The Verlinde formula in logarithmic conformal field theory

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Conformal field theory and the Verlinde formula

The two building blocks of a conformal field theory (CFT)

- A vertex operator algebra (VOA) V.
- A *physical* category $\mathcal{C}$ of V-modules which is
  - closed under conjugation C,
  - closed under fusion $\otimes$,
  - admits a modular invariant partition function.

Definition

A CFT is rational if $\mathcal{C}$ is semisimple with finitely many (isomorphism classes of) simple V-modules.

Conformal field theory and the Verlinde formula

Theorem [Zhu]

For rational CFTs, the span of characters carries a representation of the modular group $\mathsf{SL}(2,\mathbb{Z}) = \langle \mathsf{S}, \mathsf{T} | \mathsf{S}^4 = 1, (\mathsf{ST})^3 = \mathsf{S}^2 = \mathsf{C} \rangle$.

Theorem [Verlinde, Moore-Seiberg, Huang]

For rational CFTs, $\mathsf{S}^{\mathsf{T}} = \mathsf{S}$, $\mathsf{S}^{\dagger} = \mathsf{S}^{-1}$, $\mathsf{S}^2 = \mathsf{C}$, and S diagonalises the fusion coefficients through the Verlinde formula:

$$\mathcal{L}_i \otimes \mathcal{L}_j = \bigoplus_k \begin{bmatrix} & k & \\ i & & j \end{bmatrix} \mathcal{L}_k, \quad \begin{bmatrix} & k & \\ i & & j \end{bmatrix} = \sum_{\ell} \frac{\mathsf{S}_{i,\ell} \mathsf{S}_{j,\ell} \overline{\mathsf{S}_{k,\ell}}}{\mathsf{S}_{0,\ell}}.$$

Example: Yang-Lee minimal model

The central charge $c = -\frac{22}{5}$ minimal model with 2 simple modules.

- vacuum $\mathcal{L}_0$:

$$\mathsf{ch}_{\mathcal{L}_0}(\tau) = \mathrm{Tr}_{\mathcal{L}_0}\left(q^{L_0 - \frac{c}{24}}\right) = q^{\frac{11}{60}}(1 + q^2 + q^3 + q^4 + q^5 + 2q^6).$$

- conformal weight $-\frac{1}{5}$-module $\mathcal{L}_1$:

$$\mathsf{ch}_{\mathcal{L}_1}(\tau) = q^{-\frac{1}{60}}(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6).$$

Modular transformations:

$$\mathsf{S} \circ \mathsf{ch}_{\mathcal{L}_0}(\tau) = \mathsf{ch}_{\mathcal{L}_0}\left(-\frac{1}{\tau}\right) = \frac{-\sqrt{2}}{\sqrt{5-\sqrt{5}}}\mathsf{ch}_{\mathcal{L}_0}(\tau) + \frac{\sqrt{3-\sqrt{5}}}{\sqrt{5-\sqrt{5}}}\mathsf{ch}_{\mathcal{L}_1}(\tau)$$

$$\mathsf{ch}_{\mathcal{L}_1}\left(-\frac{1}{\tau}\right) = \frac{\sqrt{3-\sqrt{5}}}{\sqrt{5-\sqrt{5}}}\mathsf{ch}_{\mathcal{L}_0}(\tau) + \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}}\mathsf{ch}_{\mathcal{L}_1}(\tau)$$

Example: Yang-Lee minimal model

The Verlinde formula then yields the following fusion rules:

$$\begin{aligned}
\mathcal{L}_0 \otimes \mathcal{L}_0 &= \mathcal{L}_0, \\
\mathcal{L}_0 \otimes \mathcal{L}_1 &= \mathcal{L}_1, \\
\mathcal{L}_1 \otimes \mathcal{L}_1 &= \mathcal{L}_0 \oplus \mathcal{L}_1.
\end{aligned}$$

Why go beyond rationality?

Rational CFTs describe:

- Strings on compact (Lie group) space times.
- Local observables for critical statistical models.

What about non-local observables (e.g. crossing probabilities) and non-compact space times (e.g. $\mathbb{M}^{1,d}$ or AdS)?

In these cases non-rational (infinitely many simple modules in $\mathcal{C}$) or logarithmic ($\mathcal{C}$ has indecomposable yet non-simple modules) CFTs are required.

Open problem:

How does the formalism of rational CFT, especially Verlinde, generalise to non-rational or logarithmic CFT?

Why care about Verlinde?

The existence of a Verlinde formula is a highly non-trivial indicator that a given theory is *natural*.

Given a VOA V, one proposes a module category $\mathcal{C}$ in which one can

- Check closure under conjugation. $\rightarrow$ easy

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If the goal is computing fusion products, then a Verlinde formula is immensely helpful.

A semisimple but irrational example: The free boson

VOA= Heisenberg VOA generated by the free boson $a(z)$.

$$a(z)a(w) \sim \frac{1}{(z-w)^2} \Longleftrightarrow [a_m, a_n] = m\delta_{m+n,0}\mathbf{1}.$$

$$T(z) = \tfrac{1}{2} :a(z)^2: \Rightarrow\ c = 1.$$

Fock modules:

$$\mathcal{F}_p = \mathbb{C}[a_{-1}, a_{-2}, \dots] \left|p\right\rangle,\ p \in \mathbb{C}, \quad a_0 \left|p\right\rangle = p \left|p\right\rangle,\ a_m \left|p\right\rangle = 0, n \geq 1.$$

$$L_0 \left|p\right\rangle = \tfrac{p^2}{2} \left|p\right\rangle.$$

A semisimple but irrational example: The free boson

Module category = (sums of) Fock modules with real weights.

- Simples are Fock modules $\mathcal{F}_p, p \in \mathbb{R}$.
- $\mathsf{ch}_{\mathcal{F}_p} = \mathsf{Tr}_{\mathcal{F}_p}(z^{a_0} q^{L_0 - 1/24}) = \dfrac{z^p q^{p^2/2}}{\eta(q)}$.
- $\mathsf{S} \circ \mathsf{ch}_{\mathcal{F}_p} = \int_{-\infty}^{\infty} \mathsf{S}_{pq} \mathsf{ch}_{\mathcal{F}_q} \, \mathrm{d}q$, where $\mathsf{S}_{pq} = \exp(-2\pi i p q)$.
- $\begin{bmatrix} & r & \\ p & & q \end{bmatrix} = \displaystyle\int_{-\infty}^{\infty} \frac{\mathsf{S}_{ps} \mathsf{S}_{qs} \mathsf{S}_{rs}^*}{\mathsf{S}_{0s}} \, \mathrm{d}s = \delta(p+q-r)$,

  $\Rightarrow \mathcal{F}_p \otimes \mathcal{F}_q = \displaystyle\int_{-\infty}^{\infty} \begin{bmatrix} & r & \\ p & & q \end{bmatrix} \mathcal{F}_r \, \mathrm{d}r = F_{p+q}$.

Logarithmic CFTs with finitely many simples

The modular framework does not immediately generalise to logarithmic CFTs. For example, the simple $W(1,p)$ triplet characters do not span a representation of $\mathsf{SL}(2,\mathbb{Z})$ (τ-dependent coefficients) [Flohr].

Extending to torus amplitudes gives a representation of $\mathsf{SL}(2,\mathbb{Z})$ [Miyamoto], but finding modular invariant partition functions is harder.

The character of an indecomposable module is equal to the character of the sum of composition factors.
Example: Let M be an indecomposable module with submodule N, then

$$\mathsf{ch}_M = \mathsf{ch}_N + \mathsf{ch}_{M/N}, \text{ but } M \neq N \oplus M/N.$$

$\Rightarrow$ There is no canonical basis of characters and the Verlinde formula can at best only give the character of a fusion product.

Irrational logarithmic CFTs and the standard module formalism

In many examples, identify (indecomposable) standard modules. Partition into simple (typical) and non-simple (atypical). Characters ch_m of standard modules satisfy:

1. ch_m parametrised by measurable space (M, μ).
2. Atypical characters parametrised by $A \subset M$, with $\mu(A) = 0$.
3. $\{\mathsf{ch}_m\}$ is a (topological) basis for $\mathbb{Z}$-module of all characters.
4. Characters span an $\mathsf{SL}(2,\mathbb{Z})$-module with $S \circ \mathsf{ch}_{\mathcal{M}} = \int_M \mathsf{S}_{\mathcal{M}n} \mathsf{ch}_n d\mu(n)$, satisfying
   - Symmetry: $\mathsf{S}_{mn} = \mathsf{S}_{nm}$,
   - Unitarity: $\int_M \mathsf{S}_{mp} \mathsf{S}^*_{pn} d\mu(p) = \delta(m-n)$,
   - Conjugation: S^2 is a permutation of order ≤ 2.

Irrational logarithmic CFTs and the standard module formalism

5. $\mathsf{ch}_{\mathcal{M}} = \sum_m a_m \mathsf{ch}_m \Rightarrow \mathsf{S}_{\mathcal{M}n} = \sum_m a_m \mathsf{S}_{mn}$, which converges for all typical n ($n \notin A$).
6. The vacuum module Ω satisfies $\mathsf{S}_{\Omega,n} = 0$, for all $n \notin A$.
7. Define character fusion $\times$ by Verlinde formula:

$$\mathsf{ch}_{\mathcal{M}} \times \mathsf{ch}_{\mathcal{N}} = \int_M \begin{bmatrix} & p & \\ \mathcal{M} & & \mathcal{N} \end{bmatrix} \mathsf{ch}_p d\mu(p),$$

$$\begin{bmatrix} & p & \\ \mathcal{M} & & \mathcal{N} \end{bmatrix} = \int_M \frac{\mathsf{S}_{\mathcal{M}q}\mathsf{S}_{\mathcal{N}q}\mathsf{S}^*_{pq}}{\mathsf{S}_{\Omega q}} d\mu(q),$$

where $\begin{bmatrix} & p & \\ \mathcal{M} & & \mathcal{N} \end{bmatrix} \in \mathbb{Z}_{\geq 0}$ is the fusion coefficient.

"Trivial" example is a rational CFT:

- Standard = simple, so no atypicals ($A = \emptyset$).
- M is finite and μ is counting measure.
- Character fusion = fusion.

A logarithmic irrational example: The $\beta\psi$ ghosts

$\beta\gamma$ ghost VOA is generated by 2 fields $\beta(z), \gamma(z)$:

$$\gamma(z)\beta(w) \sim \frac{1}{z-w} \Longleftrightarrow [\gamma_m, \beta_n] = \delta_{m+n,0}\mathbf{1}.$$
$$T(z) = -:\beta(z)\partial\gamma(z): \Rightarrow\ c = 2,\ h_\beta = 1,\ h_\gamma = 0.$$

Lorentzian free boson:

$$J(z) = :\beta(z)\gamma(z):\ \Rightarrow J(z)\beta(w) \sim \frac{\beta(w)}{z-w}, \quad J(z)\gamma(w) \sim \frac{-\gamma(w)}{z-w}.$$

Spectral flow automorphisms:

$$\sigma(\beta_n) = \beta_{n-1},\ \sigma(\gamma_n) = \gamma_{n+1}$$
$$\Rightarrow \sigma^\ell(J_n) = J_n + \ell\delta_{n,0}\mathbf{1},\ \sigma(L_n) = L_n - \ell J_n - \tfrac{1}{2}\ell(\ell-1)\delta_{n,0}\mathbf{1}.$$

A logarithmic irrational example: The $\beta\psi$ ghosts

The vacuum modle is the unique highest weight module

$$\Omega = \mathbb{C}[\gamma_0, \beta_{-1}, \gamma_{-1}, \beta_{-2}, \dots]\omega, \quad L_0\omega = 0 = J_0\omega.$$

Additionally, there is a continuum of relaxed highest weigh modules

$$\mathcal{W}_\lambda = \mathbb{C}[\gamma_0, \beta_{-1}, \gamma_{-1}, \beta_{-2}, \dots]u_\lambda + \mathbb{C}[\beta_0, \beta_{-1}, \gamma_{-1}, \beta_{-2}, \dots]\beta_0 u_\lambda,$$

$$J_0 u_\lambda = \lambda u_\lambda,\ L_0 u_\lambda = 0,\ \lambda \notin \mathbb{Z}.$$

For $\lambda - \mu \in \mathbb{Z}$, $W_\lambda \cong W_\mu$.

For $\lambda \in \mathbb{Z}$ there exist two reducible indecomposable modules $\mathcal{W}^\pm$ characterised by non-split exact squences

$$0 \longrightarrow \Omega \longrightarrow \mathcal{W}^+ \longrightarrow \Omega^* \longrightarrow 0$$

$$0 \longrightarrow \Omega^* \longrightarrow \mathcal{W}^- \longrightarrow \Omega \longrightarrow 0$$

The characters of the relaxed highest weight modules are algebraic distributions.

$$\mathrm{ch}_{\mathcal{W}_\lambda} = \mathrm{Tr}_{\mathcal{W}_\lambda}(z^{J_0} q^{L_0 - \frac{1}{12}}) = \frac{z^\lambda}{\eta(q)^2} \sum_{k\in\mathbb{Z}} z^k = \frac{z^\lambda}{\eta(q)^2} \sum_{k\in\mathbb{Z}} \delta(\zeta - k),$$

where $z = \exp(2\pi i\zeta)$.

A logarithmic irrational example: The $\beta\psi$ ghosts

We take the standard modules to be $\sigma^{\ell}(\mathcal{W}_{\lambda})$, $\ell \in \mathbb{Z}$, $\lambda \in \mathbb{R}$. For $\lambda \notin \mathbb{Z}$ they are typical, while $\sigma^{\ell}(\mathcal{W}^{\pm})$ are atypical.

The Verlinde formula then predicts

$$
\begin{aligned}
\mathsf{ch}\left[\sigma^{\ell}(\Omega)\right] \times \mathsf{ch}\left[\sigma^{m}(\Omega)\right] &= \mathsf{ch}\left[\sigma^{\ell+m}(\Omega)\right], \\
\mathsf{ch}\left[\sigma^{\ell}(\Omega)\right] \times \mathsf{ch}\left[\sigma^{m}(\mathcal{W}_{\lambda})\right] &= \mathsf{ch}\left[\sigma^{\ell+m}(\mathcal{W}_{\lambda})\right], \\
\mathsf{ch}\left[\sigma^{\ell}(\mathcal{W}_{\lambda})\right] \times \mathsf{ch}\left[\sigma^{m}(\mathcal{W}_{\mu})\right] &= \mathsf{ch}\left[\sigma^{\ell+m}(\mathcal{W}_{\lambda+\mu})\right] + \mathsf{ch}\left[\sigma^{\ell+m-1}(\mathcal{W}_{\lambda+\mu})\right].
\end{aligned}
$$

As long as $\lambda + \mu \notin \mathbb{Z}$, then $\sigma^{\ell+m}(\mathcal{W}_{\lambda+\mu})$ and $\sigma^{\ell+m-1}(\mathcal{W}_{\lambda+\mu})$ cannot mutually extend to form an indecomposable because the J_0 eigenvalues are different.

A logarithmic irrational example: The $\beta\psi$ ghosts

For $\lambda + \mu \in \mathbb{Z}$ the fusion product needs to be computed explicitly.
$\Rightarrow \mathcal{W}_\lambda \otimes \mathcal{W}_{-\lambda}$ is logarithmic with rank 2 Jordan blocks for L_0:

$$\begin{tikzcd}
 & \sigma^{-1}(\Omega) \arrow[dl] \arrow[dr] & \\
\sigma^{-2}(\Omega) \arrow[dr] & & \Omega \arrow[dl] \\
 & \sigma^{-1}(\Omega) &
\end{tikzcd}$$

Back to rational logarithmic CFTs

In a rational logarithmic CFT there are only a finite number of simple modules. The measure is therefore the counting measure and logarithmic complications can therefore not be ignored.

All known examples of rational logarithmic CFTs admit an infinite order orbifold that is compatible with the standard module formalism. The rational parent theory can then be reconstructed through simple current extension.

$$\begin{tikzcd}
\text{rational log} \arrow[r, dashed, "\text{direct?}"] \arrow[d, "\text{orbifold}"] & \text{rational log fusion} \\
\text{non-rational log} \arrow[r, "\text{Verlinde}"] & \text{non-rational log fusion} \arrow[u, "\text{simple current}"']
\end{tikzcd}$$

Standard module formalism has been succesfully applied to

1. $\widehat{\mathfrak{gl}}(1|1)$, [Creutzig, Ridout]
2. $\widehat{\mathfrak{sl}}(2)$ at admissible levels, [Creutzig, Ridout]
3. Triplet and singlet $W(p,q)$ algebras, [Creutzig, Milas, Ridout, SW]
4. $\beta\gamma$ ghosts, [Ridout, SW]
5. $N=1$ Virasoro, [Canagasabey, Ridout]

A general proof should hopefully be feasable soon!

The End

Thank you!