A holographic entanglement triptych

Mukund Rangamani

KAVLI IPMU, TOKYO

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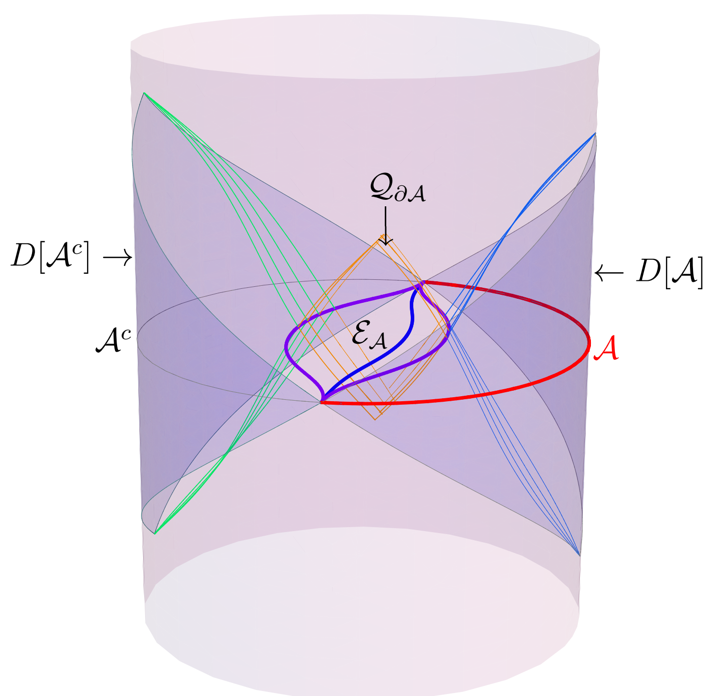
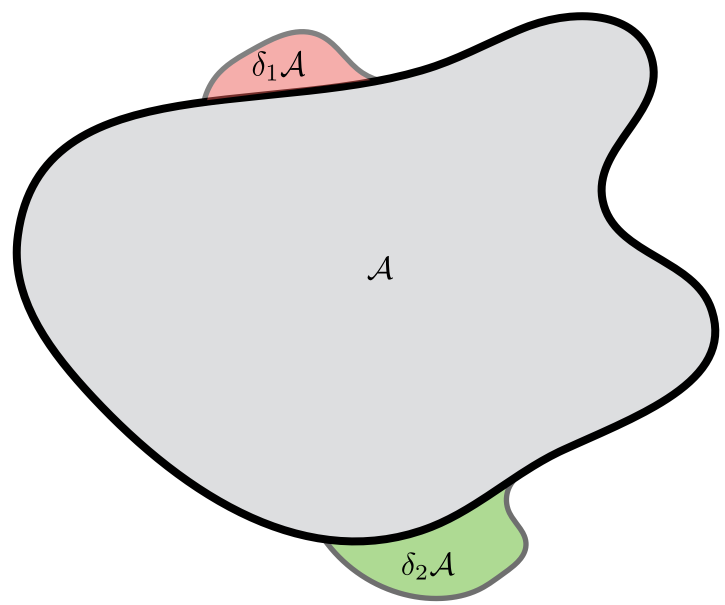
Causality



Matt Headrick,
Veronika Hubeny,
Albion Lawrence,
MR

1408.6300

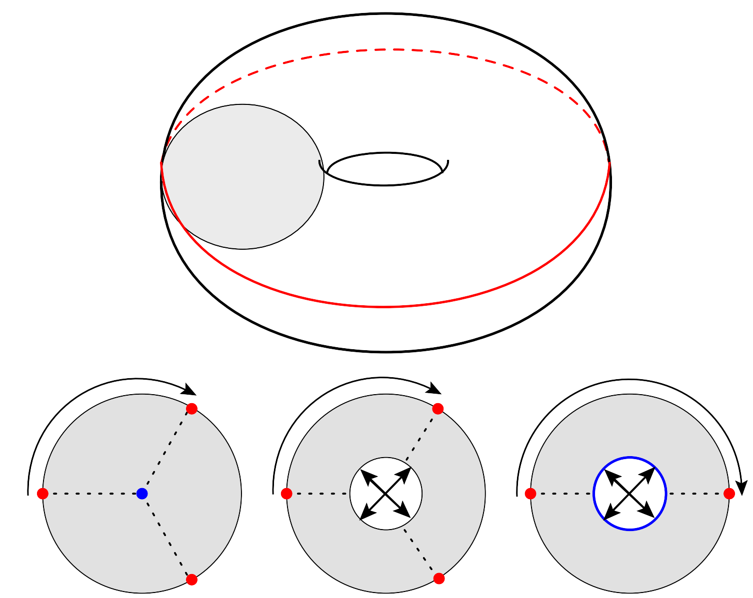
Density



Jyotirmoy Bhattacharya,
Veronika Hubeny,
MR
Tadashi Takayanagi

1412.5472

Topology



Felix Haehl,
Tom Hartman,
Don Marolf,
Henry Maxfield
MR

1412.7561

Outline

- Motivation
- Causality and entanglement entropy
- The homology constraint
- Entanglement density and gravity
- Summary

Motivation

- Holography via the AdS/CFT correspondence gives us a map between QFT and dynamics of gravity.
- The dictionary between the bulk and boundary observables should be tightly constrained by the consistency conditions of relativistic QFT.
- For eg., a pre-requisite for a sensible bulk-boundary map is that bulk dynamics respect boundary causality; this is true for sensible matter theories in the bulk.
- More generally relations/constraints satisfied by field theory quantities must be demonstrable from the bulk.
- We shall explore three such features for entanglement entropy.

Motivation

- In recent years we have come to appreciate that fundamental quantum concepts have a geometric avatar, e.g., entanglement.
- While entanglement is not a true observable in QFT, it nevertheless obeys some non-trivial consistency requirements, especially vis a vis causality.
- Holographically information about quantum entanglement information is encoded in a codimension-2 extremal surface in the bulk spacetime.

Ryu & Takayanagi (2006): RT   Hubeny, MR & Takayanagi (2007): HRT

- The static prescription of RT is by now well understood and can be derived by a path integral argument and satisfies various consistency conditions.

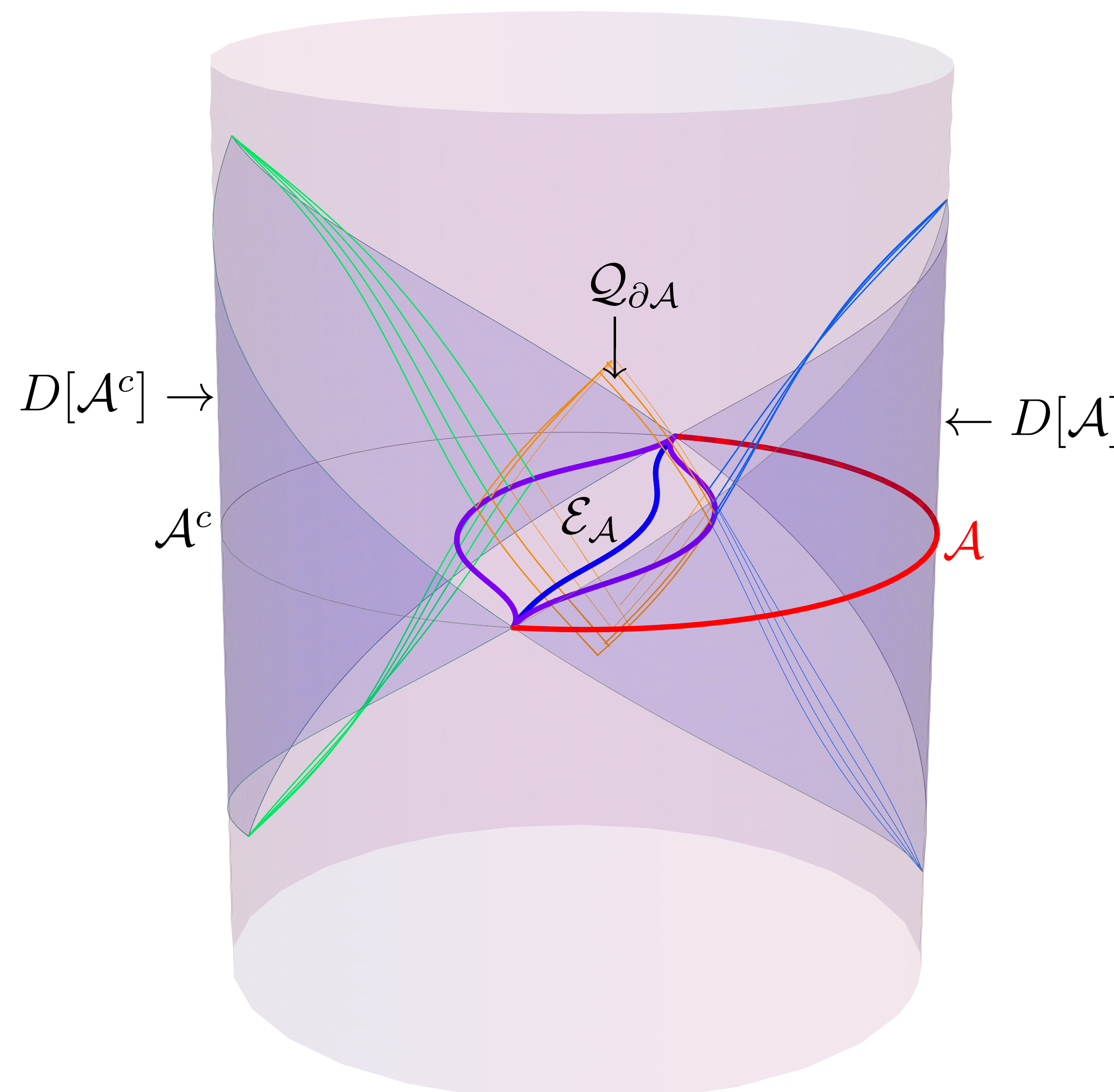
Casini, Heurta & Myers (2011): CHM   Lewkowycz & Maldacena (2013): LM

- Restriction to static situations however does not lend itself to analysis of causality conditions; one has to therefore test the covariant holographic prescription of HRT.

Motivation

- We shall ask three different questions about holographic entanglement entropy and find some interesting surprises in how the field theory expectations are upheld.
  - Part I: Holographic entanglement entropy satisfies field theory causality.
  - Part II: The homology constraint on holographic entanglement follows from a Euclidean quantum gravity path integral.
  - Part III: Constraints from strong-subadditivity suggest connections with the laws of gravitational thermodynamics.

Causality of Holographic Entanglement



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Wall **1211.3494**

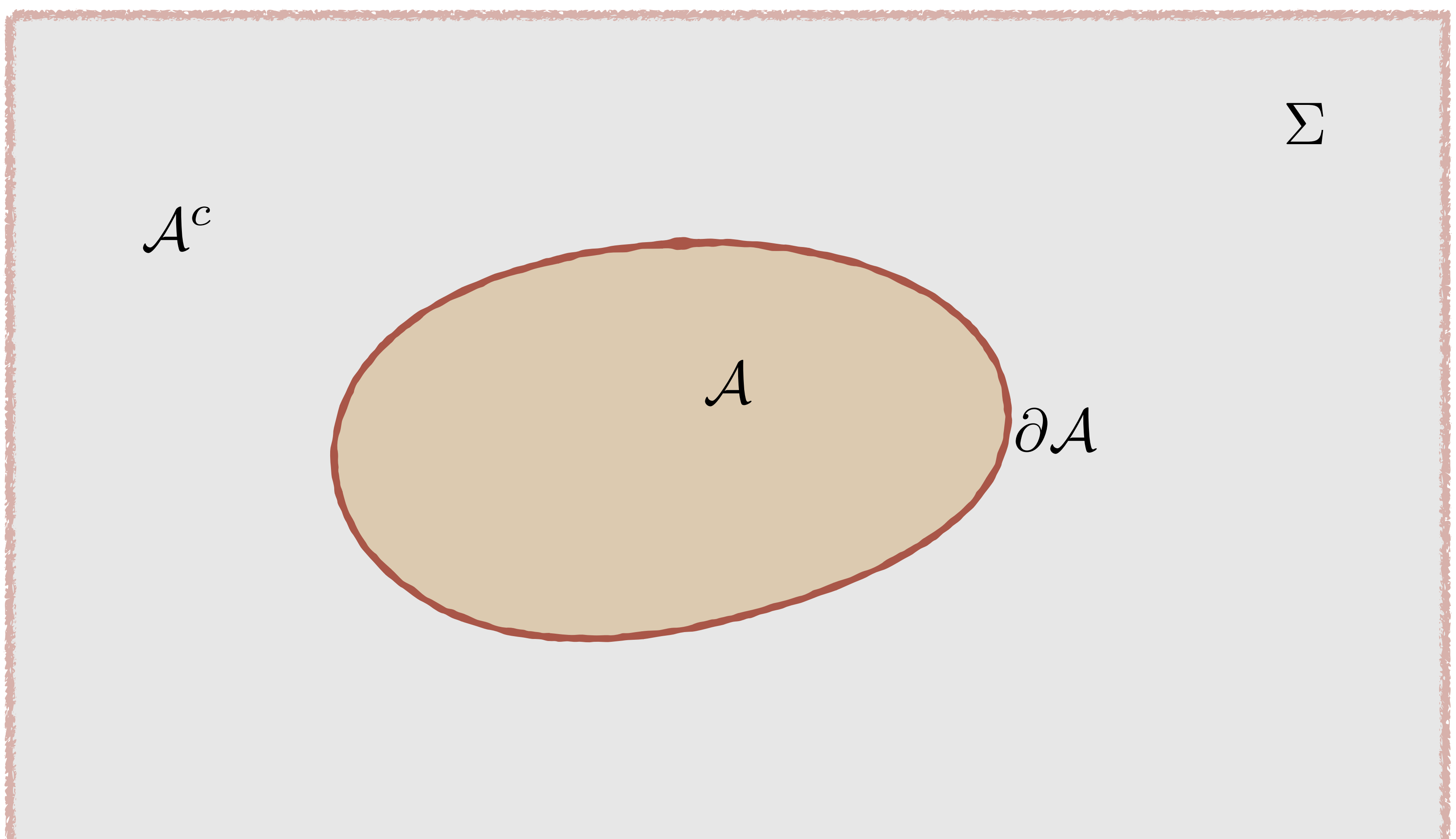
Motivation

- *No short-cuts through the bulk.*
  - For bulk matter obeying null energy condition, signals propagating through an asymptotically AdS bulk spacetime are time-delayed relative to signals propagating through the boundary. Gao & Wald (2000)
  - Fastest propagation between boundary points is along the boundary.
- Note that this statement relies on the bulk spacetime being smooth.
- Timelike singularities in the bulk can indeed result in time advance.
  - Obvious eg., negative mass AdS-Schwarzschild
  - Less obvious: charged scalar solitons with positive boundary energy.

Gentle & MR (2013)

Entanglement in QFT

- Consider a QFT in a density matrix, living on a background $\mathcal{B}$ which is globally hyperbolic spacetime with a nice time foliation (Cauchy slices Σ).
- $\mathcal{A}$ is a subregion of the Cauchy slice, with an *entangling surface* $\partial\mathcal{A}$.

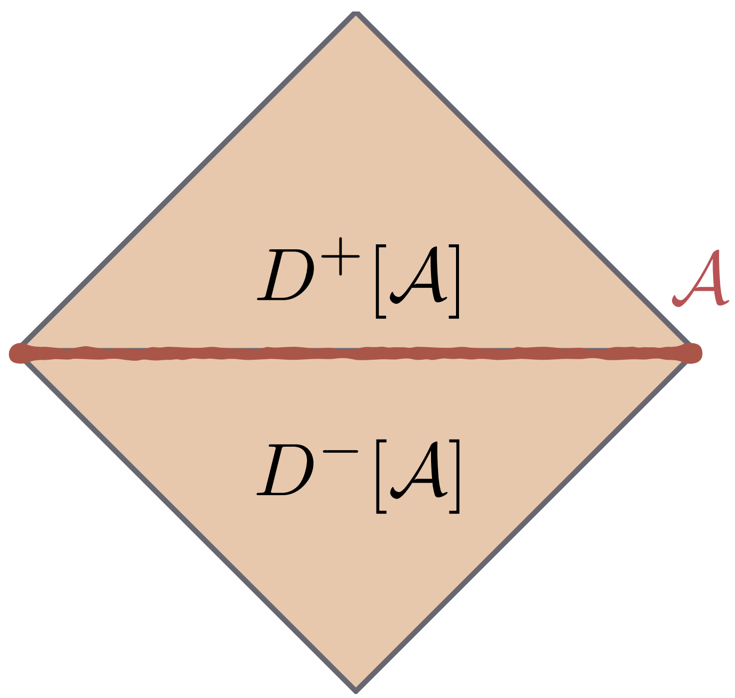


$\rho_{\mathcal{A}}$ reduced density matrix

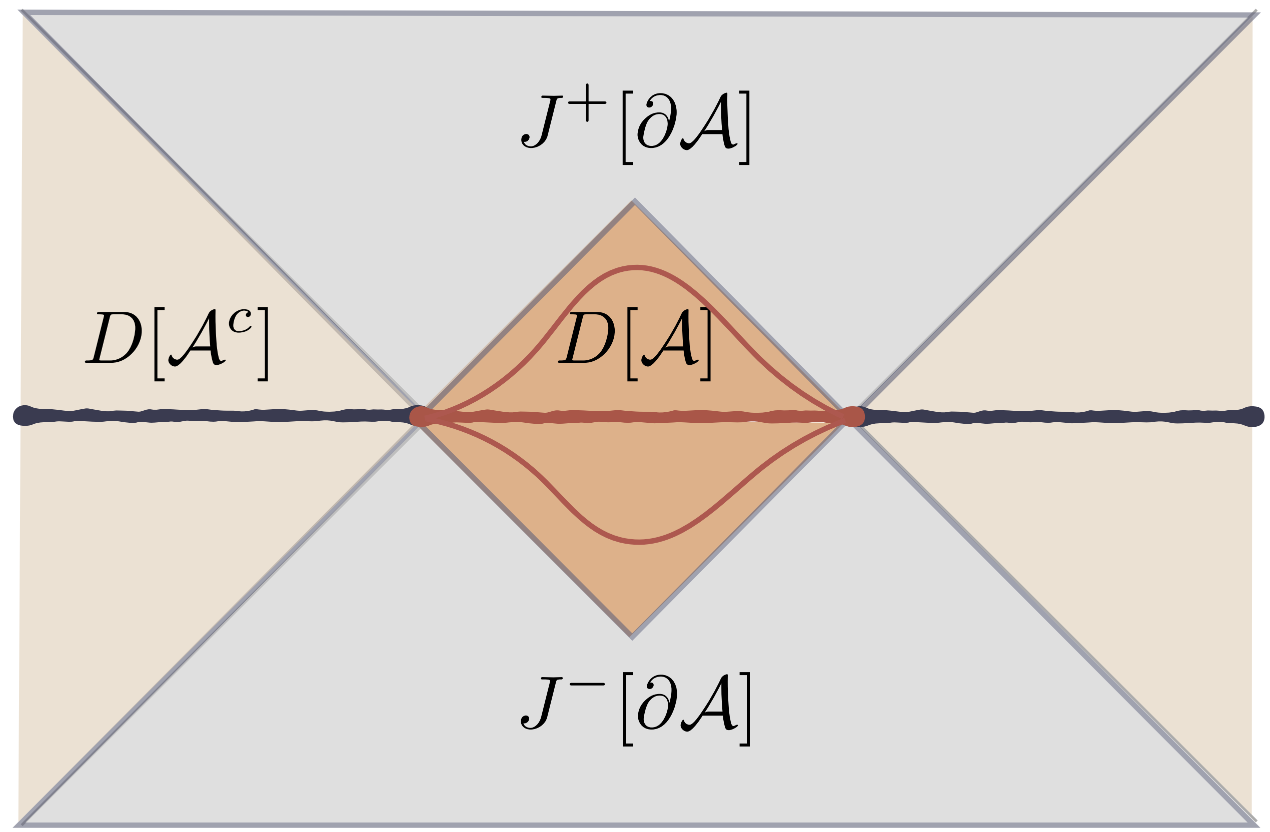
$$S_{\mathcal{A}} = -\mathrm{Tr}\left(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}\right)$$

Causality and Entanglement

✦ Entanglement entropy in QFT is a *wedge observable*.



$D[\mathcal{A}] = D^+[\mathcal{A}] \cup D^-[\mathcal{A}]$



✦ The entanglement entropy can only be influenced by changing conditions in the past domain $J^-[\partial\mathcal{A}]$.

Covariant Holographic Entanglement Entropy

- Given the boundary region $\mathcal{A}$ the prescription to compute entanglement holographically involves finding a bulk extremal surface $\mathcal{E}_{\mathcal{A}}$ which is anchored on $\partial\mathcal{A}$ and is homologous to $\mathcal{A}$.

$$S_{\mathcal{A}} = \frac{\mathrm{Area}(\mathcal{E}_{\mathcal{A}})}{4G_N}$$

- The extremal surface $\mathcal{E}_{\mathcal{A}}$ is a codimension-2 surface in the bulk asymptotically AdS spacetime $\mathcal{M}$ (nb: $\partial\mathcal{M} = \mathcal{B}$)

- The proposal has passed some basic consistency checks and gives reasonable results in many settings, but unlike the static case we don't yet have a proof.

- Progress has been made in proving various entropy inequalities (strong subadditivity), but lacking a general proof tests of consistency with QFT are desirable.

Why causality for HRT?

- To appreciate the problem, recall that a-priori causal domains seem not be a barrier to extremal surfaces. In dynamical spacetimes the extremal surface can (and often does) go behind event and apparent horizons.

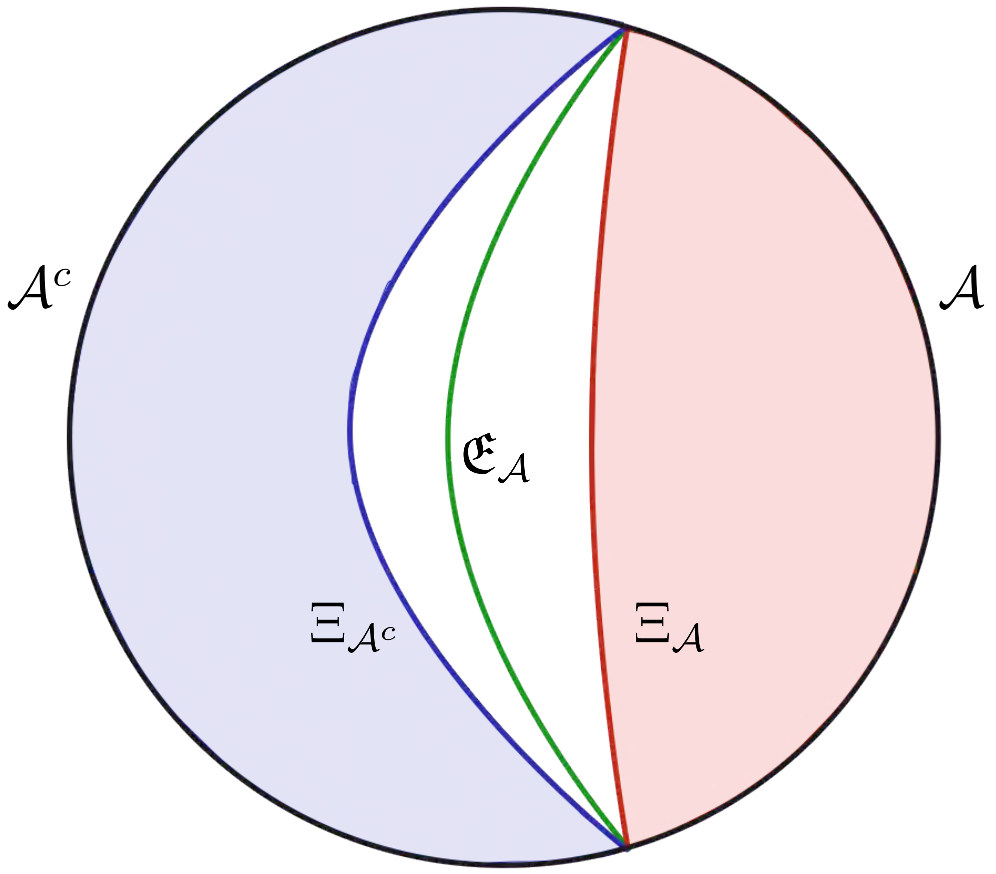
- More generally, associated with a region on the boundary, we can define a corresponding bulk causal wedge.

- Extremal surfaces can be shown to lie *outside* the causal wedge in asymptotically AdS spacetimes.

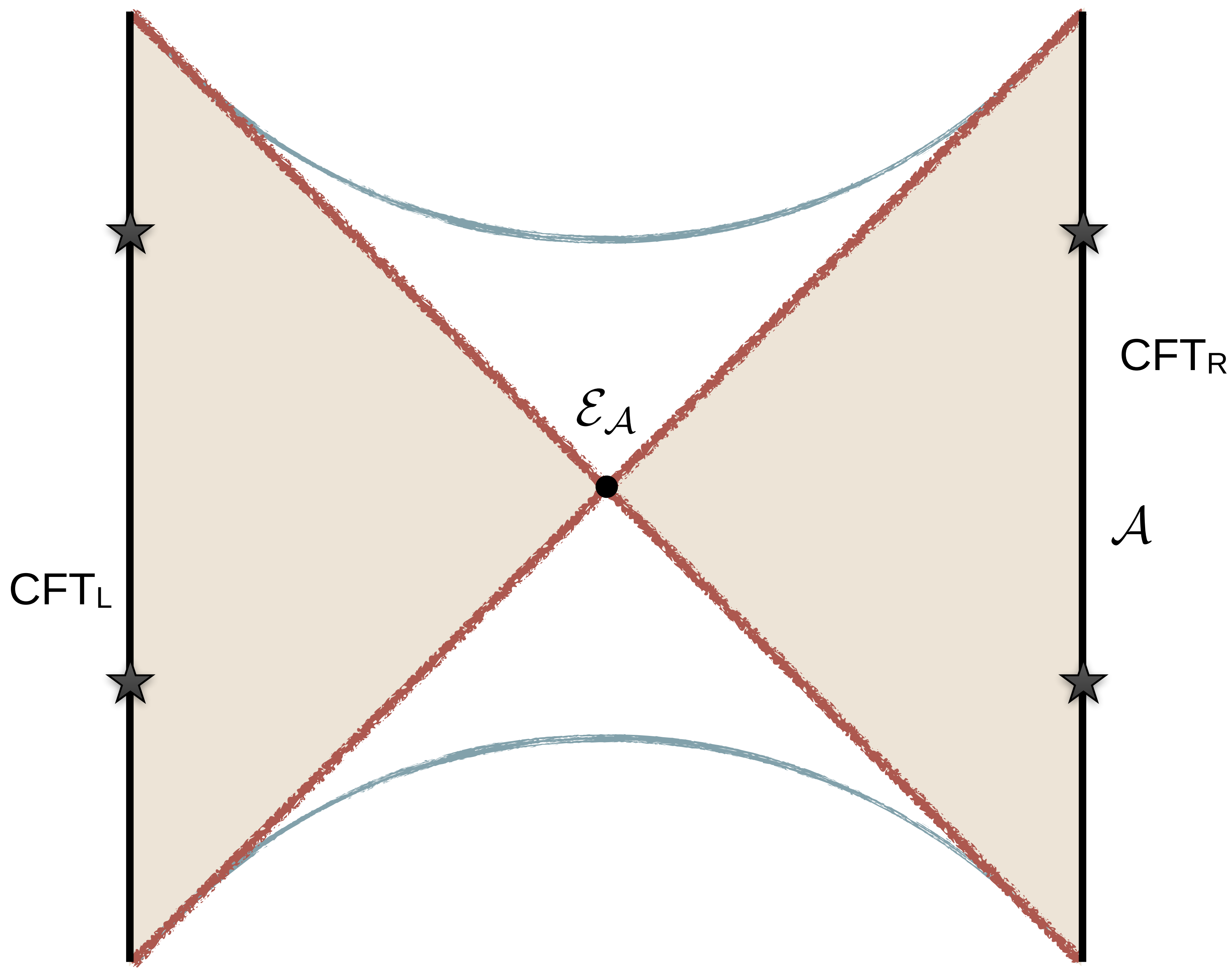
- This is in fact a consequence of the time-delay result discussed earlier.

Hubeny, MR (2012) Wall (2012)

Hubeny, MR & Tonni (2013)

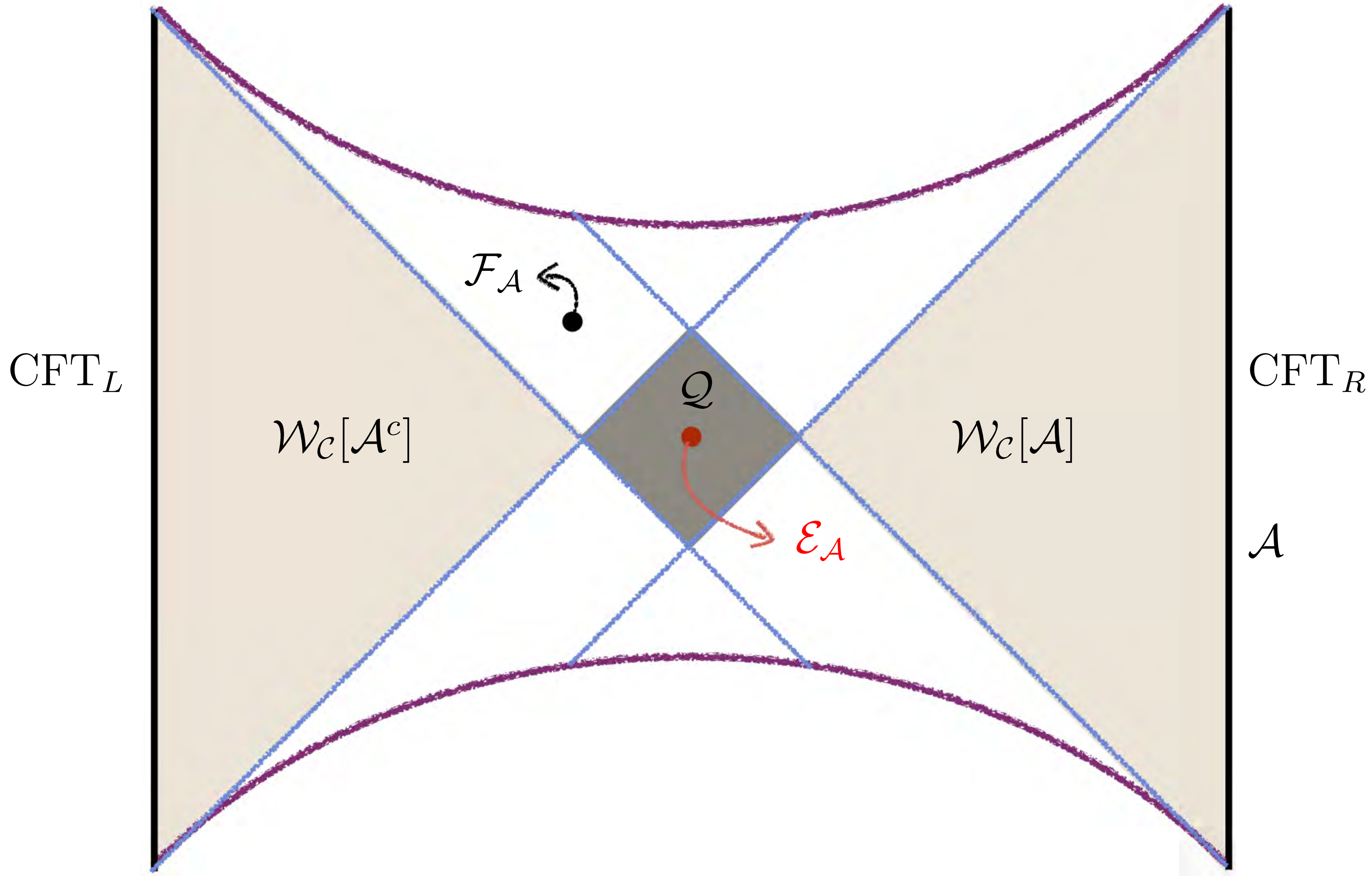


A gedanken experiment



Eternal BH in AdS = Entangled state in 2 CFTs
Perturb the two boundaries

A gedanken experiment



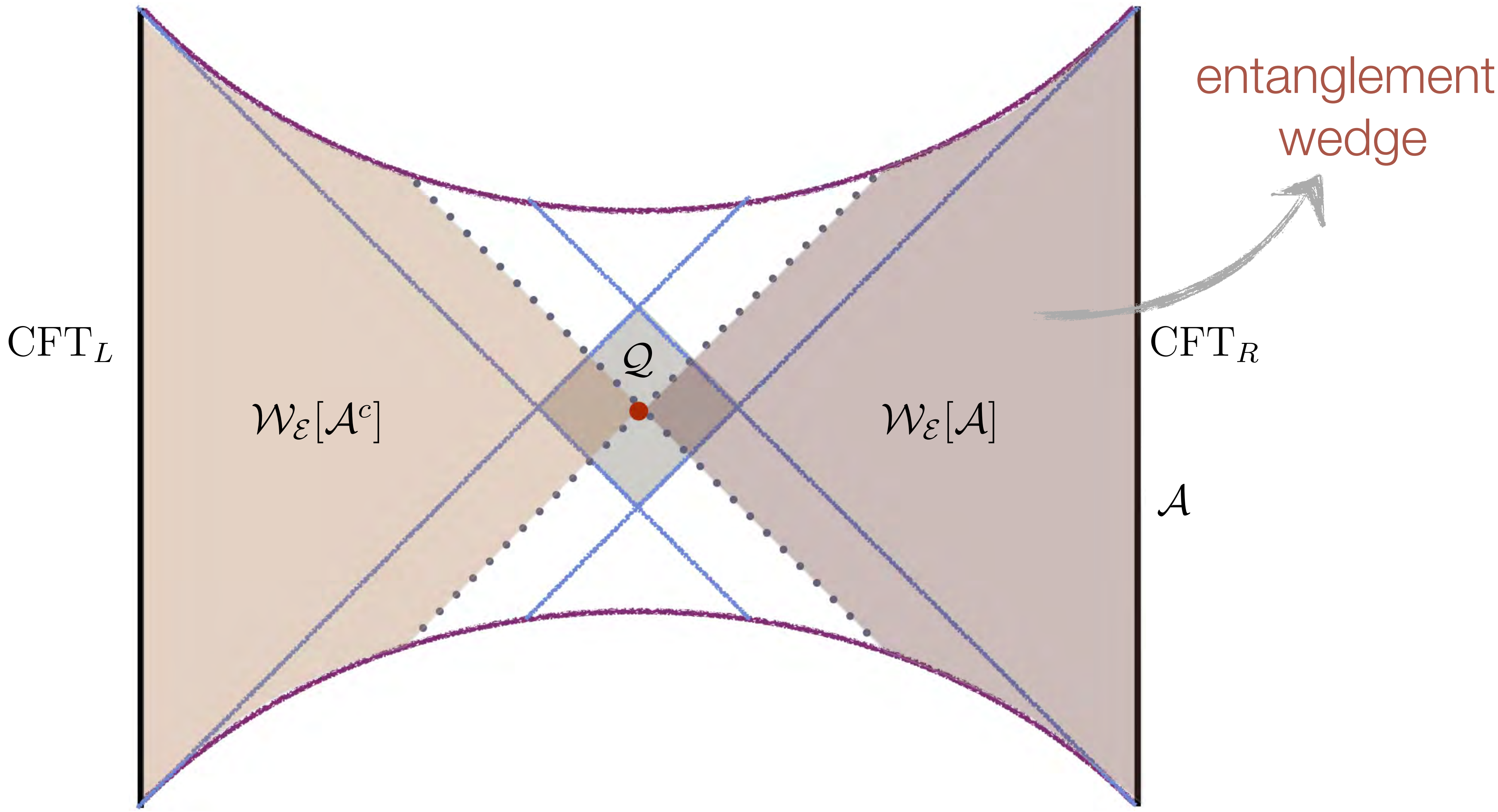
The extremal surface lies in the *causal shadow*.

The argument

- The extremal surface is by definition a codimension-2 bulk surface whose null expansions are vanishing.
- The congruence emanating from this surface will start converging & it cannot make it out to the boundary without encountering caustics/crossover points.
- If the extremal surface lies in the forbidden regions where it can be influenced by perturbations in the causal wedge of the region or its complement, then the null congruence can make it out to the boundary within the domain of dependence $D[\mathcal{A}]$.
- One can in fact show that the congruence from $\mathcal{E}_{\mathcal{A}}$ intersects the boundary precisely on $D[\mathcal{A}]$.

Entanglement wedges

Natural decomposition of the bulk spacetime in distinct domains.

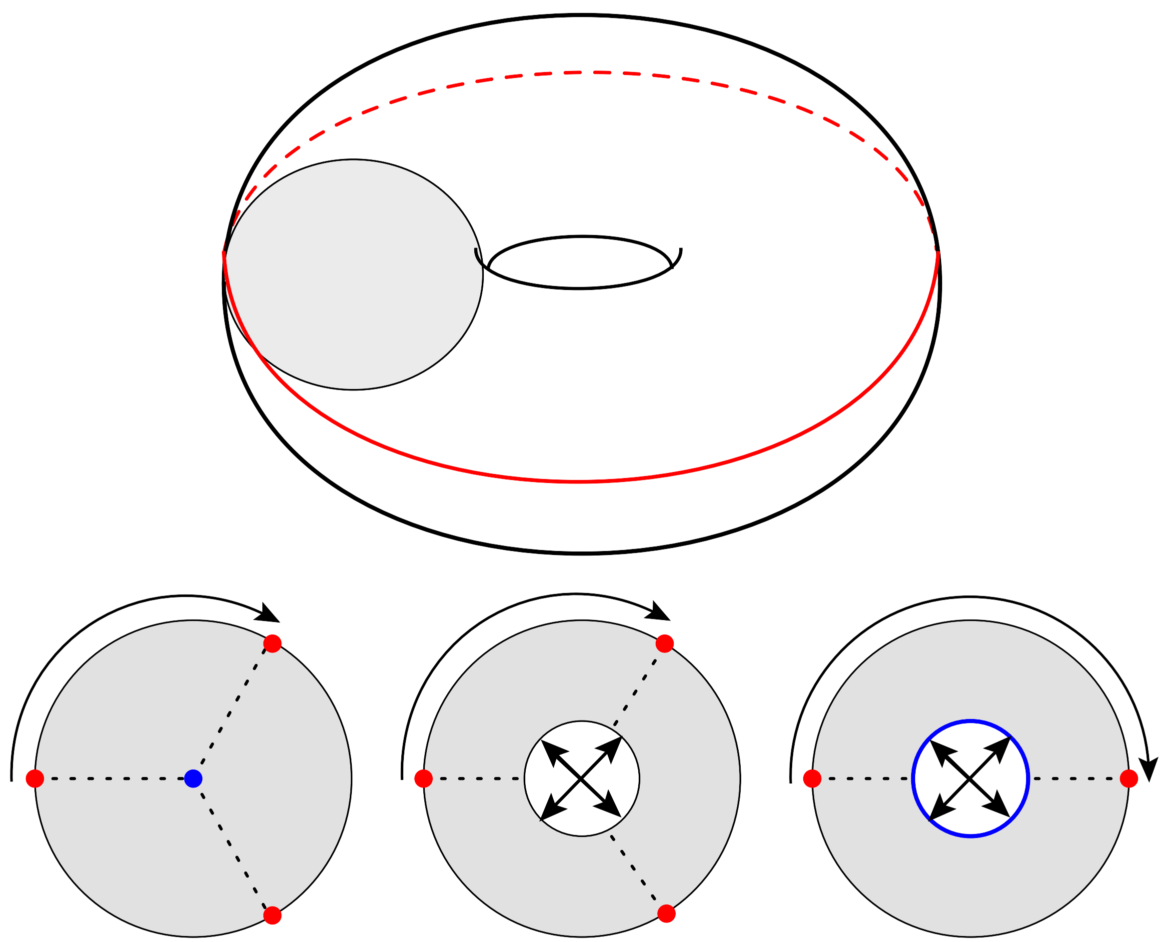


$$\mathcal{M} = \mathcal{W}_{\mathcal{E}}[\mathcal{A}] \cup \mathcal{W}_{\mathcal{E}}[\mathcal{A}^c] \cup \tilde{J}^+[\mathcal{E}_{\mathcal{A}}] \cup \tilde{J}^-[\mathcal{E}_{\mathcal{A}}]$$

Precis Part I

- The extremal surface computing the holographic entanglement entropy lies in the causal shadow in the bulk.
- The decomposition of the bulk spacetime into four casual domains, determined by the codimension-2 extremal surface, is the holographic image of the boundary causal domain decomposition.
- It is natural to conjecture that the entanglement wedge which is a bulk codimension-0 region is dual to the density matrix.
- Consequences for holographic subregion duality arising from the presence of casual shadows remain to be explored in greater detail.

Topology & Generalized Gravitational Entropy



Felix Haehl, Tom Hartman, Don Marolf, Henry Maxfield, MR **1412.7561**

Motivation

- The RT/HRT constructions requires a homology constraint.

Fursaev (2007); Headrick & Takayanagi (2007)

- Basically the extremal surface $\mathcal{E}_\mathcal{A}$ is required to be smoothly contractible onto the region $\mathcal{A}$. More precisely, we require the existence of a codimension-1 (spacelike) bulk region

$$\exists\, \mathcal{R}_\mathcal{A} \subset \mathcal{M} : \quad \partial\mathcal{R}_\mathcal{A} = \mathcal{E} \cup \mathcal{A}$$

- This condition was motivated in the context of proving strong-subadditivity of holographic entanglement.

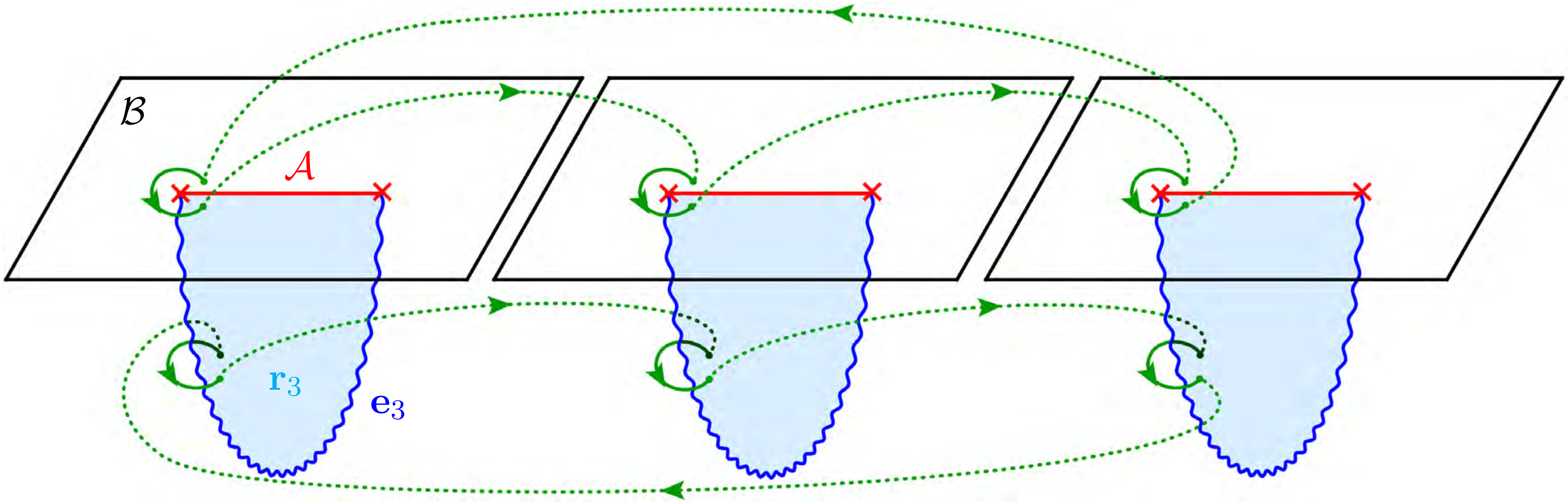
- The strongest motivation for it comes from requiring that the entanglement entropy for a subregion tend to the von Neumann entropy of the total density matrix when the region becomes the entire Cauchy slice.

cf., Headrick (2013); Hubeny, Maxfield, MR & Tonni (2013)
Headrick, Hubeny, Lawrence & MR (2014)

Wherefrom homology?

- This condition was motivated in the context of proving strong-subadditivity of holographic entanglement.
- The strongest motivation for it comes from requiring that the entanglement entropy for a subregion tend to the von Neumann entropy of the total density matrix when the region becomes the entire Cauchy slice.
- Whilst the constraint seems necessary to ensure that the holographic prescriptions is consistent with the features of quantum entanglement, its origins are murky.
- Is this automatic or should it be imposed simply to ensure consistency?
- Instructive to examine this in the case of the RT proposal using the generalized gravitational entropy prescription of LM.

Replicas in QFT and gravity



- The replica method is a useful technical tool for computing powers of the reduced density matrix and can be naturally be motivated not only in the boundary QFT, but also in the bulk gravitational theory.

Rényi entropy in QFT

- The Rényi entropy in QFT is obtained from the powers of the reduced density matrix

$$S_{\mathcal{A}}^{(q)} = \frac{1}{1-q} \log \mathrm{Tr}(\rho_{\mathcal{A}}^{q}) = \frac{1}{1-q} \log \frac{Z_q}{Z_1^q}$$

- This quantity can be obtained by considering the partition function Z_q of the QFT on a new background geometry $\tilde{\mathcal{B}}_q$.

- $\tilde{\mathcal{B}}_q$ is a singular manifold obtained by taking the q-fold branched cover over $\mathcal{B}\backslash\partial\mathcal{A}$ (cut-out the entangling surface and cyclically sew *q*-copies).

- Note that there is a natural *replica symmetry* (cyclic $\mathbb{Z}_q$ symmetry) on $\tilde{\mathcal{B}}_q$ with the fundamental domain being a copy of the original spacetime.

$$\tilde{\mathcal{B}}_q/\mathbb{Z}_q \simeq \mathcal{B}$$

Rényi entropy in gravity

- The usual rules of AdS/CFT say that the partition function Z_q is obtained as the on-shell action of gravity on the bulk geometry $\tilde{\mathcal{M}}_q$ whose boundary is the replica geometry $\partial\tilde{\mathcal{M}}_q = \tilde{\mathcal{B}}_q$.

- In fact, if we assume that the bulk gravity theory continues to respect the replica symmetry then we can consider the quotient spacetime:

$$\mathcal{M}_q = \tilde{\mathcal{M}}_q/\mathbb{Z}_q$$

- We expect $\mathcal{M}_q$ has a codimension-2 singular surface $\mathbf{e}_q$ which in $q \to 1$ limit becomes the RT surface (extension to HRT seems possible).

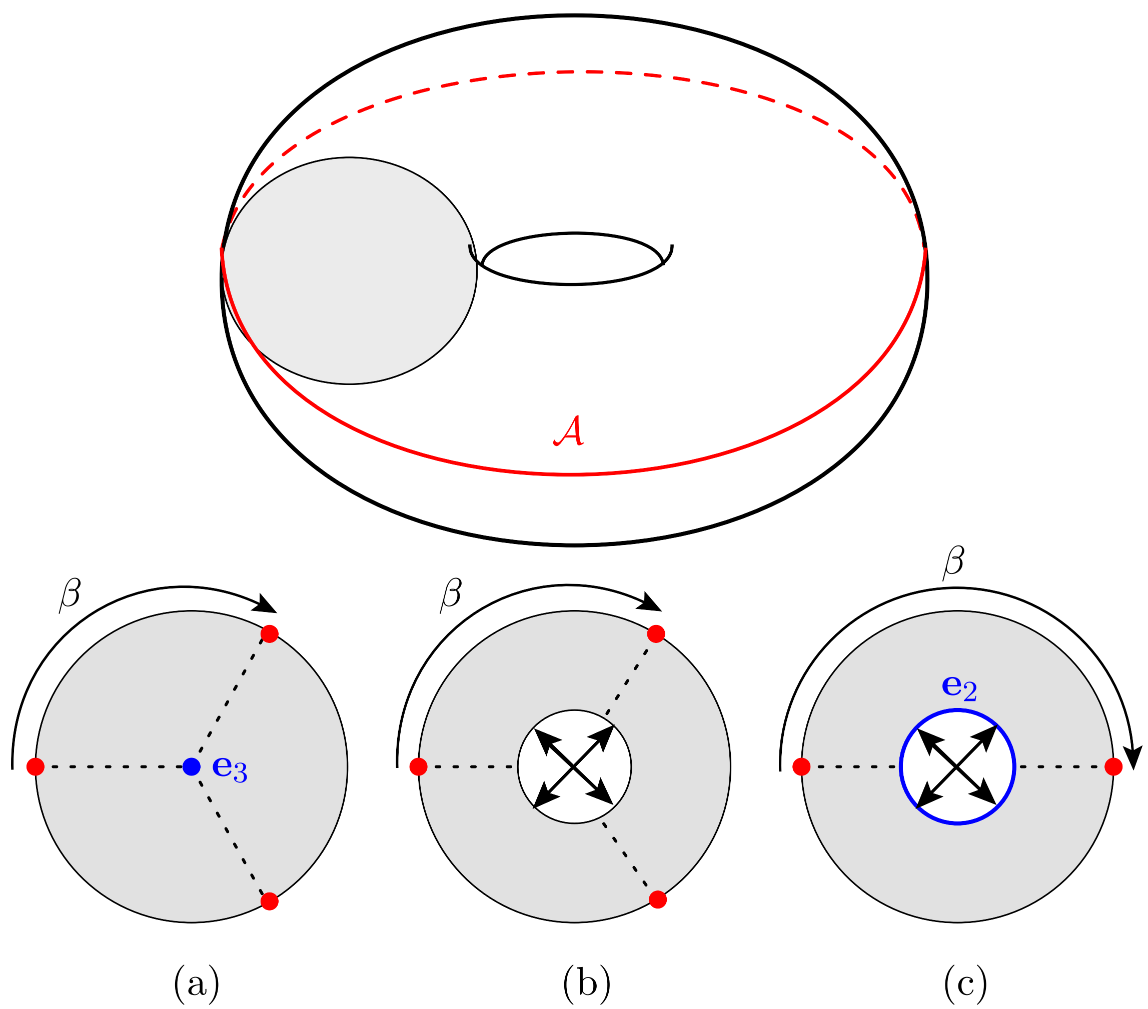
- This singular surface which is anchored on the entangling surface is usually referred to as the conical defect or cosmic brane in the literature.

Lewkowycz & Maldacena (2013)

Rényi and homology

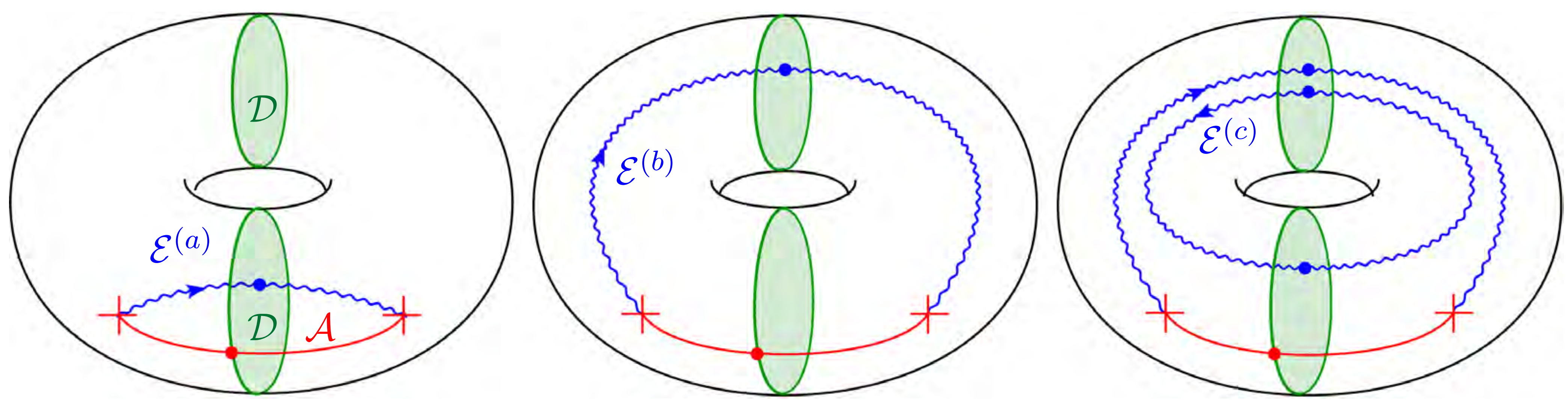
- Naively one expects a (two-sided) codimension-1 interpolating surface that connects the cosmic brane and the entangling surface.
- Assuming the codimension-1 surface exists it then follows that the quotient geometry $\mathcal{M}_q$ lifts to a branched cover $\tilde{\mathcal{M}}_q$. Roughly think of all fields having $\frac{2\pi}{q}$ monodromy about the branching surface $\mathbf{e}_q$.
- Rather surprisingly the converse statement can fail (for some values of *q*):
  - the quotient geometry does not admit an interpolating surface, or
  - the fixed point set is something other than the conical defect of interest.
- Both of these scenarios can be realized by explicit counter-examples.
- Homology is not guaranteed in the LM construction trivially at the level of topology; a precise criterion can be given, but first…

When homology fails…



- Topological example: three scenarios for computing thermal entropy using replica method. CFT on spatial circle (suppressed) with the period of Euclidean time direction setting the temperature.

And when it works…



(a) (b) (c)

- Homology respecting extremal surfaces in the solid torus geometry (BTZ).
- Thinking of the above as the fundamental domain of the branched cover we require that the sheet counting map on the boundary agree with the sheet counting map in the bulk.

Homology from gravity

- Start with original bulk manifold and put in a codimension-2 conical defect.
- We want the defect to have the correct monodromy for fields, so that going around the defect in the bulk is tantamount to going through the region on the boundary.
- Formally, we need a local bulk sheet counting map (defined in the neighbourhood of the conical defect) which lift to a global sheet counting map and restricts on the boundary to the boundary sheet counting map.
- This is guaranteed provided we have replica symmetric saddle points for every integer *q*, i.e., we need families of Rényi saddles.

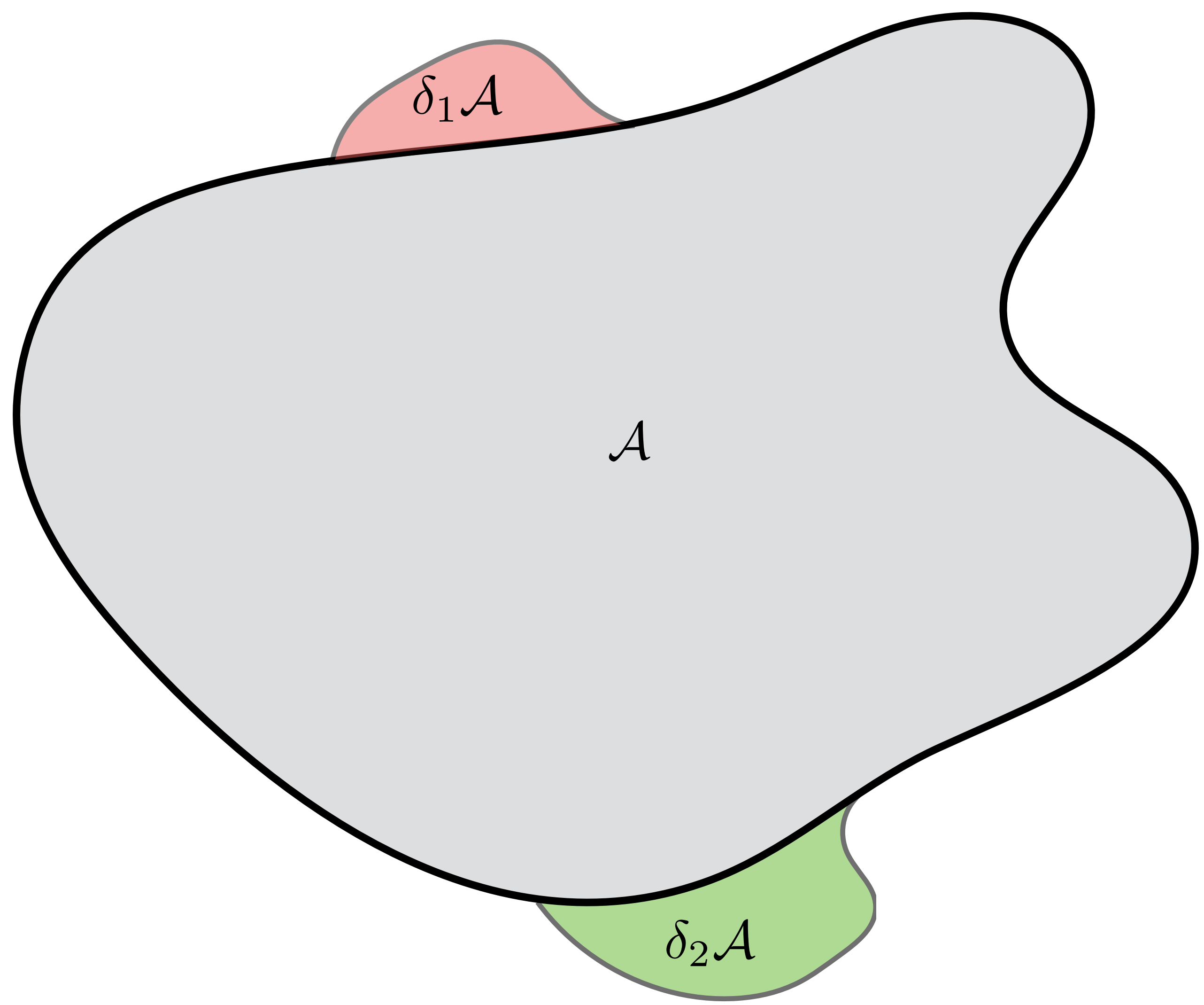
$$\begin{tikzcd} \pi_1(\mathcal{B} - \partial\mathcal{A}) \arrow[r, "\phi"] \arrow[d, "i_*"] & \mathbb{Z} \\ \pi_1(\mathcal{M} - \mathcal{E}) \arrow[ur, "\psi"] & \end{tikzcd}$$

$$H^1(\mathcal{M} - \mathcal{E}) \xrightarrow{(i^*,\delta)} H^1(\mathcal{B} - \partial\mathcal{A}) \oplus H^2(\mathcal{M}, \mathcal{M} - \mathcal{E}) \xrightarrow{\delta - j^*} H^2(\mathcal{M}, \mathcal{B} - \partial\mathcal{A})$$

Precis Part II

- The homology constraint follows from the LM in gravity iff the replica symmetric bulk geometries are branched cover for every integral Rényi index, with the branching structure commensurate with the boundary conditions.
- The argument can be made purely in topology and thus should hold for any bulk gravitational theory.
- There is an interesting challenge to extend this to the covariant case as well as to understand relevance of complex saddles in the LM construction.

Entanglement density & gravitational thermodynamics



Jyotirmoy Bhattacharya, Veronika Hubeny, MR, Tadashi Takayanagi **1412.5472**

Lashkari, Rabideau, Sabella-Garnier, Van Raamsdonk **1412.3514**

Strong-subadditivity of entanglement entropy

- The von Neumann entropy satisfies a number of important constraints, primary amongst which is strong-subadditivity (SSA).

$$S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2} \leq S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \qquad \forall\, \mathcal{A}_{1,2}$$

Lieb & Ruskai (1973)

- This is a convexity property of entanglement and has played an important role in QFT. eg., proofs of c-theorem and F-theorem in 2,3 dimensions.

Casini & Huerta (2004, 2012)

- The holographic entanglement entropy satisfies the SSA; the proof in the static case is remarkably simple, but even the covariant prescription can be shown to respect it.

Headrick & Takayanagi (2007)
Wall (2012)

Gravity and entanglement

- Recent discussions in the AdS/CFT context have tried to make precise the idea, that the bulk geometry is the encoder of the entanglement structure of the QFT state.

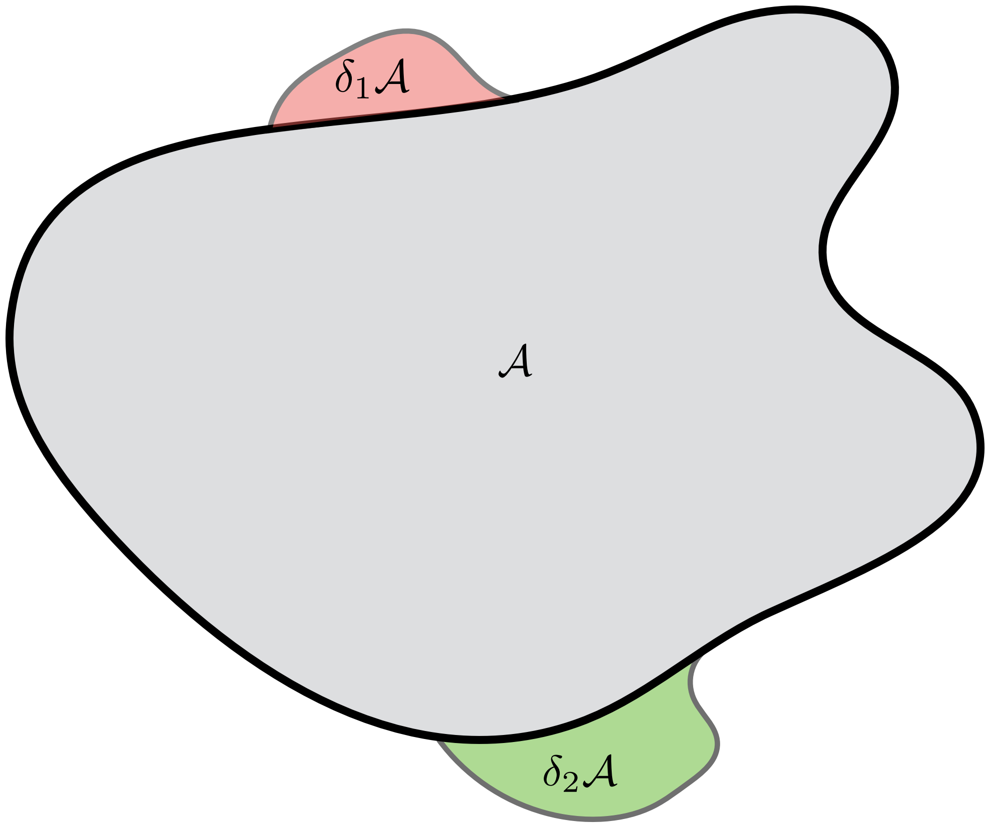
- If true, dynamics of gravity, ought to arise from some basic principle of entanglement entropy. Indeed, this has been argued for using relative entropy, and special properties of entanglement for ball-shaped regions in the vacuum state of a CFT.

Lashkari, McDermott, Van Rammsdonk(2013)

Faulkner, Guica, Hartman, Myers, Van Rammsdonk(2013)

- In general, however, entanglement is rather non-local. Could one identify a more local construct that distills its essence and gives us insight into gravity?

Entanglement density

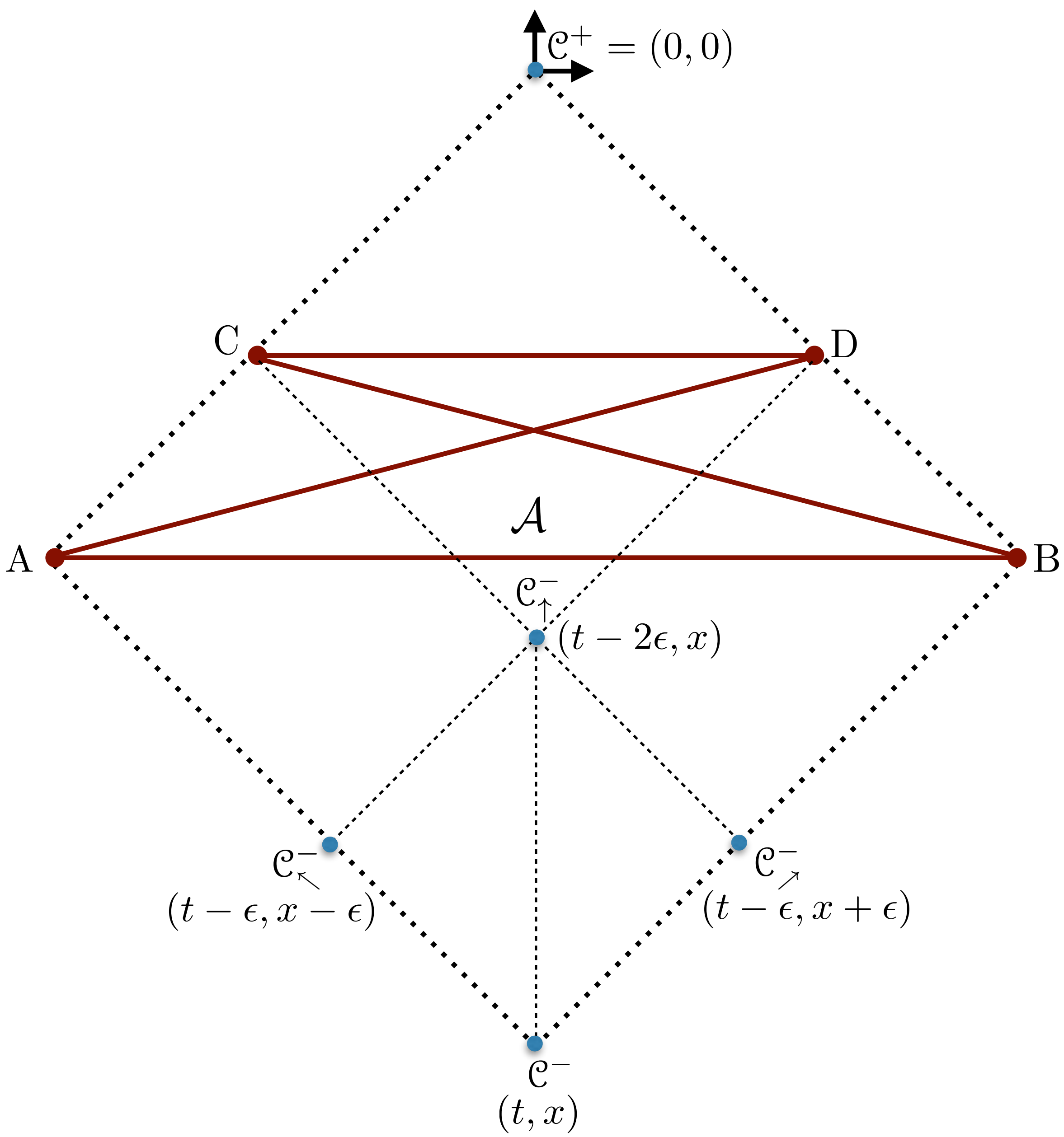


- Consider infinitesimal deformations of a given region.
- Convexity of entanglement entropy which is encoded in the statement of strong subadditivity can be distilled into a statement about the second variation.
- Inspired by this we define a notion of entanglement density which is sign-definite by SSA.

$$\hat{n}\,(\delta_1\mathcal{A}, \delta_2\mathcal{A}) = \delta_1\,\delta_2\,S_{\mathcal{A}}$$

$$\hat{n} \geq 0$$

Entanglement density in 2d QFT



$$D[\mathcal{A}] = J^-[\mathcal{C}^+] \cap J^+[\mathcal{C}^-]$$

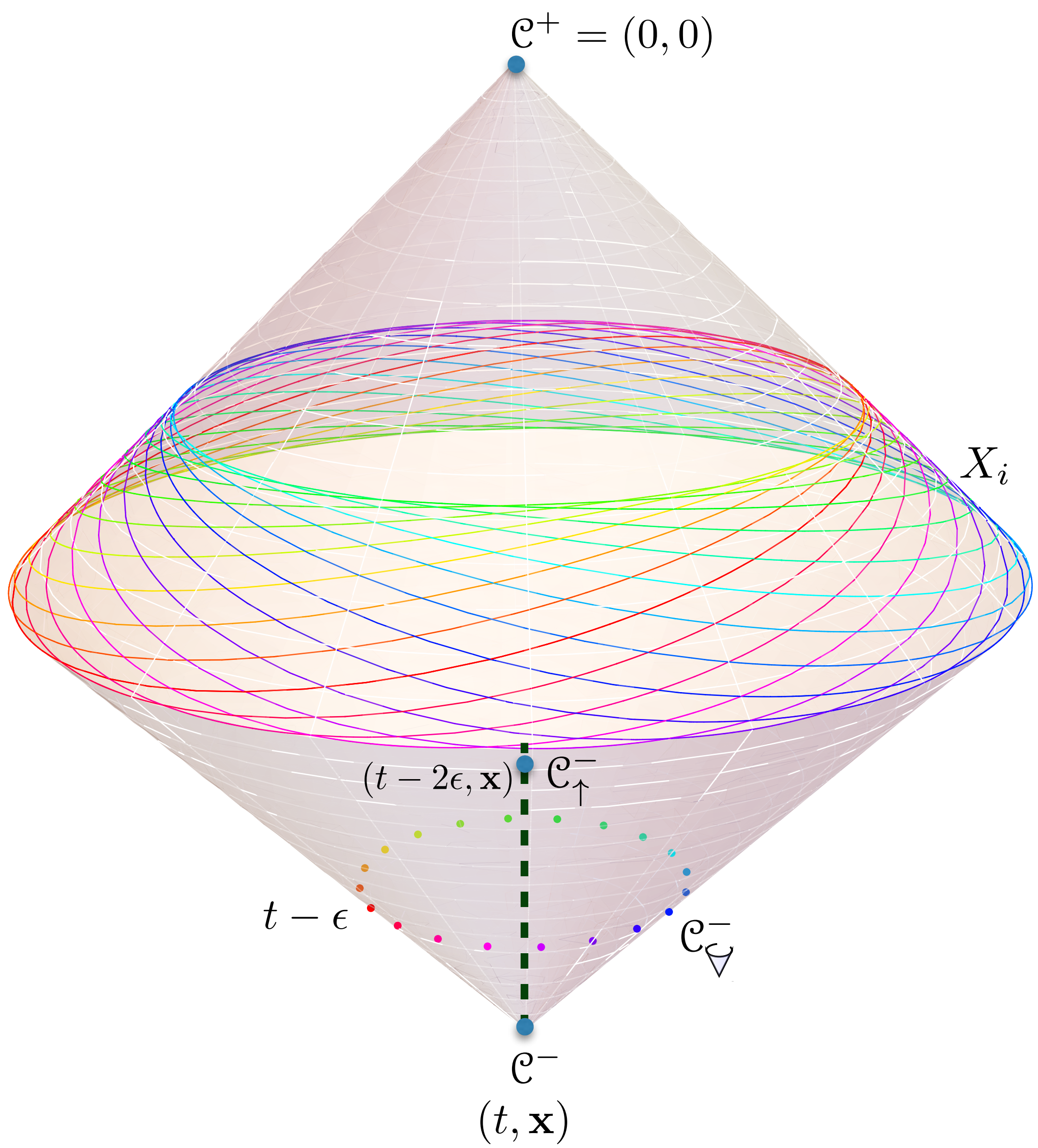
- Look at regions domain of dependence is generated by the light cone from two points $\mathcal{C}^\pm$.
- We can slide the region up and down along the light-cone to conjure a configuration where SSA can be applied.

$$S_{AD} + S_{CB} \geq S_{AB} + S_{CD}$$

$$\Downarrow$$

$$\hat{n}_- \equiv \left(-\partial_t^2 + \partial_x^2\right) S(t,x) \geq 0.$$

Entanglement density in 3d QFTs



- In higher dimensions we consider a sequence of regions & iterate the SSA to get:

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{ij}(X_i \cap X_j)) + S(\cup_{ijk}(X_i \cap X_j \cap X_k)) + \cdots + S(\cap_i X_i)$$

$$\Downarrow$$

$$\hat{n}_- \equiv \left[\Box + \frac{2}{t}\partial_t\right] S(t, \mathbf{x}) \geq 0$$

Casini & Huerta (2012)

Holographic entanglement density

- What does this translate to in gravity?
- First, we observe that the density vanishes in the vacuum of a CFT.
- This allows us to talk about the density relative to the vacuum, which is also guaranteed by SSA to be positive definite.
- Examining perturbations around linearized AdS we find that the density naturally relates to the bulk gravitational dynamics

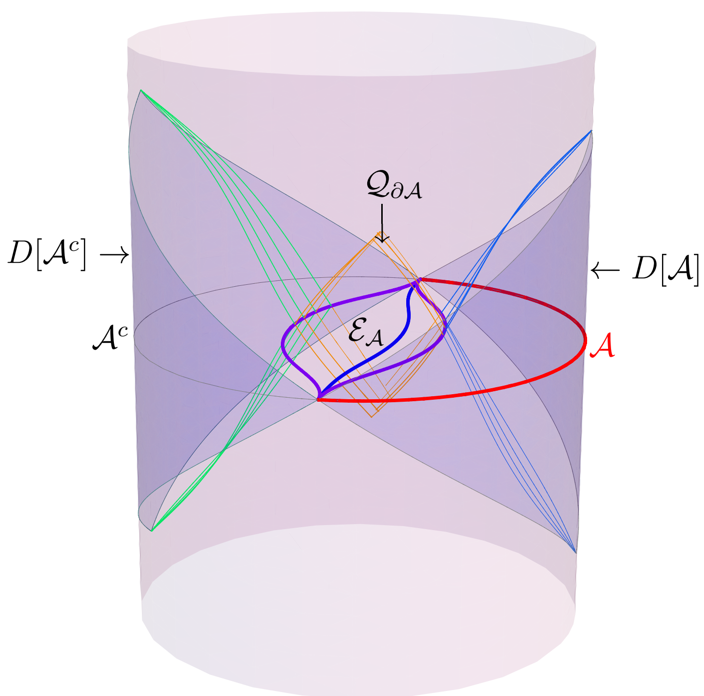
$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\pi G_N\, T_{\mu\nu}$$

$$\text{SSA} \implies \hat{n}_{\pm} \geq 0\,, \quad \hat{n}_{\pm}^{vac} = 0$$

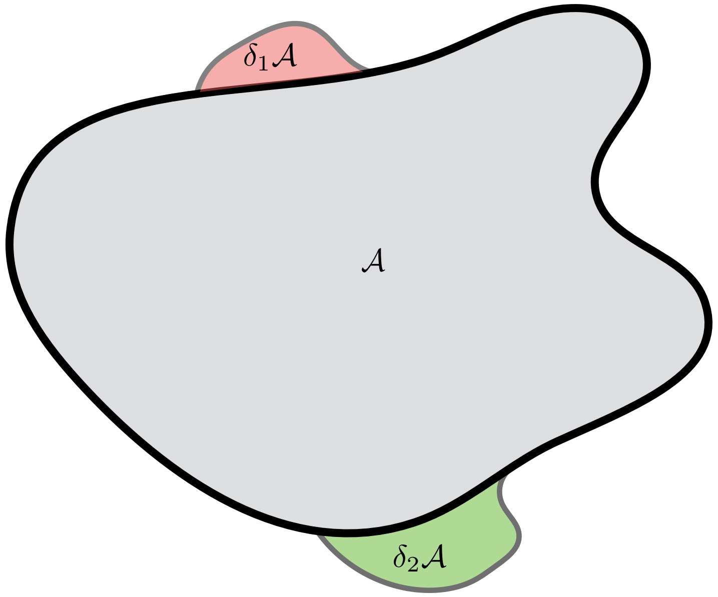
$$\implies \int_{\mathcal{E}_{\mathcal{A}}} \boldsymbol{\epsilon}\, N^{\mu}_{(\pm)} N^{\nu}_{(\pm)}\, E_{\mu\nu} \geq 0$$

Precis Part III

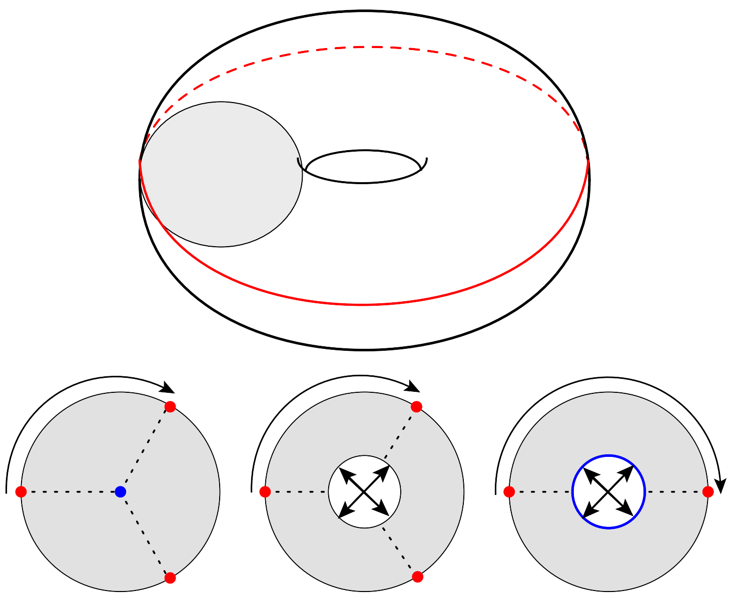
- Entanglement density distills the essence of SSA into a nice set of local differential inequalities.
- The positivity of the density maps holographically to the null energy condition.
- It is tempting to speculate that the local convexity of the entanglement density is related to a version of the second law of gravitational thermodynamics.
- Indeed when we focus on the long-wavelength fluctuations about a density matrix in the QFT, we can use the entanglement density as a proxy for the hydrodynamic entropy production, which satisfies the second law.



$\mathcal{Q}_{\partial\mathcal{A}}$
$D[\mathcal{A}^c] \rightarrow$
$\leftarrow D[\mathcal{A}]$
$\mathcal{A}^c$
$\mathcal{E}_\mathcal{A}$
$\mathcal{A}$



$\delta_1\mathcal{A}$
$\mathcal{A}$
$\delta_2\mathcal{A}$



Thank You