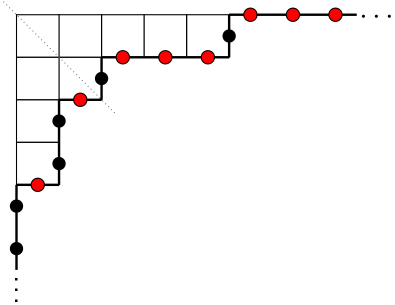


Fermionic states and Young diagrams

Define a *partition* to be a weakly decreasing finite sequence of non-negative integers: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. We usually represent partitions as *Young diagrams*: for example $\lambda = (5, 2, 1, 1)$ is depicted as

$\lambda =$ 

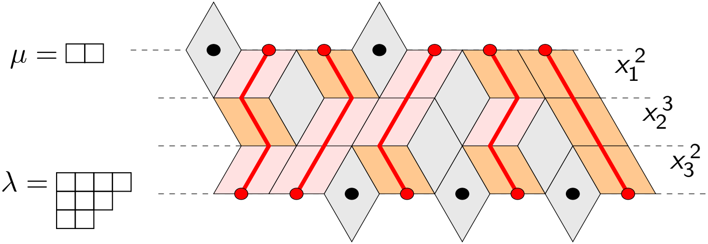
To each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ one associates a fermionic state $|\lambda\rangle$ so that the black (resp. red) sites correspond to vertical (resp. horizontal) edges:



$\mathcal{F} = \bigoplus_\lambda \mathbb{C}\,|\lambda\rangle$ is the fermionic Fock space (with charge 0).

Definition of Schur polynomials

To a pair of Young diagrams λ, μ one associates the skew Schur polynomial $s_{\lambda/\mu}(x_1, \ldots, x_n)$:



The (usual) Schur polynomial is $s_\lambda = s_{\lambda/\varnothing}$.

Remark: the number of plane partitions in $a \times b \times c$ is $s_{[a\times c]}(x_1 = \cdots = x_{a+b} = 1)$.

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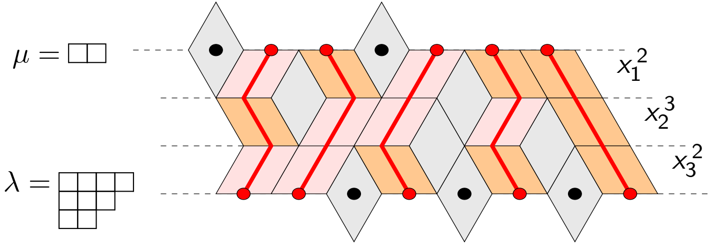

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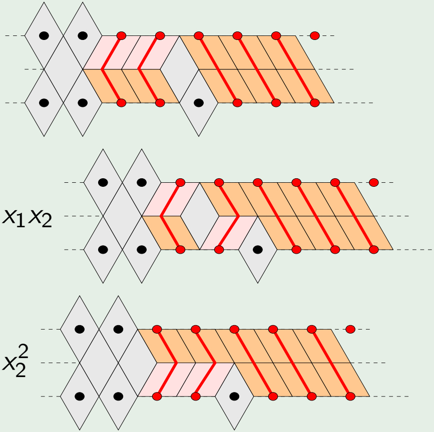
Example

$$s_{\square\square}(x_1, x_2) = \quad x_1^2 \quad + x_1 x_2 \quad + x_2^2$$



Transfer matrix formulation

Consider the operator $T(x)$ on $\mathcal{F}$ with matrix elements

$$\langle\mu|\, T(x)\, |\lambda\rangle = s_{\lambda/\mu}(x)$$

It corresponds to the addition of one row of the tiling.
In particular

$$s_{\lambda/\mu}(x_1, \ldots, x_n) = \langle\mu|\, T(x_1) \ldots T(x_n)\, |\lambda\rangle$$

Properties

- "Integrability" property:

$$[T(x), T(x')] = 0 \quad \Rightarrow \quad s_{\lambda/\mu} \text{ symmetric polynomial}$$

- Stability property:

$$T(0) = I \quad \Rightarrow \quad s_{\lambda/\mu}(x_1, \ldots, x_n, x_{n+1} = 0) = s_{\lambda/\mu}(x_1, \ldots, x_n)$$

Thus, the $s_{\lambda/\mu}$ are *symmetric functions* (symmetric polynomials in an infinite number of variables).
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Some identities

- An identity that can be derived using the formalism above:

$$\sum_{\mu} s_{\lambda/\mu}(x_1,\ldots,x_n)s_{\mu/\rho}(y_1,\ldots,y_m) = s_{\lambda/\rho}(x_1,\ldots,x_n,y_1,\ldots,y_m)$$

- Identities which remain mysterious:

$$s_{\lambda/\mu}(x_1,\ldots,x_n) = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_\nu(x_1,\ldots,x_n)$$

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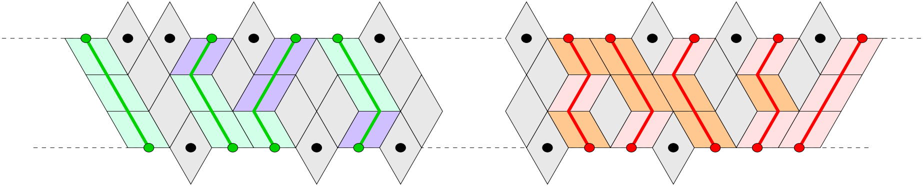
Two species of fermions



Pilings of (hyper)cubes in *four* dimensions!

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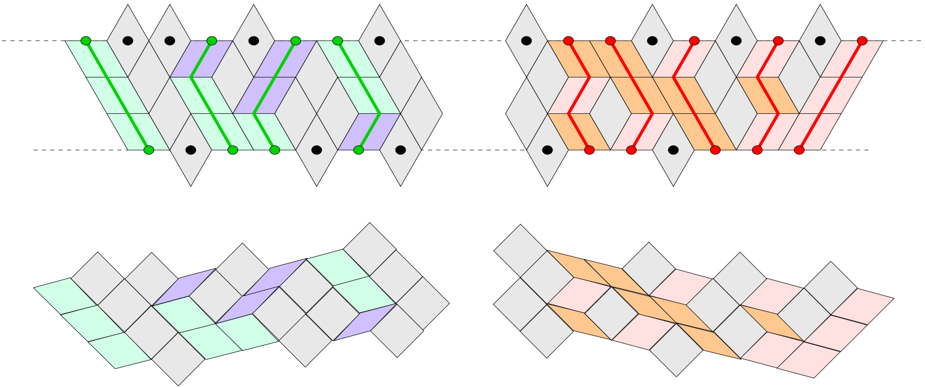

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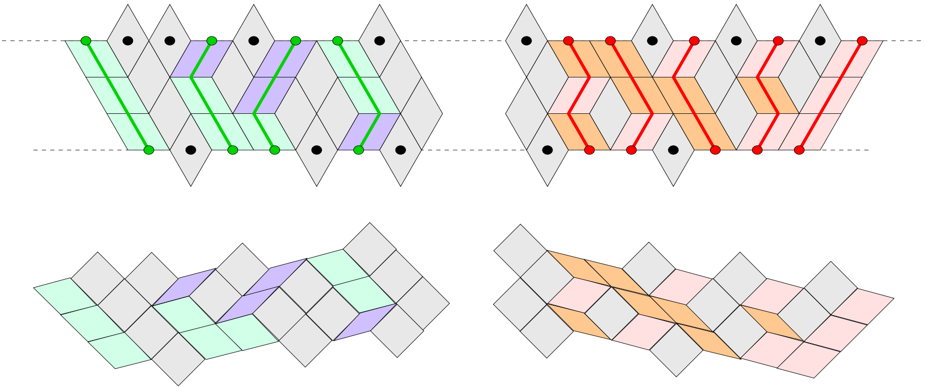

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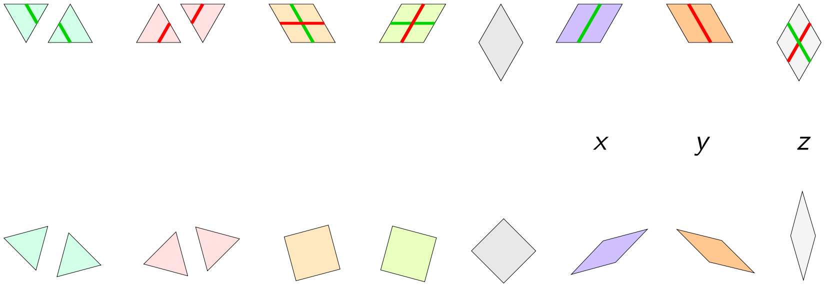
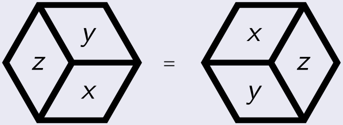
The interaction



Yang–Baxter equation

Theorem

If $x + y + z = 0$, then



for any fixed boundaries and where tile x (resp. y, z) is only allowed where marked.

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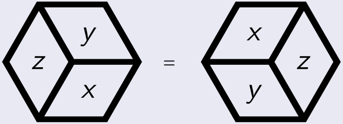

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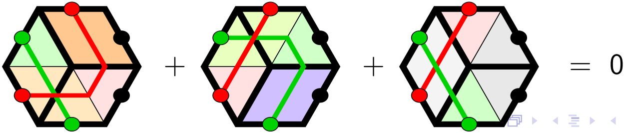

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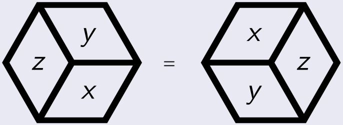

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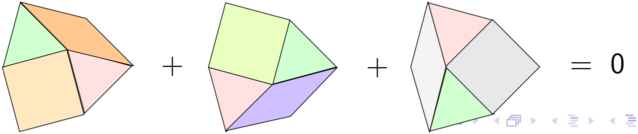
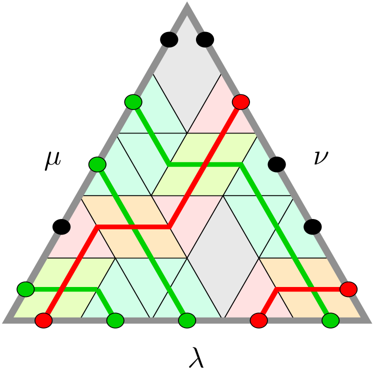
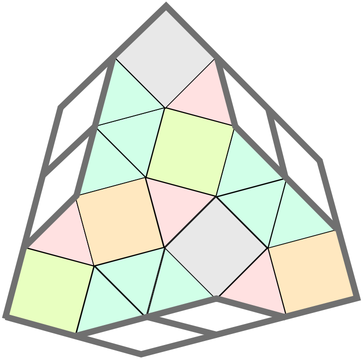


Puzzles

Remove all tiles x, y, z:





Some history...

- 1993: M. Widom introduces the square-triangle model, deforms it into a regular triangular lattice ($\sim$ puzzles) and proves integrability.
- 1994: P. Kalugin (partially) solves the Coordinate Bethe Ansatz equations (size$\to \infty$).
- 1997–2006: B. Nienhuis et al reinvestigate it: underlying algebra, commuting transfer matrices, force networks ($\sim$ honeycombs).
- 1992: Berenstein, Zelevinsky introduce a new Littlewood–Richardson rule (honeycombs).
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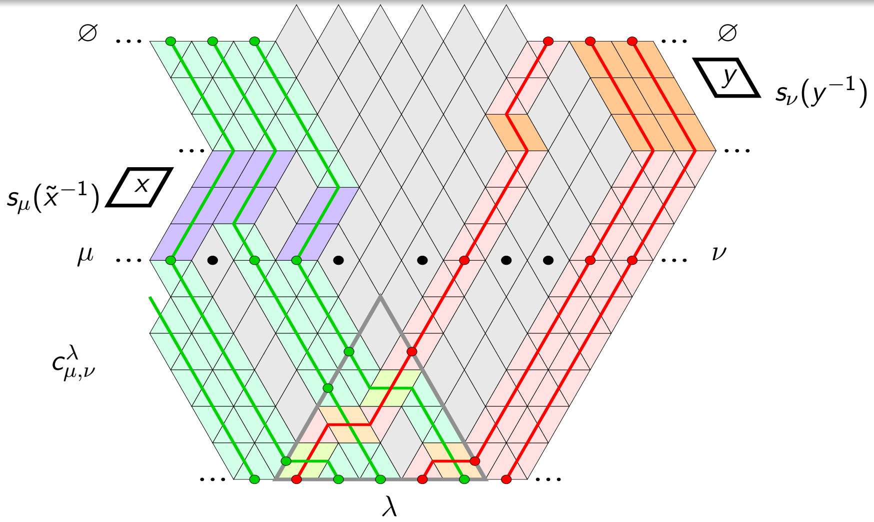
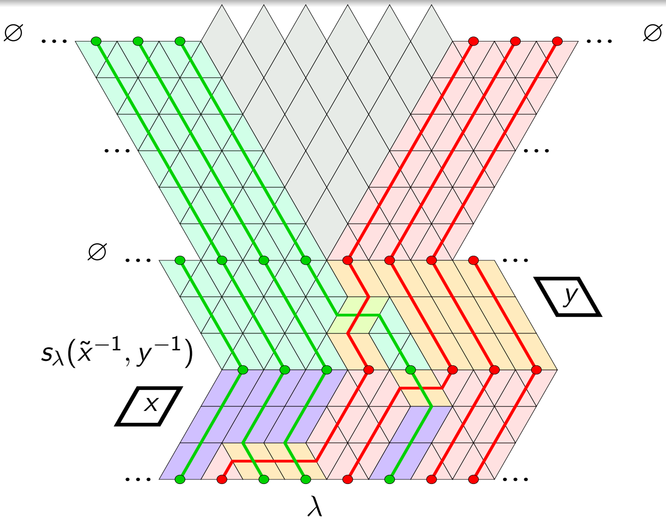


$$\sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu(\tilde{x}^{-1}) s_\nu(y^{-1})$$



$s_\lambda(\tilde{x}^{-1}, y^{-1})$

Cohomology of Grassmannians

The cohomology ring of $Gr(n,k) = \{V \subset \mathbb{C}^n,\ \dim V = k\}$ is the quotient of the ring of symmetric functions by the span of the s_λ, $\lambda \not\subset [k \times (n-k)]$.

Given a fixed flag, one can build *Schubert varieties* indexed by $\lambda \subset [k \times (n-k)]$ such that the s_λ are their cohomology classes.

There is a torus $T = (\mathbb{C}^\times)^n$ acting on $Gr(n,k)$ and a corresponding equivariant cohomology ring. It is a module over $\mathbb{Z}[y_1, \ldots, y_n]$, with basis the $\tilde{s}_\lambda$, $\lambda \subset [k \times (n-k)]$.

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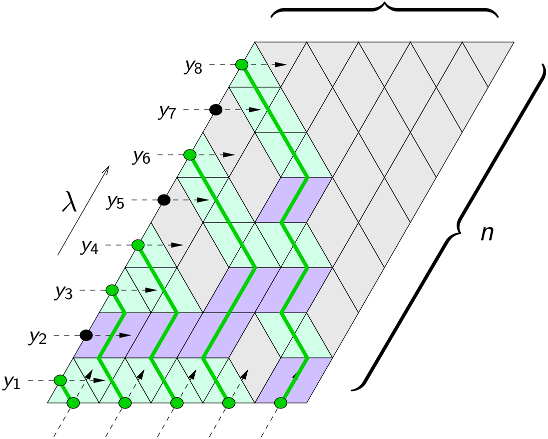
Double Schur functions

The $\tilde{s}_\lambda$ can be represented as polynomials $s_\lambda(x_1, \ldots, x_n | y_1, \ldots, y_n)$.
(such that $s_\lambda(x_1, \ldots, x_n | \underbrace{0, \ldots, 0}_{k}) = s_\lambda(x_1, \ldots, x_n)$).



Product formulae

- Knutson–Tao problem:

$$s_\lambda(x_1,\ldots,x_k|z_1,\ldots,z_n)s_\mu(x_1,\ldots,x_k|z_1,\ldots,z_n) = \sum_\nu c^\nu_{\mu,\lambda}(z_1,\ldots,z_n)s_\nu(x_1,\ldots,x_k|z_1,\ldots,z_n)$$

- Molev–Sagan problem:

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Unifying solution of these two problems!

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$$s_\lambda(x_1,\ldots,x_k|z_1,\ldots,z_n)s_\mu(x_1,\ldots,x_k|z_1,\ldots,z_n) \\ = \sum_\nu c^\nu_{\mu,\lambda}(z_1,\ldots,z_n)s_\nu(x_1,\ldots,x_k|z_1,\ldots,z_n)$$

- Molev–Sagan problem:

$$s_\lambda(x_1,\ldots,x_k|z_1,\ldots,z_n)s_\mu(x_1,\ldots,x_k|y_1,\ldots,y_n) \\ = \sum_\nu e^\nu_{\lambda,\mu}(y_1,\ldots,y_n;z_1,\ldots,z_n)s_\nu(x_1,\ldots,x_k|y_1,\ldots,y_n)$$

Unifying solution of these two problems!

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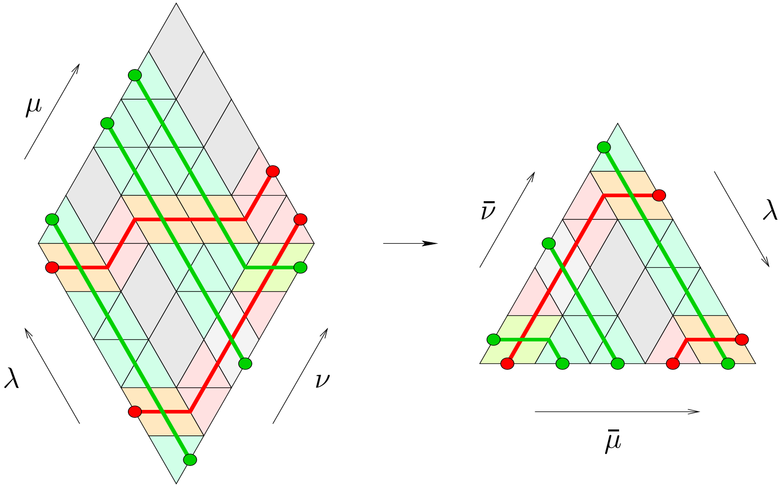
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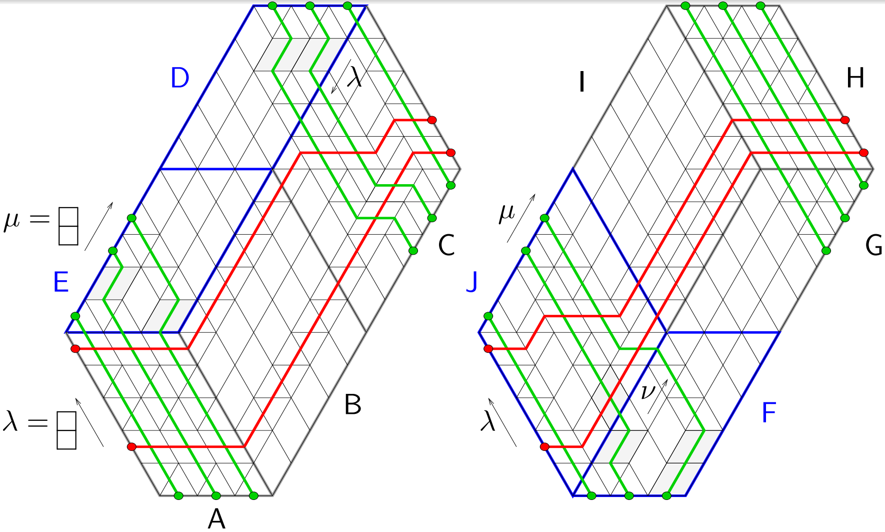

$$s_\lambda(x|z)s_\mu(x|y) = \sum e^\nu_{\lambda,\mu}(y;z)s_\nu(x|y)$$

- "Integrable" proofs of combinatorial identities?
- Coproduct formula for double Schur functions?
- Use of Bethe Ansatz?
- Connection to work of Gleizer and Postnikov?
- Generalization to other families of symmetric polynomials? (Jack, Hall–Littlewood, Macdonald)
- Generalization to other families of polynomials of geometric origin? (Schubert, Grothendieck)

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