Recent Topics in Rigorous Proof of Quark Confinement in Lattice Gauge Theory

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Outline

Quark confinement is a long-standing problem in modern physics, but its rigorous proof is not in our sight. E. T. Tomboulis published a paper in which he claims a rigorous proof of quark confinement in 4D LGT.
But this paper contains many problematic points and many physicists were confused.
We point them out and discuss if it is possible to correct his proof. If not, what could be the next step?

It started with the paper by Wilson.

$$
\begin{aligned}
< f(g_C) > &= \frac{1}{Z}\int \exp[\beta \sum_p \chi(g_p)] f(g_C) \prod db_b \\
g_p &= \prod_{p \supset b} g_b \\
f(g_C) &= f(\prod_{b \in C} g_b)
\end{aligned}
$$

where χ is a rep. of G, p are unit squares (blocks, plaquettes) and

$$
g_b = \exp\left[\frac{2}{\sqrt{\beta}} A_b\right] \in G, \quad A_b = \sum_i A_\mu^{(i)} \tau_i, \quad \tau \in \mathcal{G}
$$

Typically two phases:

$$< \chi(g_C) >= \begin{cases} \exp[-\sigma|C|] & \text{area decay} \\ & (\sigma = \text{Wilson's string tension}) \\ \exp[-\sigma|\partial C|] & \text{perimeter decay} \end{cases}$$

We expect area decay for $D = 4$ and $G = SU(N)$.

The Millenium Problem

Open Problem

1. *Does quark confinement hold for all* $\beta > 0$ *in 4D LGT of* $G = SU(N)$*?*
2. *Does the continuum limit exist and both asymptotic freedom and confinement hold there ?*

If the first claim is not affirmative, quark confinement and asymptotic freedom may depend on the methods of continuum limit.
If (1) is true, then the second one is plausible but its proof is much more difficult.

Our Present Knowledges

Theorem

1. *For any D, if $\beta > 0$ is small, the area decay law holds for $G = SU(N), U(N)$.*
2. *If $D = 3$, the area law holds for $G = U(1)$ for all β*
3. *If $D = 4$ and $G = U(1)$, there exit two phases: QED (KT) phase and confining phase in LQED (Guth, Froehlich, Spencer)*

For $G = SU(N)$, nothing is known (even in $D = 3$) except for the case of $\beta \ll 1$.

Block Spin transformation

$$\phi(x) \quad \rightarrow \quad \phi_1(x) = \frac{1}{L^{(d+2)/2}} \sum_{y \in \Box} \phi(y)$$

$$\phi(x) \quad = \quad \frac{1}{L^{(d-2)/2}} \phi_1([x/L]) + \sum (\text{small fluctuation})$$

may not work well for groups (average of groups is not a group):

$$g_b \quad = \quad \exp[(i/\sqrt{\beta}) A_{x,x+e_\mu}] \in G$$

$$\text{Block Ave of } g_b \quad = \quad \exp[\sum_{b'//b} \log g_{b'}] \in G$$

(Rivasseau - Balaban's method,

straight calculation is terrible)

Approximate RG was invented by Migdal 30 years ago!
Block plaquettes of size $b \times b$ are shifted to the walls at $x_i = b, 2b, 3b, \cdots$ (for $i = 1, 2, .., D$). (b^{D-2} block plaquettes are gathered.)

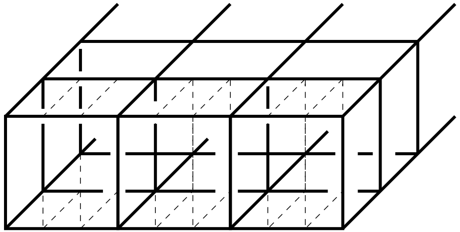


Figure: decimation of Kananoff type: b^{D-2} block plaquettes of size $b \times b$ are moved to the walls of large cubes and gathered (by putting $\beta = \infty$)

For the dimension D, b^{D-2} block plaquettes (from $x_3, \cdots x_D$ directions) of size $b \times b$ are glued together. This is calculable (2D LGT).
Each bond means $f^{(n-1)}(U)^{b^{D-2}}$ (Kadanoff type):

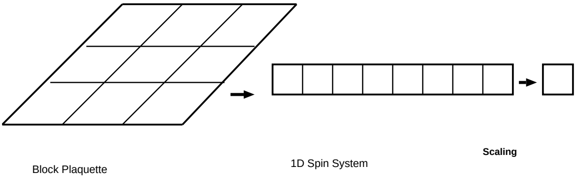


Figure: 2Dim LGT = 1D Spin Sys. Decimation and Scaling

Internal bonds are integrated out (decimated) and closed RG formula is obtained for plaquette actions $f^{(n)}(U_p)$:

$$f^{(0)}(U) = \exp[\beta\chi(U)], \quad U = \prod_{\partial p \supset b} U_b \in G$$

and

$$f^{(n)}(U) = \frac{1}{F_0(n)} \times$$
$$\int [\underbrace{f^{(n-1)}(UU_1) f^{(n-1)}(U_1^{-1}U_2) \cdots f^{(n-1)}(U_{b^2})}_{b^2}]^{b^{D-2}} \prod dU_k$$

where

$$F_0(n) = \left(\int [f^{(n-1)}(U)]^{b^{D-2}} dU \right)^{b^2}$$

In terms of the Fourier (characteristic func.) expansion,

$$\begin{aligned} f^{(n)}(U) &\equiv f^{(n)}(\{c_j(n)\}, U) = 1 + \sum_j c_j(n) d_j \chi_j(U) \\ c_j(n) &\equiv \int f^{(n)}(U) \frac{1}{d_j} \chi_j(U) dU \end{aligned}$$

we have

$$f(\{c_j(n-1)\}, U) \to f^{(n)}(U) = f(\{c_j(n)\}, U)$$

Theorem

For $D \leq 4$ and for $G = SU(N)$ or $G = U(N)$,

$$\lim_{n \to \infty} c_j(n) = 0 \qquad \text{for } j \neq 0$$

Quark confinement holds for $D \leq 4$ and $G = SU(N), U(N)$ in the MK approximation . These recursions are approximate and yield upper bounds for the partition functions.

This means that the MK fails to prove Kosterlitz-Thouless transition (QED phase).

Proof

Put $D = 4$ and define the set $\mathcal{F}$ of class functions such that

1. $f(u)$ is a class function of positive type, $f(uv) = f(vu), 1 = f(1) \geq f(u) = f(u^{-1})$
2. For $\sigma(z) = \exp[i \sum z_i \lambda_i]$ and $\tau(\omega) = \exp[i \sum \omega_i \lambda_i]$ where $\{\lambda_i\}$ are $N \times N$ hermitian matrices, $f(\sigma v \tau \tilde{v})$ is analytic in $D = \{(z, \omega); |\mathrm{Im} z_i|, |\mathrm{Im} \omega_i| < \ell\}$
3. f satisfies the bound

$$|f_{n+1}(\sigma(z) u \tau(\omega) \tilde{v})|$$
$$\leq |f_{n+1}(\sigma(\mathrm{Re} z) u \tau(\mathrm{Re} \omega) \tilde{v})| \exp[\beta C \sum ((\mathrm{Im} z_i)^2 + (\mathrm{Im} \omega_i)^2)]$$

Clearly $f_0 \in \mathcal{F}$. Then $f_n \in \mathcal{F}$ by induction since

$$f_{n+1}(\sigma(z) u \tau(\omega) \tilde{v}) = \frac{1}{\mathcal{N}} \int \Big[f_n(\sigma(z/b^2) u \tau(\omega/b^2) v_1^{-1}) \cdots$$
$$\times f_n(\sigma(z/b^2) v_{b^2-1} \tau(\omega/b^2) \tilde{v}) \Big]^{b^2} \prod dv_i$$

f_n belongs to $\mathcal{F}$ larger analytic region D. We define

$$\beta_v^n(a) \quad = \quad \frac{2}{a^2} \log \left| \frac{f_n(v\tau(ia))}{f_n(v)} \right|$$

(real analytic and even in a). Then

$$\beta_v^n \quad = \quad \beta_v^n(0) = -\frac{\partial^2}{\partial^2 \theta} \log |f_n(v\tau(\theta))|$$

Note that

$$f_{n+1}(v\tau(a))$$

$$= \frac{1}{\mathcal{N}} \int \left[f_n(v\tau(a/b^2)v_1^{-1}) \cdots \times f_n(v_{b^2-1}\tau(a/b^2)) \right]^{b^2} \prod dv_i$$

Take the absolute values of both sides and expand $|f_n(v\tau(ia/b^2))|$ in a (even in a) to find

$$\beta_{n+1} \leq \beta_n, \quad D \leq 4$$

This arguments work for both $U(N)$ and $SU(N)$.
MK does not see the difference between $U(N)$ and $SU(N)$.

To my best knowledge, nobody suceeded to find any approximate formula which distinguishes non-abelian and abelian.

But this is not a shame since nobody could solve the real non-abelian system.

Vortex Condensation

Mack, Petkova, 't Hooft and Yoneya introduced the idea of vortex condensation:
$V \subset \Lambda$ is $p = (x_0, x_0 + e_1, x_0 + e_1 + e_2, x_0 + e_2)$ in an $x_1 - x_2$ plane and its translations along the axis normal to p (say, 3rd and 4th axis)

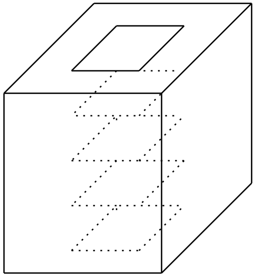


Figure: vortex container V of base size $L_1 \times L_2$, height $L_3 \times L_4$ in Λ

We consider Λ containing one vortex (vortex container). (Or called twisted boundary condition).

$$\begin{aligned} Z^{-} &= \int dU_{\Lambda} \prod_{p \subset \Lambda} f(\{c_j\}, (-1)^{\nu(p)} U_p) \\ & -1 \in \text{ center of } G = SU(2) \end{aligned}$$

where

$$\nu(p) = \begin{cases} 0 & \text{if } p \notin V \\ 1 & \text{if } p \in V \end{cases}$$

$$
\begin{aligned}
Z &= \int dU_\Lambda \prod_{p\subset\Lambda}(1+\sum_{j\neq 0} c_j d_j \chi_j(U_p)) \\
Z^- &= \int dU_\Lambda \prod_{p\subset\Lambda\setminus V}(1+\sum_{j\neq 0} c_j d_j \chi_j(U_p)) \\
&\quad \times \prod_{q\subset V}(1+\sum_{j\neq 0}(-1)^{2j} c_j d_j \chi_j(U_q))
\end{aligned}
$$

Old results

Theorem

Take $L_1, L_2 << L_3, L_4$ *and let* $F \equiv -\log \frac{Z_\Lambda^-}{Z_\Lambda}$

(1) If $G = U(1)$*,*

$$F \sim \begin{cases} L_3 L_4 / L_1 L_2 & \text{for large } \beta \quad \text{perimeter} \\ L_3 L_4 \exp[-\sigma L_1 L_2] & \text{for small } \beta \quad \text{area} \end{cases}$$

($\sigma > 0$ *= 't Hooft's string tension.)*

(2) For $G = SU(N)$

$$F \sim L_3 L_4 \exp[-\sigma L_1 L_2]$$

(area decay law) for small β*.*

(1) High-Temp Expansion: Assume $\beta << 1$. Take 2D slices $Z_{L_1 L_2}$ and $Z^{-}_{L_1 L_2}$. Then

$$Z_{L_1 L_2} = 1 + e^{-\sigma L_1 L_2}, \quad Z^{-}_{L_1 L_2} = 1 - e^{-\sigma L_1 L_2}$$

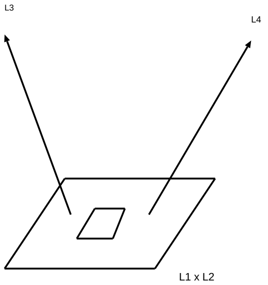


Figure: high temp exp = $L_3 L_4$ copies of 2D slices

$$\frac{Z^-_\Lambda}{Z_\Lambda} \sim \left(\frac{Z^-_{L_1L_2}}{Z_{L_1L_2}}\right)^{L_3L_4}$$
$$= \left(\frac{1-e^{-\sigma L_1L_2}}{1+e^{-\sigma L_1L_2}}\right)^{L_3L_4} \sim \exp[-L_3L_4e^{-\sigma L_1L_2}]$$

(2) The existence of KT phase uses duality transformation.

$$\exp[\beta\cos\theta] = \sum_n \exp[-n^2/2\beta + in\theta]$$

and $n = h + *d*\phi$ (Hodge decomposition)

Tomboulis' difficult to understand Trick:

$Z = Z_\Lambda$ increases by the MK formula. Two competing parameters α and t are introduced:

- $c_j(n) \to \alpha c_j(n) = \tilde{c}_j(n)$, Z increases as $\alpha \nearrow 1$,
- $[F_0(n)]^{h(\alpha,t)}$ decreases as t increases

$$h(\alpha, t) = \exp[-t(1-\alpha)/\alpha]$$

Choose t and α cleverly so that $'' ='' $ holds:

$$Z_{n-1} = \int \prod (1 + \sum c_j(n-1) d_j \chi_j(U_p)) dU_{n-1}$$
$$= [F_0(n)]^{|\Lambda_n| h(\alpha(t),t)} \int \prod (1 + \sum \alpha(t) c_j(n) d_j \chi_j(U_p)) dU_n$$

Then there exist functions α and α^+ and values t and t^+ such that

$$\begin{aligned} Z_\Lambda(\{c_j\}) &= [F_0(1)]^{h(\alpha(t),t)|\Lambda^{(1)}|} Z_{\Lambda^{(1)}}(\{\tilde{c}_j(\alpha(t))\}) \\ Z^+_\Lambda(\{c_j\}) &= [F_0(1)]^{h(\alpha^+(t),t)|\Lambda^{(1)}|} Z^+_{\Lambda^{(1)}}(\{\tilde{c}_j(\alpha^+(t^+))\}) \end{aligned}$$

where

$$\begin{aligned} Z^+ &= \frac{1}{2}(Z + Z^-) \\ \tilde{c}_j(\alpha(t))(n) &= \alpha(t) c_j(n) \end{aligned}$$

Then we have

$$\frac{Z_\Lambda^+(c_j)}{Z_\Lambda(c_j)} = \underbrace{\frac{F_0(n)^{|\Lambda_n|h(\alpha^+(t^+),t^+)}}{F_0(n)^{|\Lambda_n|h(\alpha(t),t)}}}_{\text{phase factor}} \times \underbrace{\frac{Z_{\Lambda_n}^+(\alpha^+(t^+)c_j)}{Z_{\Lambda_n}(\alpha(t)c_j)}}_{\text{next oredr } Z}$$

To continue this step, he wants to find α, t, α^+ and t^+ such that

(i) at first n: "phase factor$=1$"

(ii) at later steps: $\alpha^+(t^+) = \alpha(t)$

This means

$$\frac{Z_\Lambda^+(c_j)}{Z_\Lambda(c_j)} = \frac{Z_{\Lambda_n}^+(\alpha^+(t^+)c_j)}{Z_{\Lambda_n}(\alpha(t)c_j)} = \frac{Z_{\Lambda_n}^+(\alpha(t)c_j)}{Z_{\Lambda_n}(\alpha(t)c_j)}$$

Since $c_j(n) \to 0$, as $n \to \infty$ we have

$$\sigma_{'\mathrm{tHooft}} > 0 \to \sigma_{\mathrm{Wilson}} > 0 \to \text{Confinement}$$

But

1. this does not distinguish abelian and non-abelian
2. this is the ratio of the partition functions of two different systems. It is usually 0 or ∞.
3. why α and α^+? One α and t and t^+ are enough?

It seems that he intentionally introduced a maze so that the reader (and the author) would be confused.

His argumet depernds on

Claim

(1) There exist $t \geq 0$ and $t^+ \geq 0$ such that

$$1 + \frac{Z_\Lambda^-(\{c_j\})}{Z_\Lambda(\{c_j\})} = 1 + \frac{Z_{\Lambda^{(1)}}^-(\{\tilde{c}_j^{(1)}(\alpha^+(t_+))\})}{Z_{\Lambda^{(1)}}(\{\tilde{c}_j^{(1)}(\alpha(t))\})}$$

where $\tilde{c}_j^{(1)}(\alpha(t)) = \alpha(t) c_j(1)$

(2) For small β, there exists $t \geq 0$ such that $\alpha_n^+(t) = \alpha_n(t)$:

$$1 + \frac{Z_\Lambda^-(\{c_j\})}{Z_\Lambda(\{c_j\})} = 1 + \frac{Z_{\Lambda^{(1)}}^-(\{\tilde{c}_j^{(1)}(\alpha(t))\})}{Z_{\Lambda^{(1)}}(\{\tilde{c}_j^{(1)}(\alpha(t))\})}$$

For the calims (1) and (2), he assumes
(1) $\alpha(t)$ and $\alpha^+(t)$ change significantly as $t \in [0,1]$ changes.
For this he proves

$$\frac{\partial}{\partial t}\alpha(t) > \eta_1 > 0, \quad \frac{\partial}{\partial t}\alpha^+(t) > \eta_2 > 0$$

(He introduces additional dimension r to $D = 4$ to prove. This changes the recursion formula completely and unreliable.)

(2) Put (do not ask me what this is)

$$\Psi(t,\lambda) = h(\alpha(t),t) + (1-\lambda)\Phi^+(\alpha^+(t_l)) + \lambda\Phi^+(\alpha^+(t))$$

He obtains $t(\lambda)$ which satisfies $\Psi(\lambda,t) = 0$ and shows that $t(\lambda = 1)$ exits. For this, he uses the implicit function theorem but wrongly:

if $\Psi(\lambda,t) \in C^1$ *and* $\Psi_t(\lambda,t) \neq 0$, *there exists* $t = t(\lambda)$ *satisfying* $\Psi(\lambda,t) = 0$ *for all* λ

What is $r \in (0, 1]$? In his paper, $f^{(n)}(U)$ is redefined by

$$\int [\underbrace{f^{(n-1)}(UU_1)f^{(n-1)}(U_1^{-1}U_2)\cdots f^{(n-1)}(U_{b^2\times r})}_{b^2\times r}]^{b^2}\prod dU_k$$

This is a risky parameter which increaeses dimension D.
Put $f = \exp[-\beta\theta^2/2]$, $\theta \in R$, (non-compact QED):

$$\int [\underbrace{f(UU_1)f(U_1^{-1}U_2)\cdots f(U_{b^2r})}_{b^2\times r}]^{b^2}\prod dU_k$$

$$= \int \exp[-\frac{\beta b^2}{2}\Big((\theta-\theta_1)^2+(\theta_1-\theta_2)^2+\cdots+\theta_{b^2r}\Big)]\prod d\theta_i$$

$$= (\mathrm{const.})\exp[-\frac{\beta}{2\mathrm{r}}\theta^2]$$

Theorem

In the previous recursion formulas, if $r < 1$ and $D = 4$, $f^{(n)}(U)$ ($G = U(1)$) converges to the delta funcation as $n \to \infty$ if $\beta > \beta_c$. (An ordered phase takes place.)

This is presumably true for $G = SU(N)$.
By the way, the implicit function $t = t(\lambda)$ of $\Psi(\lambda, t)$ satisfies

$$t(\lambda) = -\int_0^{\lambda} \frac{\Psi_\lambda(x, t(x))}{\Psi_t(x, t(x))} dx$$

But $\Psi_t \neq 0$ does not mean $t(\lambda)$ can be defined for all λ.
Example (T.Kanazawa): $\Psi(\lambda, t) = e^{-t} - 1 + 2\lambda$

Remedy?

Sorry, I cant give you any remedy in this direction.

I think
it is impossible to use methods or formula which cannot distinguish $U(N)$ and $SU(N)$.

Though the MK RG formulas cannot distinguish non-abelian groups from abelian ones, the velocities of the convergences of $\{c_j(n)\}_{j=1/2}^{\infty}$ to 0 as $n \to \infty$ are very different. So there may be something.

Ising, Sigma and LGT

Non-abelan LGT is more non-linear than 2D sigma model (Heisenberg model) and domain wall problem is most serious in this game.
In the case of the Ising model, the domain wall is easily defined and this leads us to the famous Peiers argumet of the spontaneous magnetization.

$$\exp[\beta \sum_{nn} s_i s_j], \quad s_i = \pm 1$$

$\rightarrow$ domain wall of length $\ell < 3^\ell e^{-\beta\ell}$

$\rightarrow$ Peiels famous argument

large domain wall energy means an ordered phase

How does this work for 2D O(N) Heisenberg and for 4F LGT?
We decompose $\phi_n(x) \in R^N$ into block spin $\phi_{n+1}(x)$ and zero-average fluctuations $Q\xi_n$

$$\phi_{n+1}(x) \in R^N, \xi_n \in R^N, \quad \sum_{\zeta \in \Delta_0} (Q\xi)(x+\zeta) = 0$$
$$\phi_n(x) = (A\phi_{n+1})(x) + (Q\xi)(x)$$

integrate over fluctuations in the wine bottle:

$$\exp[-W_{n+1}(\phi_{n+1})]$$
$$= \int \exp\left[-\frac{g}{2N} \sum_x \left(\underbrace{[(A\phi_{n+1})(x) + (Q\xi)(x)]^2}_{\phi_n^2(x)} - N\beta_n \right)^2 - \frac{1}{2}\xi_x^2 \right] \prod d\xi$$

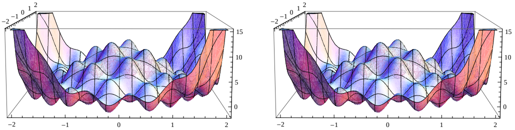
where

$$\begin{aligned} \phi_n^2(x) &= N\beta_n + \text{fluct.} \\ \phi_{n+1}^2(x) &= N\beta_{n+1} + \text{fluct.} = N\underbrace{(\beta_n - \kappa)}_{\beta_{n+1}} + \text{fluct.} \end{aligned}$$

and

$$\begin{aligned} \phi_n^2(x) - N\beta_n &= [(A\phi_{n+1})^2(x) - N\beta_{n+1}] + q(x) \\ q(x) &= 2(A\phi_{n+1})(Q\xi) + :(Q\xi)^2: \\ &= \text{Wick ordered fluctuations affected by DW} \end{aligned}$$

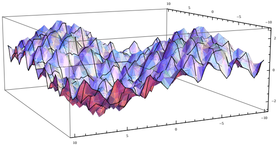
The fluctuation integral $d\xi$ is very much influenced by the back ground field $=$ spin waves ϕ_{n+1}.

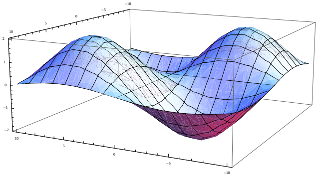


The subtle is the definition of domain wall. In the present system, domain walls are very thick and consists of rotating spins:

$$\begin{aligned} D_W(\phi) &= \{\Delta \subset \Lambda; \forall \Delta \subset D_W, \exists x \in \Delta, \exists \Delta' \subset D_W, \exists y \in \Delta', \\ &\quad | :\phi_n(x)\phi_n(y):_{G_n} | \geq N^{1/2+\varepsilon} \exp[(c/10)|x-y|] \end{aligned}$$

This is an extension of the domain walls in the Ising model and very implicit. The right hand is indep. of β except of $:\cdots:_{G_n}$





Lemma

(i) the energy of the domain wall D_w is large. g^{D_i} in below are small
(ii) $q(\xi)$ contains ξ^4. But distribution of $q(\xi)$ is approximately Gaussian of variance N

Theorem

For given $D(\phi_n) = \cup D_i$, the n'th Gibbs measure is given by

$$\exp[-W_n] = \prod_i g^{D_i}(\phi_n)$$
$$\times \exp\left[-\frac{1}{2}\langle\phi_n, (-\Delta)\phi_n\rangle - \frac{g^*}{2N}\sum_{x\in D^c}\left(\phi_n^2(x) - N\beta_n\right)^2\right]$$
$$\beta_n = \beta_0 - n\kappa \to 0, \kappa > 0$$

where g^* is the fixed point of running coupling constant g_n.

So we apply this idea to LGT giving up that the block spin transf keeps group property.

$$\exp[-W_{n+1}] = \int \exp\left[\beta \sum_p \mathrm{Tr} \prod_{b\in\partial p} g_b - \lambda \sum_b \mathrm{Tr}(g_b^t g_b - 1)^2\right] \prod d\xi$$

$$\lambda \gg 1$$

where

$$g_b \in \mathrm{Mat}(N\times N, R),$$

$$g_b = (Ag_b^1) + Q\xi_n \in \mathrm{Mat}(N\times N, R), \quad \sum_{\zeta\in\Delta_0} (Q\xi)(x+\zeta) = 0$$

Here we will again encounter the problem of domain walls which is very much complicated. Without solving this, it may be very hard to prove our millenium problem.