(I) 20 min : General Introduction

(II) 50 min : more details

TITLE : "Surprises in gauge theories: temperature-reflection symmetry, hidden modular properties, and vacuum energies."

A

ABSTRACT : "I will discuss the presence of a previously unrecognized $+T \to -T$ ~~symm~~ temperature-reflection ~~symmetry~~ (T-reflection) symmetry for the partition functions of a number of physically interesting quantum field theories. T-reflection symmetry turns out to have several striking consequences. For instance, it helps reveal that the confining-phase partition functions of infinite-N gauge theories on $S^3 \times S^1$ have very simple modular properties, and suggests a remarkable 4d-2d spectral equivalence. T-reflection symmetry also plays a key role in the discovery of a universal spectral sum rule in these confining gauge theories, which forces their infinite-N Casimir vacuum energies to vanish."

Talk @ IPMU, for the Math-Strings Group Meeting / Seminar, 12 May 2015 [Tuesday]

Part (I): General Introduction:

(B)

- Thank you for coming today!
- Overall structure of whole talk:

(a) Part I: the broad sweep of the ideas

(a1) Idea of T-reflection & Eqs. (1) & (2), i.e.:

$$\boxed{(1)}\ Z_r(\beta) = e^{i\gamma} Z(-\beta), \text{ w/ } Z(\beta) = \sum_n d_n e^{-\beta E_n} \equiv \mathrm{tr}(e^{-\beta H})$$

$$\boxed{(2)}\ Z_\Delta(\beta) = \sum_n d_n e^{-\beta(E_n+\Delta)} = e^{-\beta\Delta} Z(\beta)$$

$$\Downarrow$$

$$Z_\Delta(\beta) = e^{i\gamma} Z_\Delta(-\beta) \text{ iff } \Delta = 0. \text{ FIXES } E_{vac}!!$$

~~NB/ Generalities~~ ...

(a2) Simplest examples in QM: $\boxed{\text{2-level system}}^{(3)}$ & $\boxed{\text{SHO}}^{(4)}$

MORE DETAIL IN PART-(II).

(a3) Simplest examples in QFT:
- ∞ # of decoupled 2-level / SHO
- 2d CFTs: free fermi, free boson
- Fermion $\leftrightarrow M(4,3)$ or interesting CFT!

(a4) Mathematics of T-reflection of modular forms ...

- $T: \tau \to -\tau$, so $T: \text{UHP} \to \text{LHP}$. Okay?
- 24 free Bosons have $\boxed{(6)}\ Z(\tau) = \dfrac{1}{\Delta(\tau)} \overset{\text{Physics}}{=} \left(\dfrac{1}{\eta(\tau)}\right)^{24} \overset{\text{Various Mathematics}}{=} \left(\dfrac{1728}{E_6(\tau)^2 - E_4(\tau)^3}\right)$

$$\boxed{(7)}\ E_{2k}(\tau) = \frac{1}{2\zeta(2k)} \sum_{m,n \in \mathbb{Z}^2} \frac{1}{(m+n\tau)^{2k}} = E_{2k}(-\tau)$$

Part (I): General Introduction:

— Overall structure of whole talk (contd...):

ⓐ Part I: (contd...)

(a5) T-reflection in Q.C.D.*! Saw

* "QCD" = "YM w/ possibly N_f, N_s adjoint massless fermions & scalars", placed on $S^3_R \times S^1_\beta$ w/ $\beta \gg R$, $N_c = \infty$, and $\Lambda R \ll 1$."

$$\boxed{(8)}\quad Z_{QCD} = \prod_{n\geq 1}^{\infty} \frac{1}{1 - z_B(x^n) + (-1)^n z_F(x^n)}, \quad x = e^{-\beta/R}$$

$$\boxed{(9)}\quad 1 - z_B(x) = (1 - z_V(x)) - N_s\, z_s(x)$$

$$= \frac{(x^3+1) - 3(x^2+x)}{(1-x)^3} - N_s \frac{x(1+x)}{(1-x)^3}$$

$$+ z_F(x) = 4N_f \frac{x^{3/2}}{(1-x)^3}$$

$$z_F\left(\frac{1}{x}\right) = -z_F(x)$$

MANY LESSONS

(a6) From this, we can see that

$$\boxed{(10)}\quad P_{N_f, N_s}(x^n) = \frac{(1-x^n)^3}{[(x^{3n}+1) - (x^{2n}+x^n)3] + N_s[x^{2n}+x^n] - (-1)^n N_f\, x^{3n/2}} = -P_{N_f N_f}\left(\frac{1}{x}\right)$$

(...)

...

Part I: General Introduction:

(D)

- Overall structure of whole talk (contd...):

ⓐ Part I: (contd...)

(a6) (contd...) This tells us that, if $P(x)=0$, then $P(\frac{1}{x})=0$ too.

Thus, we can rewrite $P(x) \equiv p(y)$, w/ $y^2 \equiv x$ (trivial), or

$$\boxed{(11)} \quad p(y^n) \equiv \prod_{i=1}^{3} \frac{(1-y^{2n})}{(1-y^n r_i)(1-y^n/r_i)} = P(x^n)\Big|_{x=y^2}$$

$$\boxed{(12)} \quad Z_{QCD}^{N_f,N_s}(y) = \prod_{i=1}^{3}\left(\prod_{n=1}^{\infty} \frac{(1-y^{2n})}{(1-y^n r_i)(1-y^n/r_i)}\right) = \prod_{i=1}^{3} \frac{\eta(y^2)\eta(y)}{\theta_1(y, r_i)}$$

↳ A modular form!

- In some cases of interest, these roots simplify & the modular weight changes...

* Modular forms ⟷ modular group ↳ STRINGS in (quasi) NORMAL QCD??? A different realm where "strong" dynamics are literally described by a theory of strings???

WORK IN PROGRESS... WITH AC, KD & GB. WILL DISCUSS PRIVATELY, BUT NOT IN PART-(II))!

(a7) Finally, (13) $Z_{QCD}(y) = (-1)^{\delta(0)} Z_{QCD}(1/y)$, &

(14) $Z_{QCD}(y) = 1 + \# y^2 + \ldots \Rightarrow E_{vac} = 0$!

Part I : General Introduction:

E

- Overall structure of whole talk (contd...):

(a) Part I : (contd...)

(a7) (contd...) * THIS MEANS THE VALUE OF E_{vac} SELECTED BY T-REFLECTION/MOD INV/ CONFINEMENT IS ZERO.

I WILL DISCUSS [IF TIME] IN PART-(II)

* Independent arguments show this.

(b) Part II :

(b1) Discussion of ~~proof~~ physics-proof" that $\frac{1}{\eta(+\tau)} = (-1)^{s(0)} \frac{1}{\eta(-\tau)}$, and extension to ALL minimal models & indeed ALL partition functions (& observables) written in terms of modular forms.

(b2) Restricting to case where $Z(\beta)$ $[\mathcal{O}(\beta)]$ can be expressed in terms of ~~modular forms~~ Eisenstein series, T-reflection is a "triviality". But a triviality with TEETH.

ALSO $\exists$ MATH "PREDICTIONS"!!!

(b3) Modular forms' casting of $Z_{QCD}(\beta)$ tells us AUTOMATICALLY that as $\beta \to 0$ $Z_{QCD}(\beta) \sim e^{-(1/\beta + \# + g(\beta)) 1/\beta}$. This neatly explains results of this type in the literature.

Not all free theories have this property. Will discuss simple examples (free scalar on $S^3 \times S^1$ $Z(\beta) = e^{\int d\beta \, E_{cl}(\beta)}$

Part I: General Introduction:

(F)

- Overall structure of whole talk (cont'd...):
~~Overview of~~

(b) Part II: (cont'd...)

(b4) [Would like to discuss relationship between these results & the older ideas of asymptotic/misaligned SUSY, if time...]

(b5) Revisiting the assertion that $E_{vac} = 0$ for these large-N gauge theories, we see that the argument can be rather surprisingly broadened & shown to survive introduction of many IR scales into the problem. I.e., $E_{vac} = 0$ is NOT a (direct) consequence of any conformal symmetry, but, rather, more closely resembles a UNIVERSAL SUM RULE for confine(ed/ing) gauge theories!

CAN PICK & CHOOSE DEPENDING ON AUDIENCE INTEREST...

(b6) Would like to discuss ubiquity & underlying nature of T-reflection. This also seems very ubiquitous, deep/trivial, and is not any obvious corollary of a physical principle like TRI, (non)unitarity, exact solvability, SUSY, conformal symmetry, or SR/GR.

$\Downarrow$

MANY EXAMPLES, w/ no obvious pattern (other than some vague ties to modular forms)...

Part I: General Introduction:

- To be explicit:

  * T-reflection: ~~DAM~~ GB, AC, DAM & MY
    1406.6329

  * Casimir paper I: GB, AC, DAM & MY
    1408.3120

  * QCD paper I: GB, AC & DAM
    1409.1617

  * Casimir paper II: GB, AC, DAM & MY
    TBD (soon!)

  * QCD paper II: GB, AC, KD & DAM
    TBD (soon!)

  * T-reflection II: GB, AC, DAM & MY
    TBD (this summer!)

Part II : Physics & Mathematics of T-reflection & Vacuum Energies & Modularity

A T-reflection's physicality ∈ Field Theory:

(A1) First an aside into QM:

- $Z_{\text{2-level}}(\beta) = e^{+\beta\Delta/2} + e^{-\beta\Delta/2}$
- $Z_{SHO}(\beta) = e^{-\beta\Delta/2}\sum_{n=0}^{\infty} e^{\beta\Delta n} = \dfrac{e^{-\beta\Delta/2}}{1-e^{-\beta\Delta}} = \dfrac{1}{e^{\beta\Delta/2}-e^{-\beta\Delta/2}}$
- $Z_{\sqcup}(\beta) = \sum_{n\in\mathbb{Z}} e^{-\beta n^2/2L^2 m} = \Theta(\beta/L^2 m) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n+\frac{1}{2}})^2$

  w/ $q = e^{-\beta/L^2 m}$

  A product of ∞ # of decoupled fermionic SHOs.

- $\boxed{\text{Random levels fail T-reflection, of course...}}$

(A2) How I want to think of Field Theory:

- Consider N decoupled SHOs w/ characteristic frequencies $\{\omega_1, \ldots, \omega_N\}$
- Think of simplest QFT as N SHOs w/ integer spaced frequencies, as $N \to \infty$.
- Clearly, for finite-N, $\tilde{Z}^{\text{SHO}}_{(N)}(\beta) = (-1)^{\sum_{k=1}^{N} 1}\, \tilde{Z}^{\text{SHO}}_{(N)}(-\beta)$

Part II : [Details]

A T-reflection's physicality ∈ Field Theory:

(A 2) How I want to think of Field Theory (contd...):

- As $N \to \infty$, this infinite # of modes will encounter divergences.

  These divergences must be understood, isolated, regulated, THEN renormalized.

$\boxed{\text{IN PRECISELY THAT ORDER.}}$

- How to see them?

$$\boxed{F(\beta) = -\ln Z(\beta)}$$

$$* \; -\ln Z_N(+\beta) = -\ln\left(\prod_{a=1}^{N} \frac{e^{-\beta a/2}}{1-e^{-\beta a}}\right) \quad \longleftarrow \quad w_a \equiv a$$

$$= +\frac{\beta}{2}\sum_{a=1}^{N} a + \sum_{a=1}^{N} \ln\left(1-e^{-\beta a}\right)$$

$$* \; -\ln Z_N(-\beta) = -\frac{\beta}{2}\sum_{a=1}^{N} a + \sum_{a=1}^{N} \ln\left(1-e^{+\beta a}\right)$$

$$= -\frac{\beta}{2}\sum_{a=1}^{N} a + \sum_{a=1}^{N} \ln\left(e^{+\beta a}\left(1-e^{-\beta a}\right) * (-1)\right)$$

$$= \left[-\frac{\beta}{2}\sum_{a=1}^{N} a + \beta\sum_{a=1}^{N} a\right] + \sum_{a=1}^{N} \ln\left(1-e^{-\beta a}\right) + \sum_{a=1}^{N} \ln(-1)$$

$$= -\ln Z_N(+\beta) + \sum_{a=1}^{N} \ln(-1) \; !$$

→ But we already knew that...

Part II : [Details] ③

[A] T-reflection's physicality ∈ Field Theory:

(A3) • The point ~~to the~~ — the REASON why we do things this way — is that we want to see where the (UV) divergences come into

$-\ln[Z_N(+\beta)]$ VS. $-\ln[Z_N(-\beta)]$ as $N \to \infty$. [Why justified?]

VS

$$* \; -\ln Z_N(+\beta) = \boxed{+\frac{\beta}{2}\sum_{a=1}^{N} a} + \boxed{\sum_{a=1}^{N} \ln(1-e^{-\beta a})}$$

$$* \; -\ln Z_N(-\beta) = \boxed{-\frac{\beta}{2}\sum_{a=1}^{N} a} + \boxed{\sum_{a=1}^{N} \ln(1-e^{+\beta a})}$$

UV divergent!

This is the generating function for the degeneracy of states @ a given energy. This is physical...

This is its T-reflected cousin.

* Reason we rewrite $\boxed{\sum_{a=1}^{N} \ln(1-e^{+\beta a})} = \boxed{\beta\sum_{a=1}^{N} a} + \sum_{a=1}^{N} \ln(1-e^{-\beta a}) + \boxed{\sum_{a=1}^{N} \ln(-1)}$

is to isolate the UV-divergences in the QM-well-defined $Z_N(-\beta)$ from the usual generating function of degeneracies @ a given energy.

Part II: [Details]

A T-reflection's physicality ∈ Field Theory:

(A4) So: (1) Insisting on T-reflection ∀ SHO ⇒ T-reflection for regulated theory

(2) Isolating divergences, we have:

(2a) $E_{vac} = \frac{1}{2}\sum_{n} d_n \omega_n \equiv E_{cas}$!

I.e. ∄ shifts...

(2b) T-reflection phase $e^{i\gamma} = \exp\left[\sum_{n=1} d_n \ln(-1)\right]$

(A5) Now, send $N \to \infty$ in $Z_N = \prod_{a=1}^{N} \frac{e^{-\beta a/2}}{1-e^{-\beta a}}$:

* $$\lim_{N\to\infty} Z_N(+\beta) = \exp\left[\frac{\beta}{2}\sum_{a=1} a\right] * \prod_{n=1}^{\infty} \frac{1}{1-e^{-\beta n}}, \quad \boxed{q \equiv e^{-\beta} \equiv e^{2\pi i \tau}}$$

$$= \frac{1}{q^{\frac{1}{24}}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{\eta(\tau)}$$

* $$\lim_{N\to\infty} Z_N(-\beta) = \lim_{N\to\infty} (-1)^{\sum_{a=1}^{N}(-1)} Z_N(+\beta)$$

$$= (-1)^{\zeta(0)} Z_\infty(\beta) = \frac{(-1)^{\zeta(0)}}{q^{\frac{1}{24}}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{i}{\eta(\tau)}$$

* This is the holomorphic $Z(\tau)$ for a 2d free scalar CFT!

Part II: [Details]

A T-reflection's physicality ∈ Field Theory:

(A6) Re-capping: we T-reflect the 2d free scalar boson CFT FIRST through isolating the UV-divergent parts of $F_\Lambda(\tau)$ & $F_\Lambda(-\tau)$ from the parts which are the generating function for the d_n's, THEN by removing the regulator:

$$\begin{cases} -\ln Z_\Lambda(+\beta) = +\frac{\beta}{2}\sum_\Lambda n + \sum_\Lambda \ln\left(1-e^{-\beta n}\right) \\ -\ln Z_\Lambda(-\beta) = -\frac{\beta}{2}\sum_\Lambda n + \sum_\Lambda \ln\left(1-e^{+\beta n}\right) \\ \qquad = \cdots = +\frac{\beta}{2}\sum_\Lambda n + \sum_\Lambda \ln\left(1-e^{-\beta n}\right) + \sum_\Lambda \ln(-1) \end{cases}$$

$\boxed{\Lambda \to \infty}$

$$-\ln\left(Z(+\beta)\right) = -\ln\left(Z(-\beta)\right) - \frac{1}{2} \iff \boxed{\frac{1}{\eta(+\tau)} = \frac{e^{i\pi/2}}{\eta(-\tau)}}$$

(A7) Note: w/ this argument, can see all fermionic characters in 2d CFT, e.g. $\chi_{NS-NS}(\tau) = q^{-\frac{1}{48}}\prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}}\right)$, are invariant under T-reflection. → The minimal model $\mathcal{M}(4,3)$ is invariant.

→ (... by same/isomorphic manipulations)

ALL VIRASORO MIN. MODELS INVARIANT!

(?)

Part II : [Details]

[B] T-reflection in QM :

(B1) $Z_{\sqcup}(\beta) = \cdots = \Theta(\beta / L m^2)$ $\hookrightarrow$ Invariant.

(B2) $Z^{(\phi)}(\beta) = \sum_{n \in \mathbb{Z}} e^{\beta(n+\phi)^2/2Lm^2} = \Theta(\beta / L m^2, \beta\phi / L m^2)\, e^{\phi^2/2mL^2}$

$\hookrightarrow$ Invariant. { Note: B-field breaks TRI...
Note: not REMOTELY conformal...

[C] T-reflections & mathematics:

(C1) Not much to say formally. But in nearly all cases of interest, T-reflection relies on some vestige of modularity... and modular forms are only ever discussed in the UHP! $\hookrightarrow$ What's happening?

(C2) Not 100% clear, though $\exists$ some discussion in the math literature — e.g. Don Zagier's "Quantum modular forms" — which begin to probe "continuation" to LHP.

(C3) Note: this can't be a "continuation" in any simple sense:

$$\frac{1}{\eta(\tau)} \begin{cases} \to \infty & \text{for } \tau \to p/q \in \mathbb{Q} \\ \to 0 & \text{for } \tau \to x \notin \mathbb{Q} \end{cases}$$

$\Rightarrow$ Not even continuous on real axis...

Part II: [Details]

$\boxed{C}$ T-reflections & mathematics:

(C4) BUT, for now, we can exploit a triviality,

$$E_{2k}(\tau) \equiv \frac{1}{\zeta(2k)} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^{2k}} =$$

$$= \frac{1}{\zeta(2k)} \sum_{(m,\tilde{n})} \frac{1}{(m-\tilde{n}\tau)^{2k}} \overset{!}{=} \frac{1}{\zeta(2k)} \sum_{m,n} \frac{1}{(m-n\tau)^{2k}} \equiv E_{2k}(-\tau)$$

... and another well known identity,

$$\frac{1}{\Delta(\tau)} = \frac{1728}{E_6(\tau)^2 - E_4(\tau)^3} = \left(\frac{1}{\eta(\tau)}\right)^{24} = \left(\frac{1}{q^{\frac{1}{24}}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}\right)^{24}$$

$\longrightarrow$ Trivially, $\dfrac{1}{\Delta(+\tau)} = \dfrac{1}{\Delta(-\tau)}$

$\longrightarrow$ Thus, $\left[\dfrac{1}{(q^{-1})^{\frac{1}{24}}} \displaystyle\prod_{n=1}^{\infty} \dfrac{1}{1-(q^{-1})^n}\right]^{24} = \left[\dfrac{1}{q^{\frac{1}{24}}} \displaystyle\prod_{n=1}^{\infty} \dfrac{1}{1-q^n}\right]^{24}$

(C5) More generally, $\forall$ eff. $\in \tilde{\mathcal{M}}_k$, $T: \tilde{\mathcal{M}}_k \to \tilde{\mathcal{M}}_k^{*}$ $\ni: e^{i\gamma} = (-1)^k$

(C6) MacMahon function, and periods, too. E.g.: free scalar on $S^3 \times S^1$ has

$$Z(q) = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n}\right)^{n^2} = \exp\left\{ \int d\tau \left[\sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} \right] \right\} = \exp\left[\int E_4(\tau)\right] = Z\left(\frac{1}{q}\right)!$$

Part II: [Details]

C T-reflection & mathematics:

(C5) Holds for Jacobi forms, etc...

↳ This math "justification/triviality" depends only on the fact that the lattice of points in $\mathbb{C}$ identified by $(1, \tau)$ is the same as that identified by $(1, -\tau)$. So ALMOST vacuous. Save for ~~two~~ TWO reasons:

→ Save FIRST: $Z(\tau) = e^{i\gamma} Z(-\tau)$ fixes vacuum energies uniquely!

→ SECOND: in QFT @ finite temperature, the lattice is defined by S^1_β, w/ $\beta \sim \beta + 2\pi$. So looks like all QFTs have this. But then... how? Not obvious (even "looks" violated) in many examples.

↳ (If so) An unrecognized redundancy in QM which fixes E_{vac}?

ALSO, Product forms: If $M_k = q^{\Delta} \prod (1-q^n)^{3a(n^2)}$ [Borcherds], $\in \mathbb{C}[E_4, E_6]$, then $\exists$ sum rules on $\begin{cases} \sum 3a(n^2)\cdot n = \Delta/2 \\ \sum 3a(n^2) = 0 \end{cases}$!

D Now for E_{vac}'s, w/ no appeal to T-reflection... [...]

Part II : [Details] 15 "9"

$\boxed{E}$ Asymptotic/misaligned SUSY & QCD[Adj]...:

(E1) Both, prediction on confining gauge theories that are low-E & non-SUSY limits of superstring theory. Modular inv. is key!

(E2) For us, when $M_F = M_S = 0$, on $S^3_R \times S^1_L$

$$Z_{QCD}(q = e^{-\beta/R}) = \prod_{a=1}^{3} \frac{\eta(2\tau)\,\eta(\tau)}{\theta(\tau, z_a)}$$

! A modular form, but not a modular inv.

(E3) Issue of $E_{vac} = 0$, clearly $\lim_{T\to\infty} Z_{QCD}(q = e^{-1/RT}) \sim e^{\#T + \cdots}$

$\sim$ a 2d CFT!

(E4) Further, the growth of d_n for $[N_f, N_s - \text{any}]$ is

$$d_n \sim e^{n\beta_H}, \text{ w/ } e^{\beta_H} = X_* = e^{2\pi i z_a},$$ for the largest magnitude $z_a \Rightarrow$ trivial Hagedorn growth...

(E5) ...yet, for twisted fermion BCs, $\nexists$ any pole on the interval $\beta \in (0, \infty] \Leftrightarrow T \in [0, \infty)$. Thus, $\nexists$ a phase transition for any (real) radius of the "twisted thermal" circle.

$\Downarrow$ Volume independence $\Rightarrow$ Fermionic Bosonic level-matching! $\Uparrow$ Asymptotic/Misaligned SUSY!

Part II : [Details]

16 NS

"10"

E Asymptotic/misaligned SUSY & QCD [Adj]...:

(E6) Thus, even for thermal BCs, we have a matching of Hagedorn-rise of bosonic & fermionic states...

(E7) All the ingredients for Asymptotic SUSY are there, save a concrete tie to a perturbative string dual to large-N YM/QCD.

Under active investigation. ~~Would love to discuss privately~~.

[w/ KD, AC & GB]

F...inally done.

THANK YOU!

FIGURE FOR (E4)-(E6)