Correlation Functions in Stochastic Inflation

Based on VV & A Starobinsky, arXiv:1506.04732

Vincent Vennin

IPMU Tokyo, 23 June 2015

Outline

- Quantum State of Cosmological Perturbations
- The Stochastic Inflation Formalism
- Correlation Functions in Stochastic Inflation
- The δN-stochastic formalism
- The First Passage Time Problem
- Results and Conclusion

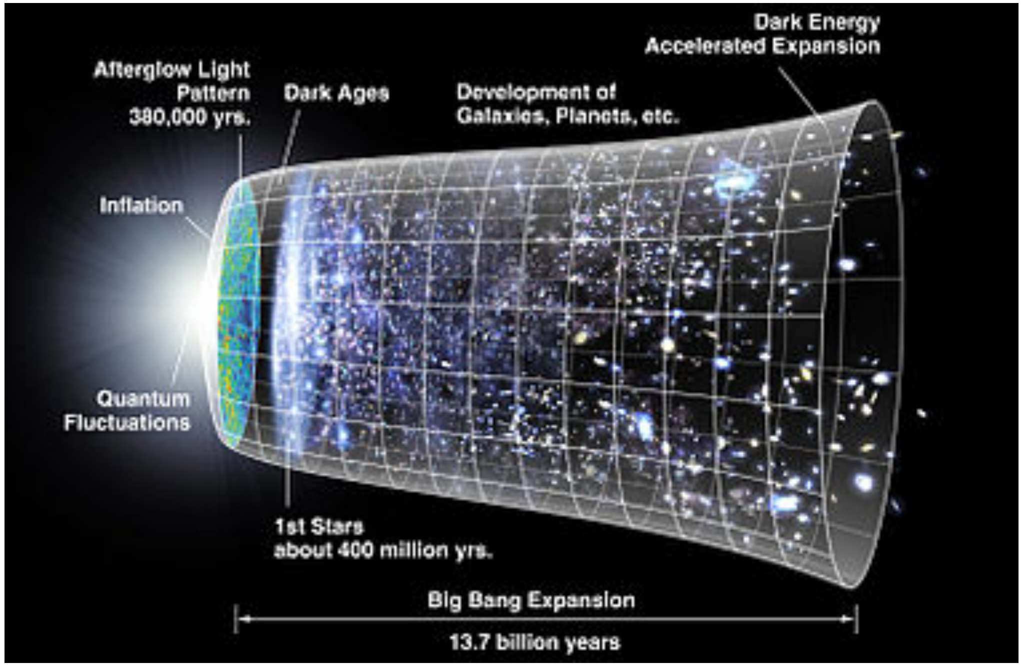
Cosmological Inflation

Starobinsky (1980)
Sato (1981)
Guth (1981)
Mukhanov & Chibisov (1981)
Linde (1982)
Albrecht & Steinhardt (1982)

Cosmological Inflation

Starobinsky (1980)
Sato (1981)
Guth (1981)
Mukhanov & Chibisov (1981)
Linde (1982)
Albrecht & Steinhardt (1982)

- Is a high energy phase of accelerated expansion in the early Universe $\ddot{a} > 0$

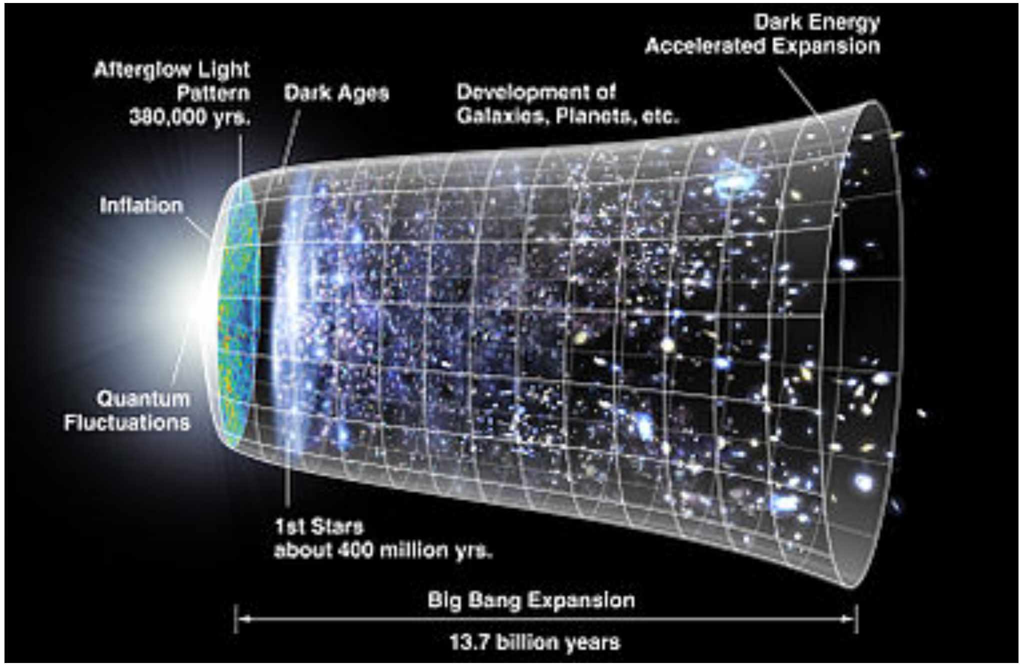


Cosmological Inflation

Starobinsky (1980)
Sato (1981)
Guth (1981)
Mukhanov & Chibisov (1981)
Linde (1982)
Albrecht & Steinhardt (1982)

- Is a high energy phase of accelerated expansion in the early Universe $\ddot{a} > 0$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\vec{x}^2$$



Cosmological Inflation

Starobinsky (1980)
Sato (1981)
Guth (1981)
Mukhanov & Chibisov (1981)
Linde (1982)
Albrecht & Steinhardt (1982)

- Is a high energy phase of accelerated expansion in the early Universe $\ddot{a} > 0$

$$\rho_{\rm now} \simeq \left(10^{-12}{\rm GeV}\right)^4$$

$$\rho_{\rm BBN} \simeq (10\,{\rm MeV})^4$$

Inflation

$$\rho_{\rm GUT} \simeq \left(10^{16}{\rm GeV}\right)^4$$

Cosmological Inflation

Starobinsky (1980)
Sato (1981)
Guth (1981)
Mukhanov & Chibisov (1981)
Linde (1982)
Albrecht & Steinhardt (1982)

- Is a high energy phase of accelerated expansion in the early Universe $\ddot{a} > 0$

- Solves the Hot Big Bang horizon and flatness problems

Cosmological Inflation

Starobinsky (1980)
Sato (1981)
Guth (1981)
Mukhanov & Chibisov (1981)
Linde (1982)
Albrecht & Steinhardt (1982)

- Is a high energy phase of accelerated expansion in the early Universe $\ddot{a} > 0$
- Solves the Hot Big Bang horizon and flatness problems
- Requires a fluid with negative pressure $\frac{\ddot{a}}{a} = -\frac{1}{6M_{\rm Pl}^2}(\rho + 3p)$

Cosmological Inflation

Starobinsky (1980)
Sato (1981)
Guth (1981)
Mukhanov & Chibisov (1981)
Linde (1982)
Albrecht & Steinhardt (1982)

- Is a high energy phase of accelerated expansion in the early Universe $\ddot{a} > 0$
- Solves the Hot Big Bang horizon and flatness problems
- Requires a fluid with negative pressure $$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\rm Pl}^2}(\rho + 3p)$$
- Needs to be connected to the subsequent radiation era through a phase of reheating (driven by the coupling between the inflationary content and other fields)

Cosmological Inflation

Starobinsky (1980)
Sato (1981)
Guth (1981)
Mukhanov & Chibisov (1981)
Linde (1982)
Albrecht & Steinhardt (1982)

- Is a high energy phase of accelerated expansion in the early Universe $\ddot{a} > 0$

- Solves the Hot Big Bang horizon and flatness problems

- Requires a fluid with negative pressure $\frac{\ddot{a}}{a} = -\frac{1}{6M_{\rm Pl}^2}(\rho + 3p)$

- Needs to be connected to the subsequent radiation era through a phase of reheating (driven by the coupling between the inflationary content and other fields)

- Combined with QM, accounts for the production of cosmological perturbations whose features depend on the underlying inflationary model.

Cosmological Perturbations

Lifshitz (1946), Grishchuk (1974)
Starobinsky (1979, 1982)
Bardeen (1980)
Mukhanov and Chibisov (1981)
Hawking (1982)
Guth and Pi (1982)
Kodama & Sasaki (1984)

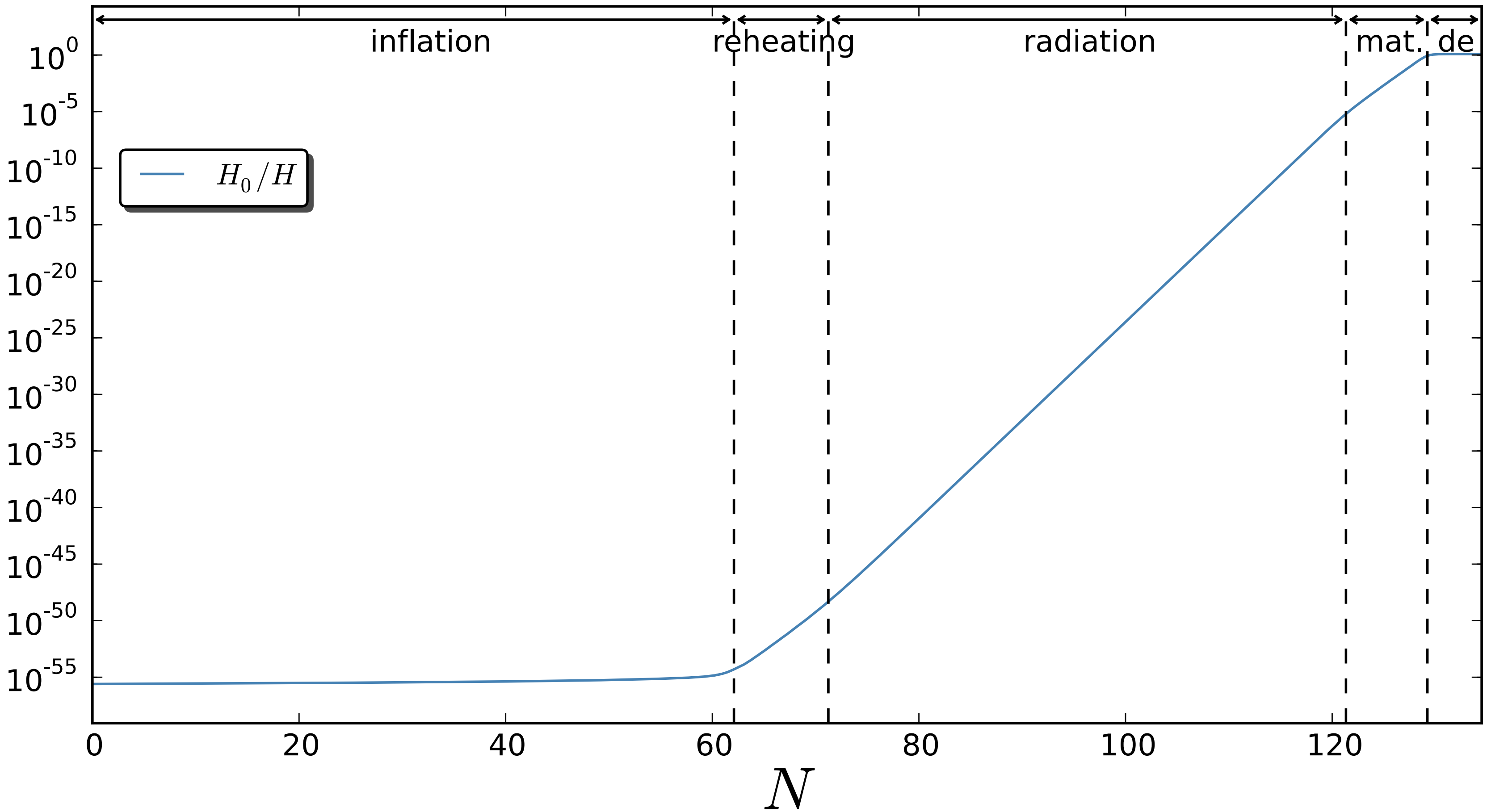
Quantized fluctuations evolved over an expanding, homogeneous and isotropic background

They are:

- Combined perturbations v of the metric and of the inflaton scalar field
- Of quantum nature $\hat{v}$
- Amplified through parametric oscillations
- The seeds of the CMB temperature fluctuations $\dfrac{\delta\hat{T}}{T} \propto \hat{\zeta} \propto \hat{v}$

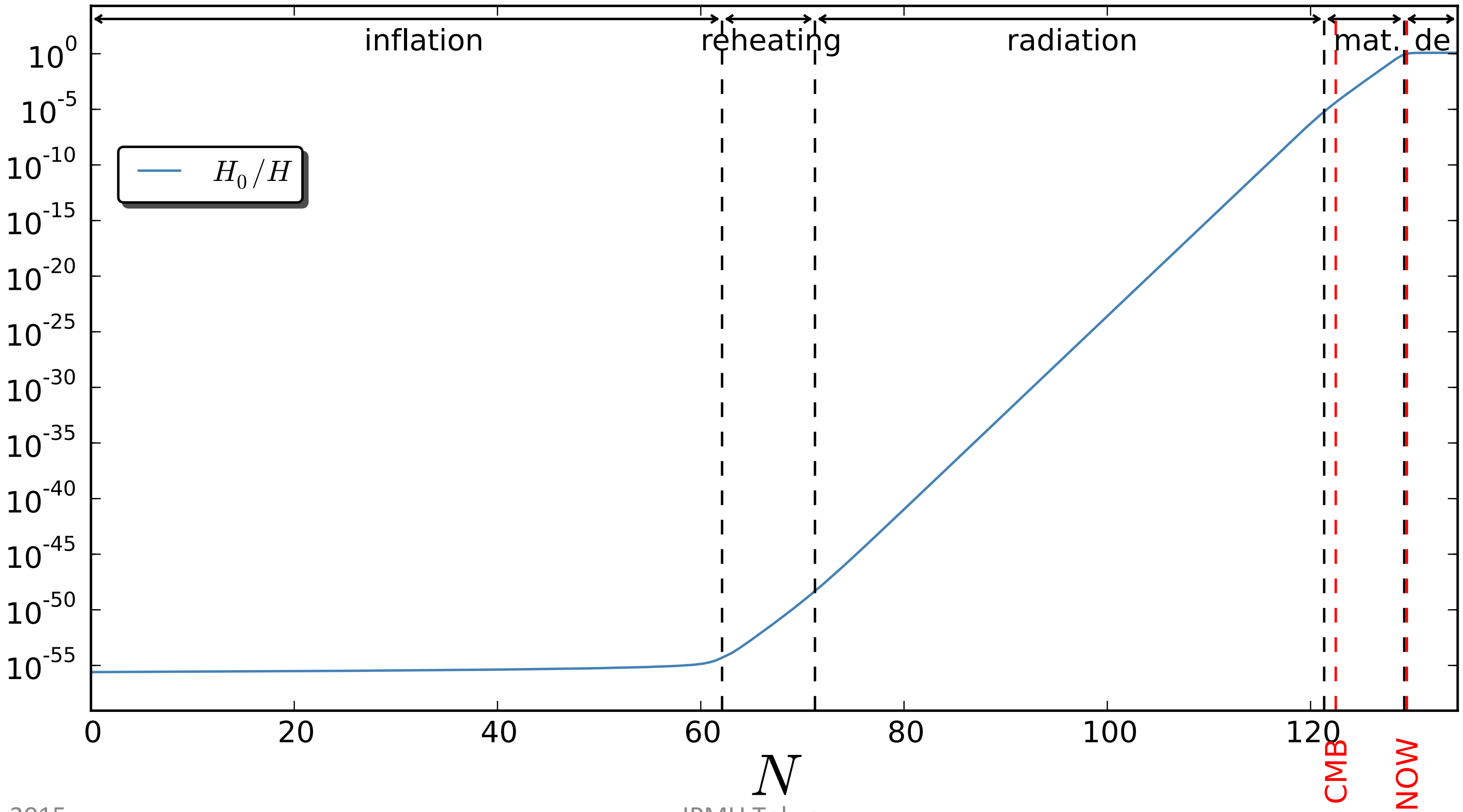
Cosmological Perturbations

Lifshitz (1946), Grishchuk (1974)
Starobinsky (1979, 1982)
Bardeen (1980)
Mukhanov and Chibisov (1981)
Hawking (1982)
Guth and Pi (1982)
Kodama & Sasaki (1984)



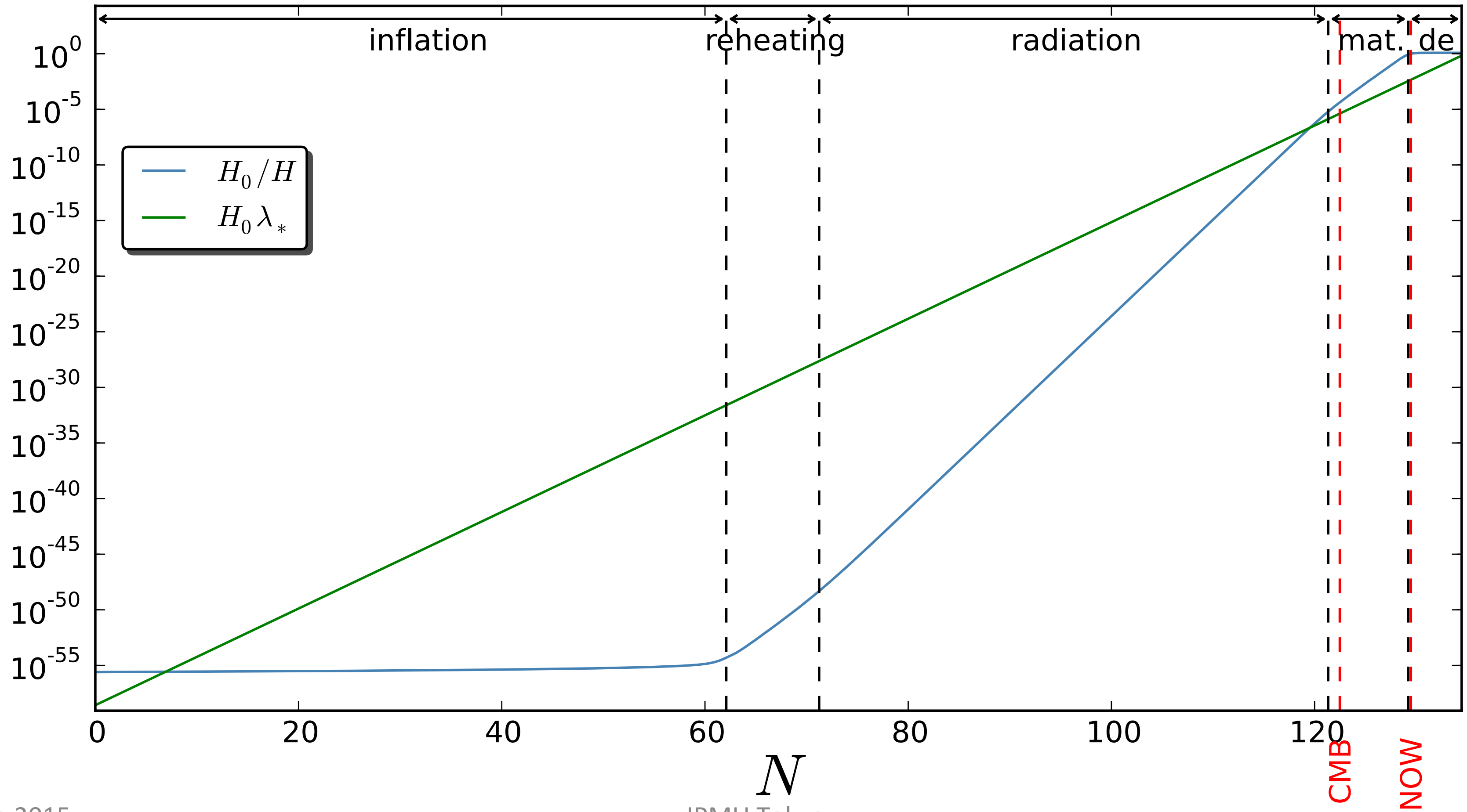
Cosmological Perturbations

Lifshitz (1946), Grishchuk (1974)
Starobinsky (1979, 1982)
Bardeen (1980)
Mukhanov and Chibisov (1981)
Hawking (1982)
Guth and Pi (1982)
Kodama & Sasaki (1984)



Cosmological Perturbations

Lifshitz (1946), Grishchuk (1974)
Starobinsky (1979, 1982)
Bardeen (1980)
Mukhanov and Chibisov (1981)
Hawking (1982)
Guth and Pi (1982)
Kodama & Sasaki (1984)



Quantum State of Cosmological Perturbations

Quantization in the Schrödinger picture
(reciprocal space)

$$\Psi(\eta, v_{\boldsymbol{k}}^{\mathrm{R,I}}) = \left[\frac{2\,\Re\mathrm{e}\Omega_{\boldsymbol{k}}(\eta)}{\pi}\right]^{1/4} \mathrm{e}^{-\Omega_{\boldsymbol{k}}(\eta)\left(v_{\boldsymbol{k}}^{\mathrm{R,I}}\right)^2}$$

$$i\frac{\mathrm{d}|\Psi_{\boldsymbol{k}}\rangle}{\mathrm{d}\eta} = \hat{\mathcal{H}}_{\boldsymbol{k}}\,|\Psi_{\boldsymbol{k}}\rangle$$

with

$$\hat{\mathcal{H}}_{\boldsymbol{k}} = \frac{\hat{p}_{\boldsymbol{k}}^2}{2} + \omega^2(\boldsymbol{k},\eta)\hat{v}_{\boldsymbol{k}}^2$$

$$\omega^2(\boldsymbol{k},\eta) = k^2 - \frac{(a\sqrt{\epsilon_1})''}{a\sqrt{\epsilon_1}}$$

$$\Omega_{\boldsymbol{k}}' = -2i\Omega_{\boldsymbol{k}}^2 + \frac{i}{2}\omega^2(\eta,\boldsymbol{k})$$

$$\Omega_{\boldsymbol{k}} = -\frac{i}{2}\frac{f_{\boldsymbol{k}}'}{f_{\boldsymbol{k}}}$$

$$f_{\boldsymbol{k}}'' + \omega^2(\boldsymbol{k},\eta)f_{\boldsymbol{k}} = 0$$

Quantum State of Cosmological Perturbations

Number of Particles and Initial State

Sub-Hubble limit: $\omega^2(\boldsymbol{k},\eta) = k^2 - \frac{(a\sqrt{\epsilon_1})''}{a\sqrt{\epsilon_1}}$

harmonic oscillator

$$f_k = A_k e^{-ik\eta} + B_k e^{ik\eta}$$

$$\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k = -\frac{1}{2} + \frac{w}{2}\hat{v}_k^2 + \frac{\hat{p}_k^2}{2w}$$

$$\langle \hat{n}_k \rangle = \int \mathrm{d}v_k \Psi_k^*(v_k)\hat{n}_k \cdot \Psi_k(v_k) = \frac{|A_k|^2}{|B_k|^2 - |A_k|^2}$$

vacuum state $\rightarrow A_k = 0 \rightarrow f \propto e^{ik\eta} \rightarrow \Omega_k = k/2$

Quantum State of Cosmological Perturbations

Quantization in the Schrödinger picture
(reciprocal space)

$$\Psi(\eta, v_{\boldsymbol{k}}^{\mathrm{R,I}}) = \left[\frac{2\,\Re\mathrm{e}\Omega_{\boldsymbol{k}}(\eta)}{\pi}\right]^{1/4} \mathrm{e}^{-\Omega_{\boldsymbol{k}}(\eta)\left(v_{\boldsymbol{k}}^{\mathrm{R,I}}\right)^2}$$

Wigner Function

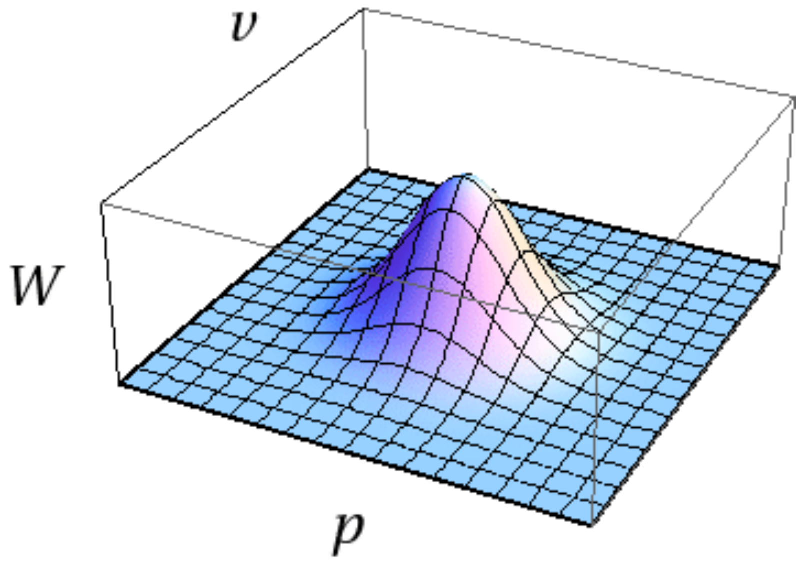
$$W(v_{\boldsymbol{k}}, p_{\boldsymbol{k}}) = \int \frac{\mathrm{d}x}{2\pi^2}\Psi^*(v_{\boldsymbol{k}} - \frac{x}{2})\,\mathrm{e}^{-ip_{\boldsymbol{k}}x}\;\Psi(v_{\boldsymbol{k}} + \frac{x}{2})$$

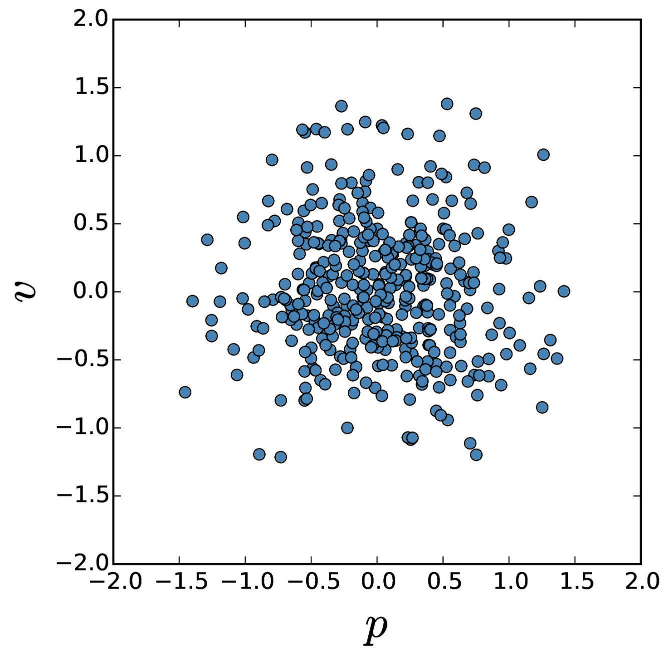
Quantum State of Cosmological Perturbations

Wigner Function $$W(v_{\boldsymbol{k}}, p_{\boldsymbol{k}}) = \int \frac{\mathrm{d}x}{2\pi^2} \Psi^*(v_{\boldsymbol{k}} - \frac{x}{2}) \, \mathrm{e}^{-ip_{\boldsymbol{k}}x} \; \Psi(v_{\boldsymbol{k}} + \frac{x}{2})$$

- Evolution Equation $$\frac{\partial}{\partial t} W(v,p,t) = -\left\{W(v,p,t)\ ,\ H(v,p,t)\right\}_{\text{Poisson Bracket}}$$

for quadratic Hamiltonian





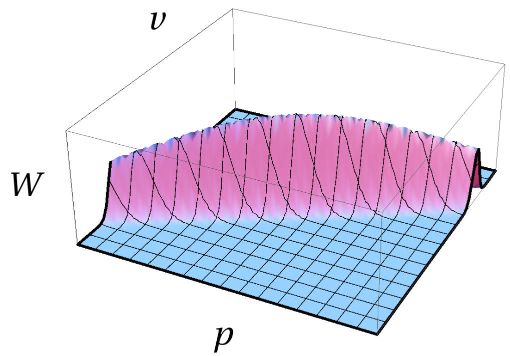
Quantum State of Cosmological Perturbations

Wigner Function
$$W(v_{\boldsymbol{k}}, p_{\boldsymbol{k}}) = \int \frac{\mathrm{d}x}{2\pi^2} \Psi^*(v_{\boldsymbol{k}} - \frac{x}{2})\ \mathrm{e}^{-ip_{\boldsymbol{k}}x}\ \Psi(v_{\boldsymbol{k}} + \frac{x}{2})$$

- Evolution Equation $$\frac{\partial}{\partial t} W(v,p,t) = -\left\{W(v,p,t)\ ,\ H(v,p,t)\right\}_{\mathrm{Poisson\ \ Bracket}}$$

  for quadratic Hamiltonian

- Quantum Mean Value and Stochastic Average



$$\left\langle \hat{\mathcal{O}}(\hat{v},\hat{p}) \right\rangle_{\mathrm{quant}} \simeq \int W(v,p)\, \mathcal{O}(v,p)\, \mathrm{d}v\, \mathrm{d}p$$

in the high squeezing limit

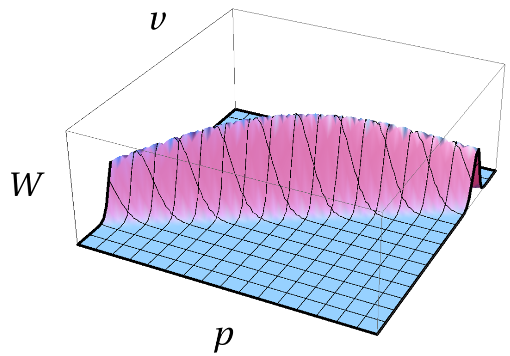
Quantum State of Cosmological Perturbations

Wigner Function $$W(v_{\boldsymbol{k}}, p_{\boldsymbol{k}}) = \int \frac{\mathrm{d}x}{2\pi^2} \Psi^*(v_{\boldsymbol{k}} - \frac{x}{2})\ \mathrm{e}^{-ip_{\boldsymbol{k}}x}\ \Psi(v_{\boldsymbol{k}} + \frac{x}{2})$$

- Evolution Equation $$\frac{\partial}{\partial t} W(v,p,t) = -\left\{W(v,p,t)\ ,\ H(v,p,t)\right\}_{\mathrm{Poisson\ \ Bracket}}$$

  for quadratic Hamiltonian

- Quantum Mean Value and Stochastic Average



$$\left\langle \hat{\mathcal{O}}\left(\hat{v},\hat{p}\right)\right\rangle_{\mathrm{quant}} \simeq \int W(v,p)\, \mathcal{O}(v,p)\, \mathrm{d}v\, \mathrm{d}p$$

in the high squeezing limit

Example: $\langle vp \rangle \xrightarrow[\mathrm{squeezed}]{} e^{\Delta N_*} + \frac{i}{2}\hbar$

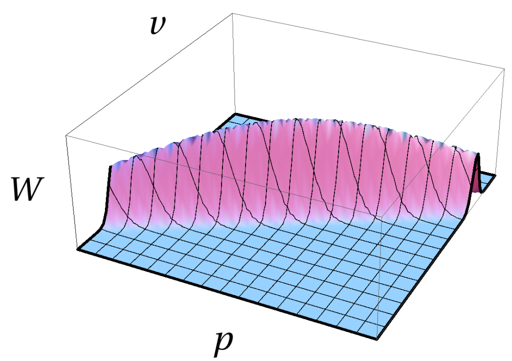
Quantum State of Cosmological Perturbations

Wigner Function $$W(v_{\boldsymbol{k}}, p_{\boldsymbol{k}}) = \int \frac{\mathrm{d}x}{2\pi^2} \Psi^*(v_{\boldsymbol{k}} - \frac{x}{2})\ \mathrm{e}^{-ip_{\boldsymbol{k}}x}\ \Psi(v_{\boldsymbol{k}} + \frac{x}{2})$$

- Evolution Equation $$\frac{\partial}{\partial t} W(v,p,t) = -\{W(v,p,t)\ ,\ H(v,p,t)\}_{\mathrm{Poisson\ \ Bracket}}$$

for quadratic Hamiltonian

- Quantum Mean Value and Stochastic Average



$$\left\langle \hat{\mathcal{O}}(\hat{v},\hat{p}) \right\rangle_{\mathrm{quant}} \simeq \int W(v,p)\,\mathcal{O}(v,p)\,\mathrm{d}v\,\mathrm{d}p$$

in the high squeezing limit

Stochastic distribution of classical processes

Stochastic Formalism

The physical scales probed in the CMB are **super-Hubble** at the end of inflation

$$\hat{\phi}(x) = \hat{\phi}_{\rm cg} + \int \frac{\mathrm{d}k}{(2\pi)^{3/2}} W\left(\frac{k}{\sigma a H}\right)\left[\phi_k(t)\, e^{-ikx}\hat{a}_k + \mathrm{h.c.}\right]$$

Upshot: derive a (stochastic and classical) **effective theory** for the coarse-grained part of the field, integrating out the small wavelength modes.

At the level of the action, this can be done using the **Schwinger-Keldysh formalism**

Morikawa, 1990
Hu & Sinha, 1995
Matarrese, Musso & Riotto, 2003

Stochastic Formalism

Heuristically, this can be done at the level of the **equation of motion**

Starobinsky, 1984, 1986, see also 1982
Rey, 1987
Goncharov, Linde & Mukhanov, 1987
Nakao, Nambu & Sasaki, 1988

Let us insert the decomposition

$$\hat{\phi}(x) = \hat{\phi}_{\mathrm{cg}} + \int \frac{\mathrm{d}k}{(2\pi)^{3/2}} W\left(\frac{k}{\sigma a H}\right)\left[\phi_k(t)\, e^{-ikx}\hat{a}_k + \mathrm{h.c.}\right]$$

In the Klein-Gordon equation of motion

$$\ddot{\hat{\phi}} + 3H\dot{\hat{\phi}} + V'\left(\hat{\phi}\right) = 0$$ and expand in $\phi - \phi_{\mathrm{cg}}$

Stochastic Formalism

At leading order in slow roll:

$$\dot{\hat{\phi}}_{\mathrm{cg}} + \frac{V'\left(\hat{\phi}_{\mathrm{cg}}\right)}{3H^2} = \hat{\xi}_1$$

with

$$\hat{\xi}_1 = -\int \frac{\mathrm{d}k}{(2\pi)^{3/2}} \frac{\partial}{\partial t}\left[W\left(\frac{k}{\sigma a H}\right)\right]\left[\phi_k\left(t\right) e^{-ikx}\hat{a}_k + \mathrm{h.c.}\right]$$

Modes smaller than the coarse-graining scale are constantly escaping the Hubble radius and **source the coarse-grained sector**.

Stochastic Formalism

Large Squeezing Approximation: $\hat{\xi}_1 \to \xi_1$

quantum operator ($\hat{\xi}_1$) — **stochastic variable** (ξ_1)

ξ_1 is a Gaussian stochastic variable with two-point correlation

$$\langle \xi_1(x,t)\xi_1(x',t')\rangle \equiv \left\langle \hat{\xi}_1(x,t)\hat{\xi}_1(x',t')\right\rangle_{\rm qu}$$

$$= \frac{\sin\left(\sigma a H|x-x'|\right)}{\sigma a H|x-x'|}\frac{\sigma^3 H^5}{2\pi^2 a^3}\left|\phi_k\right|^2_{k=\sigma a H}\delta\left(t-t'\right)$$

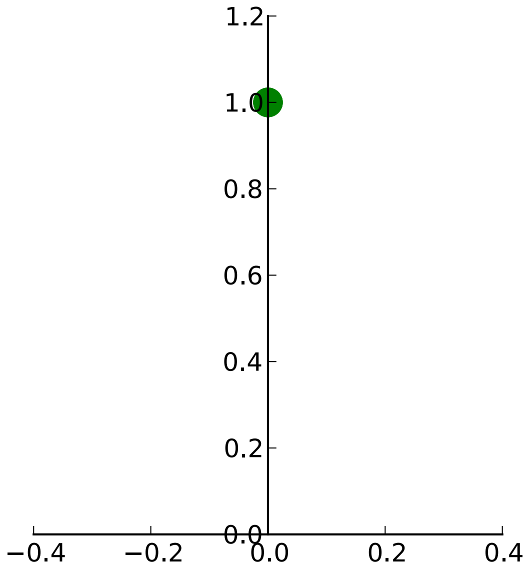
→ 1 if x and x' are in the same Hubble patch
→ 0 if x and x' are in different Hubble patches

$(H/2\pi)^2$ in de Sitter

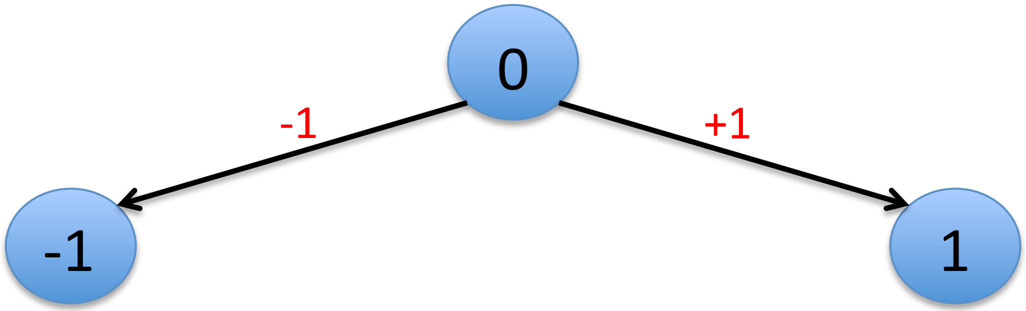
for a step window function (Markovian)

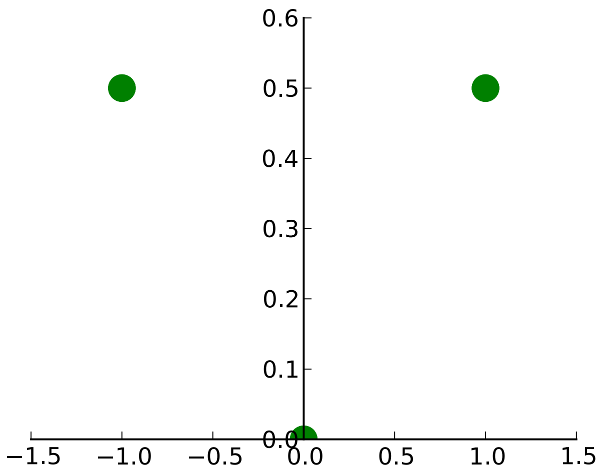
$$\frac{\partial \phi_{\rm cg}}{\partial N}+\frac{V'}{3H^2}=\frac{H}{2\pi}\xi \qquad \text{with} \qquad \langle \xi(N)\xi(N')\rangle = \delta(N-N')$$

Separate Universe Picture

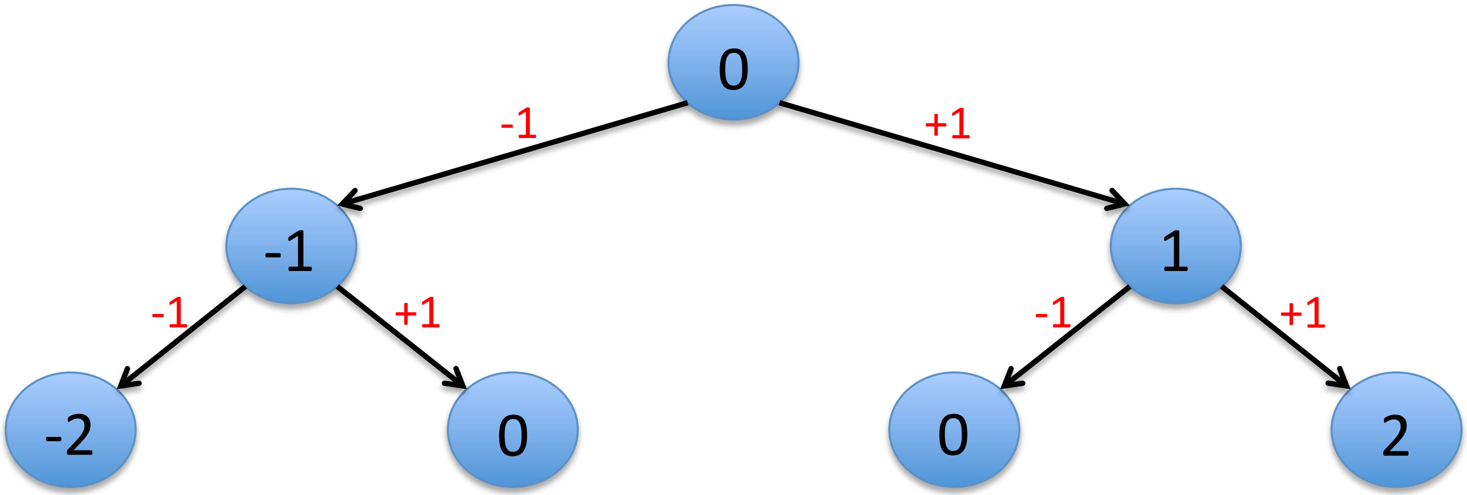


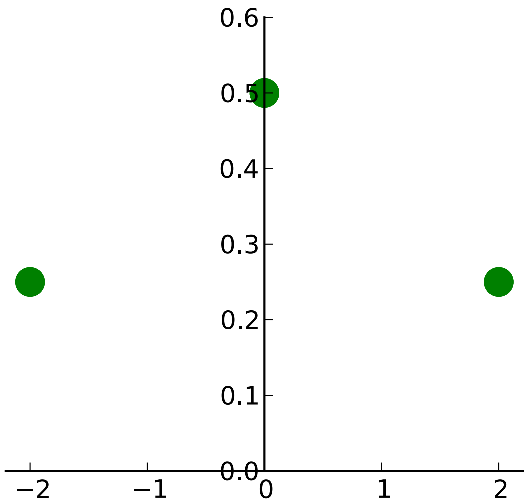
Separate Universe Picture



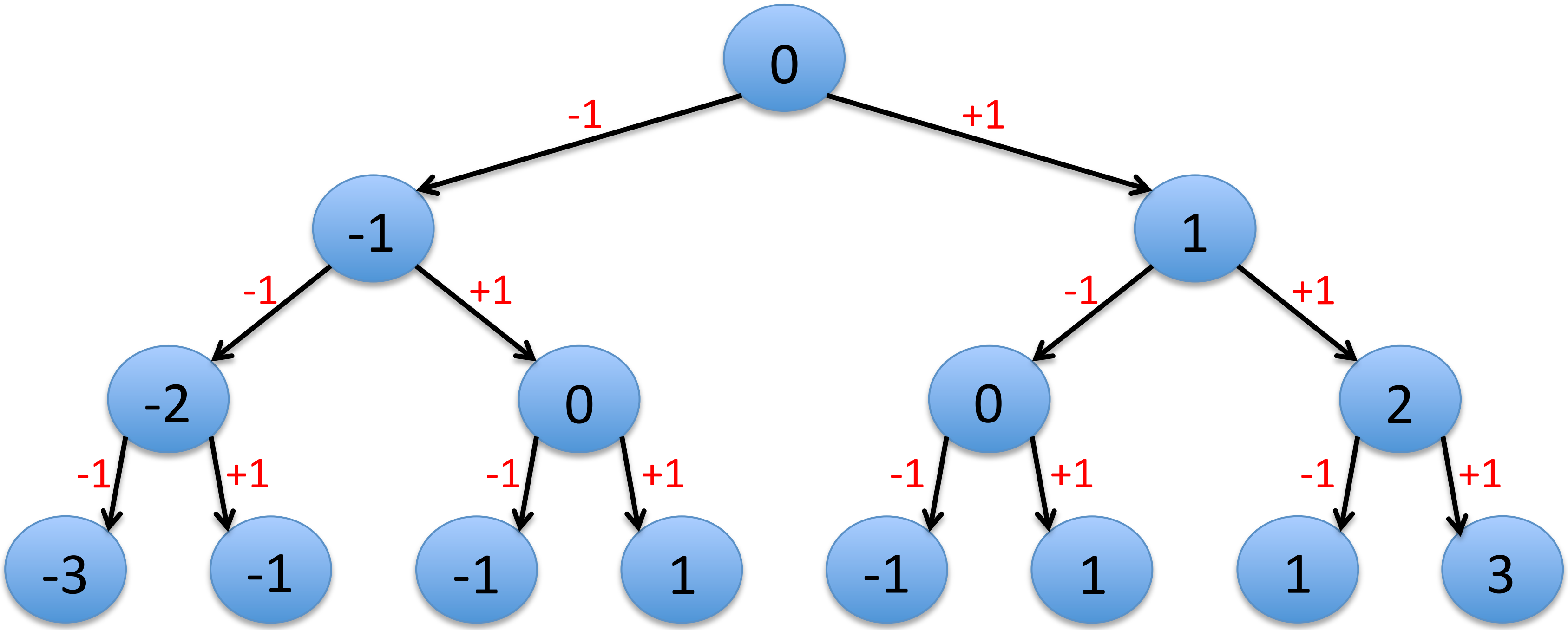


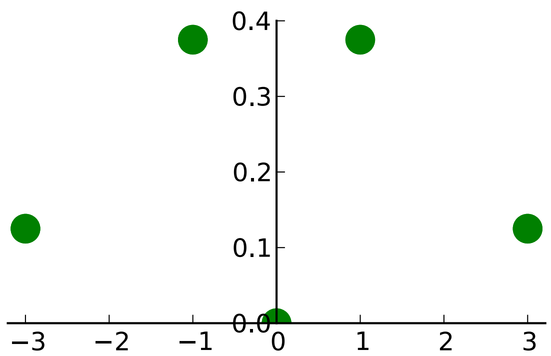
Separate Universe Picture



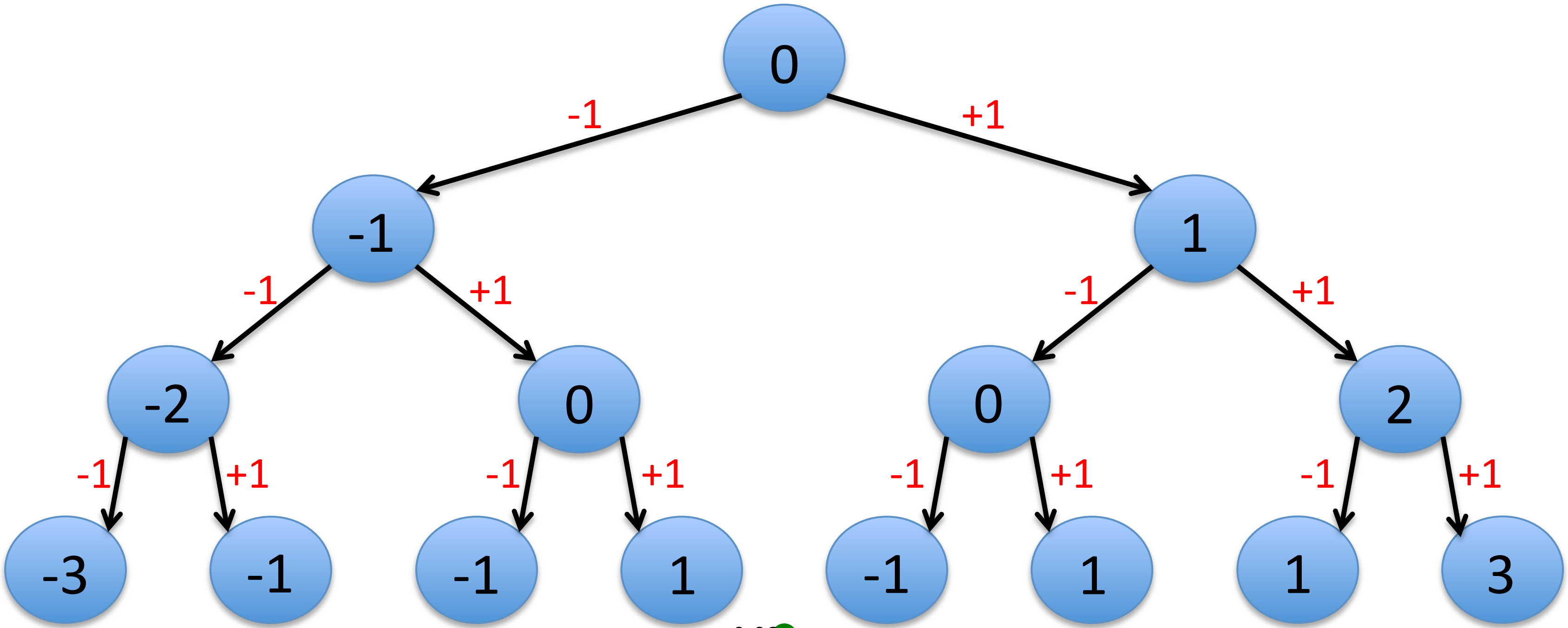


Separate Universe Picture

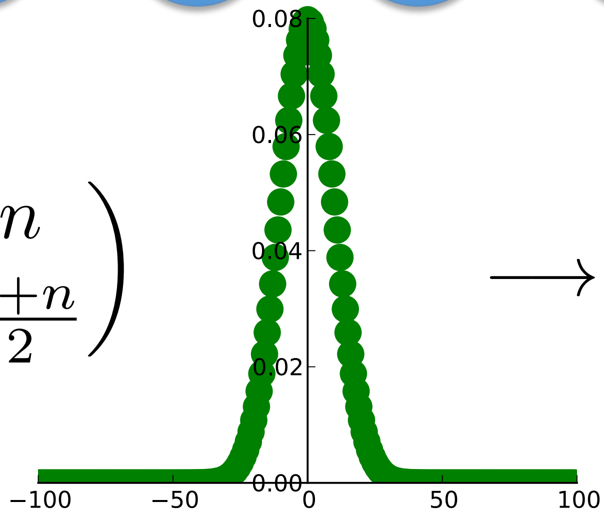




Separate Universe Picture



$$p_n(k) = 2^{-n} \binom{n}{\frac{k+n}{2}} \longrightarrow \sqrt{\frac{2}{n\pi}} \exp\left(-\frac{k^2}{2n}\right)$$



Stochastic Inflation

$$\frac{\partial\phi}{\partial N} + \frac{V'}{3H^2} = \frac{H}{2\pi}\xi$$

What Physics does it model?

The **quantum correction** to the super-horizon dynamics
sourced by the sub-horizon modes, collected in an effective noise term

What observational effect does this have?

- During one efold, $\Delta\phi_{cl} \approx V'/3H^2$ and $\Delta\phi_{qu} \approx H/2\pi$

$$\frac{\Delta\phi_{\rm qu}}{\Delta\phi_{\rm cl}} \simeq \frac{3H^3}{2\pi V'} = \sqrt{\mathcal{P}_\zeta} \sim 10^{-4} \longrightarrow$$

Small effect in
the observational window?

- **Shifts** the location of the **observational window**

Stochastic Inflation

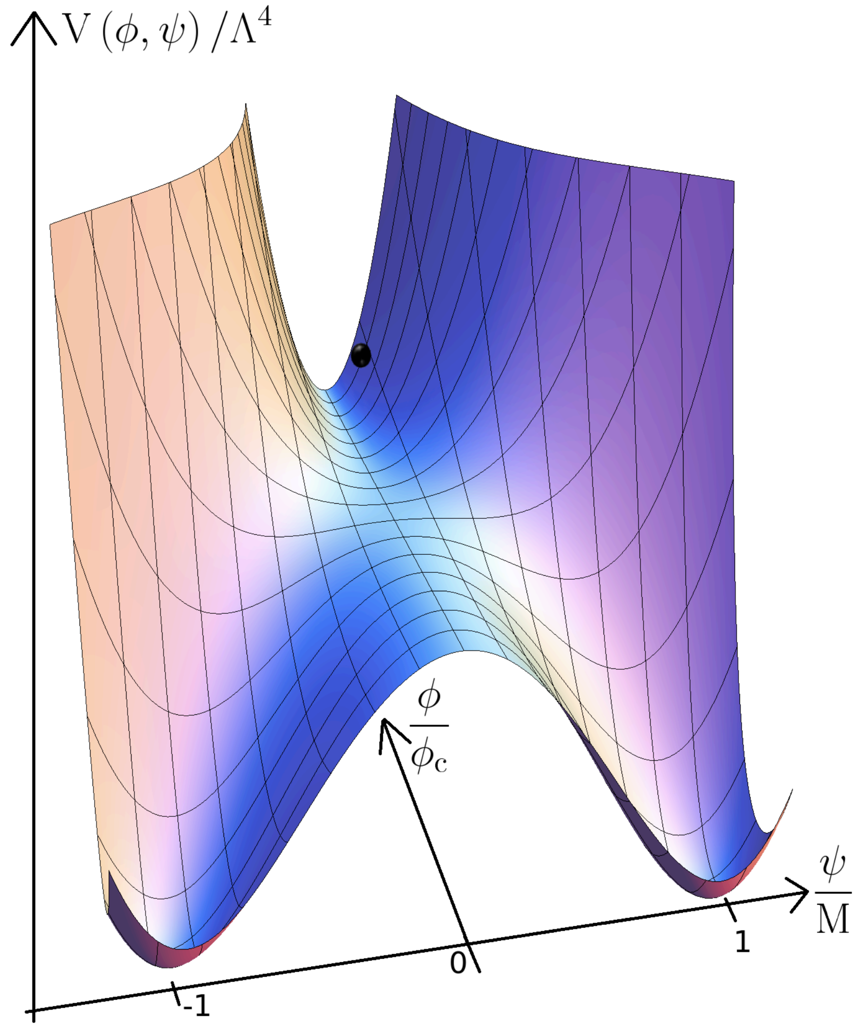
Hybrid Inflation

$$V(\phi,\chi) = \Lambda^4\left[\left(1-\frac{\psi^2}{M^2}\right)^2+\frac{\phi^2}{\mu^2}+2\frac{\phi^2\psi^2}{\phi_c^2 M^2}\right]$$

Linde, 1994
Copeland, Liddle, Lyth, Stewart, Wands, 1994

$$\frac{d\phi}{dN} = -\frac{2\Lambda^4\phi}{3H^2\mu^2}\left(1+\frac{2\psi^2\mu^2}{\phi_c^2 M^2}\right)$$

$$\frac{d\psi}{dN} = -\frac{4\Lambda^4}{3H^2M^2}\psi\left(\frac{\phi^2-\phi_c^2}{\phi_c^2}+\frac{\psi^2}{M^2}\right)$$



Stochastic Inflation

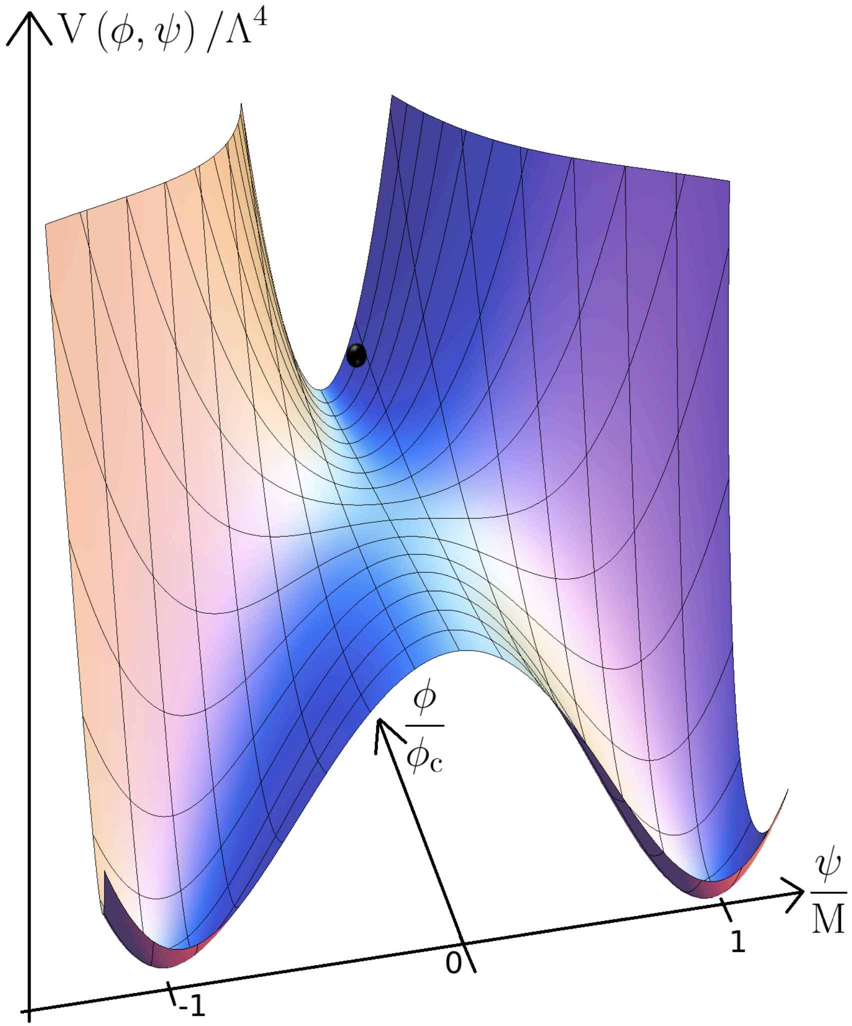
Hybrid Inflation

$$V(\phi,\chi) = \Lambda^4 \left[\left(1 - \frac{\psi^2}{M^2}\right)^2 + \frac{\phi^2}{\mu^2} + 2\frac{\phi^2\psi^2}{\phi_c^2 M^2} \right]$$

Linde, 1994
Copeland, Liddle, Lyth, Stewart, Wands, 1994

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{2\Lambda^4\phi}{3H^2\mu^2}\left(1 + \frac{2\psi^2\mu^2}{\phi_c^2 M^2}\right) + \frac{H}{2\pi}\xi_\phi(N),$$

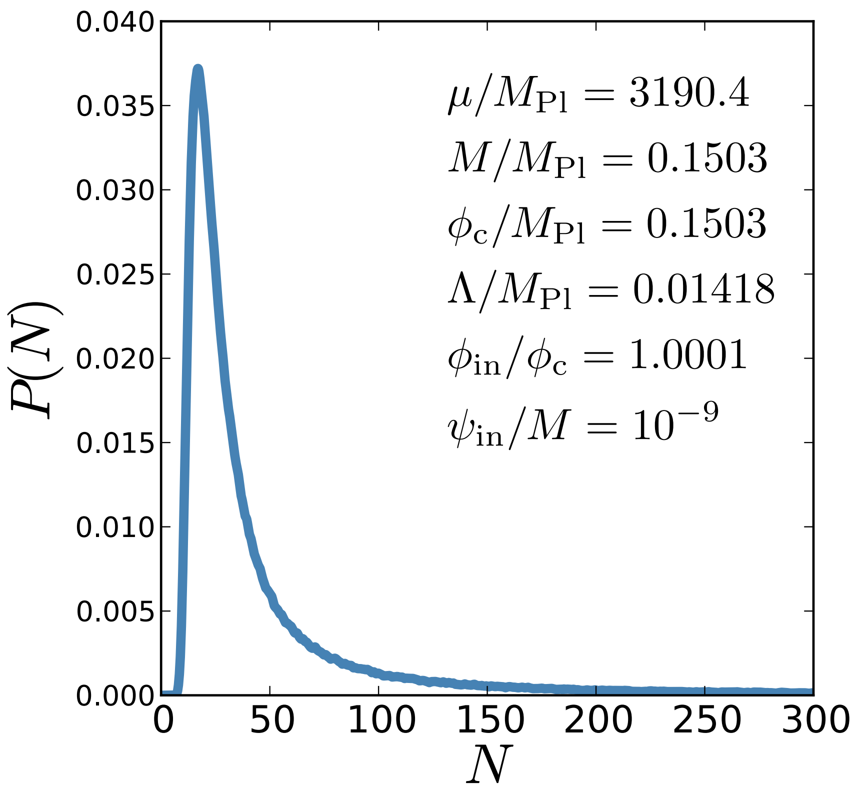
$$\frac{\mathrm{d}\psi}{\mathrm{d}N} = -\frac{4\Lambda^4}{3H^2M^2}\psi\left(\frac{\phi^2-\phi_c^2}{\phi_c^2} + \frac{\psi^2}{M^2}\right) + \frac{H}{2\pi}\xi_\psi(N).$$



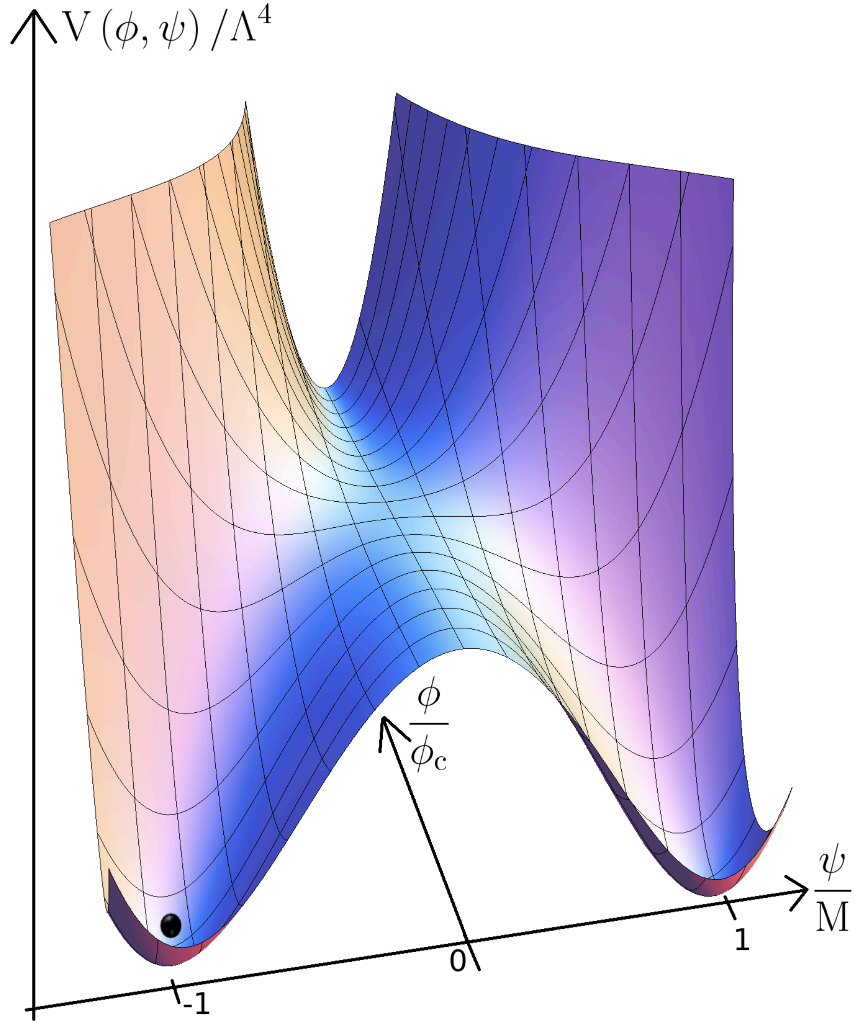
Stochastic Inflation

Hybrid Inflation

$$V(\phi,\chi) = \Lambda^4\left[\left(1-\frac{\psi^2}{M^2}\right)^2 + \frac{\phi^2}{\mu^2} + 2\frac{\phi^2\psi^2}{\phi_c^2 M^2}\right]$$



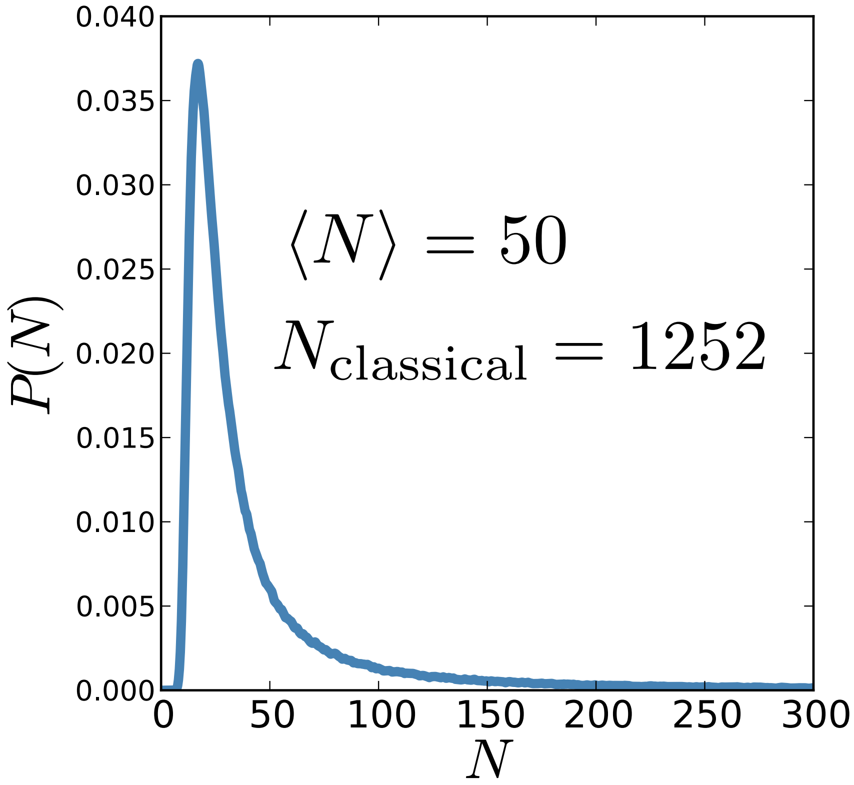
J.Martin & VV, 2011

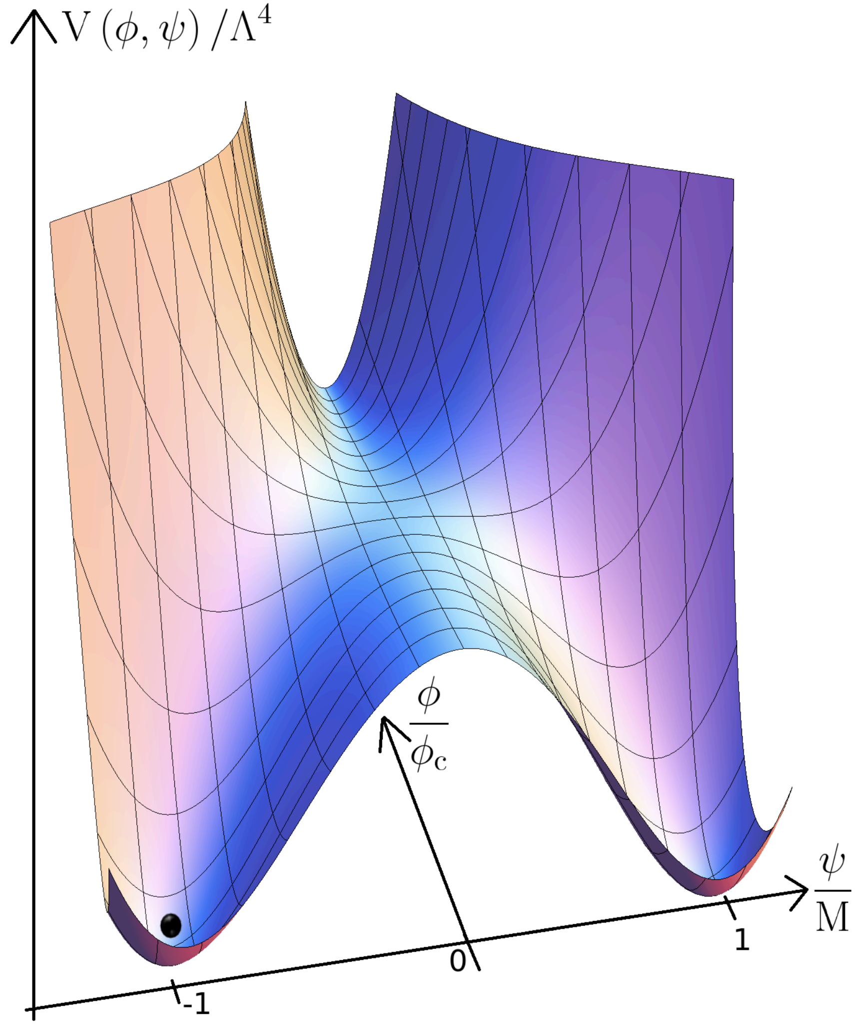


Stochastic Inflation

Hybrid Inflation

$$V(\phi,\chi) = \Lambda^4\left[\left(1-\frac{\psi^2}{M^2}\right)^2 + \frac{\phi^2}{\mu^2} + 2\frac{\phi^2\psi^2}{\phi_c^2 M^2}\right]$$





J.Martin & VV, 2011

Stochastic Inflation

$$\frac{\partial \phi}{\partial N} + \frac{V'}{3H^2} = \frac{H}{2\pi}\xi$$

What Physics does it model?

The **quantum correction** to the super-horizon dynamics
sourced by the sub-horizon modes, collected in an effective noise term

What observational effect does this have?

- During one efold, $\Delta\phi_{cl} \approx V'/3H^2$ and $\Delta\phi_{qu} \approx H/2\pi$

$$\frac{\Delta\phi_{\rm qu}}{\Delta\phi_{\rm cl}} \simeq \frac{3H^3}{2\pi V'} = \sqrt{\mathcal{P}_\zeta} \sim 10^{-4} \longrightarrow$$

Small effect in
the observational window?

- **Shifts** the location of the **observational window**

- Concretely, what features does it add to the standard predictions?

↳ **How do we calculate correlation functions of cosmological perturbations in stochastic inflation?**

Correlation Functions in Stochastic Inflation

Test Fields

- Scalar field on inflationary background: Starobinsky, Yokoyama, 1994
  Finelli, Marozzi, Starobinsky, Vacca, Venturi, 2008 & 2010
  Garbrecht, Rigopoulos and Zhu, 2013
- Purely Gravitational Systems: Tsamis, Woodard, 2005
- Scalar electrodynamics: Prokopec, Tsamis, Woodard, 2007 &2008

→ Standard QFT results recovered for $\langle\phi^2\rangle$

Perturbative Expansion

Martin, Musso, 2005
Kunze, 2006
Finelli, Marozzi, Starobinsky, Vacca, Venturi, 2008

$$\phi = \phi_{\rm cl} + \delta\phi$$

$$\langle \delta\phi^2 \rangle = \int^{\sigma a H} \mathcal{P}_{\delta\phi}(k) \mathrm{d}\log k$$

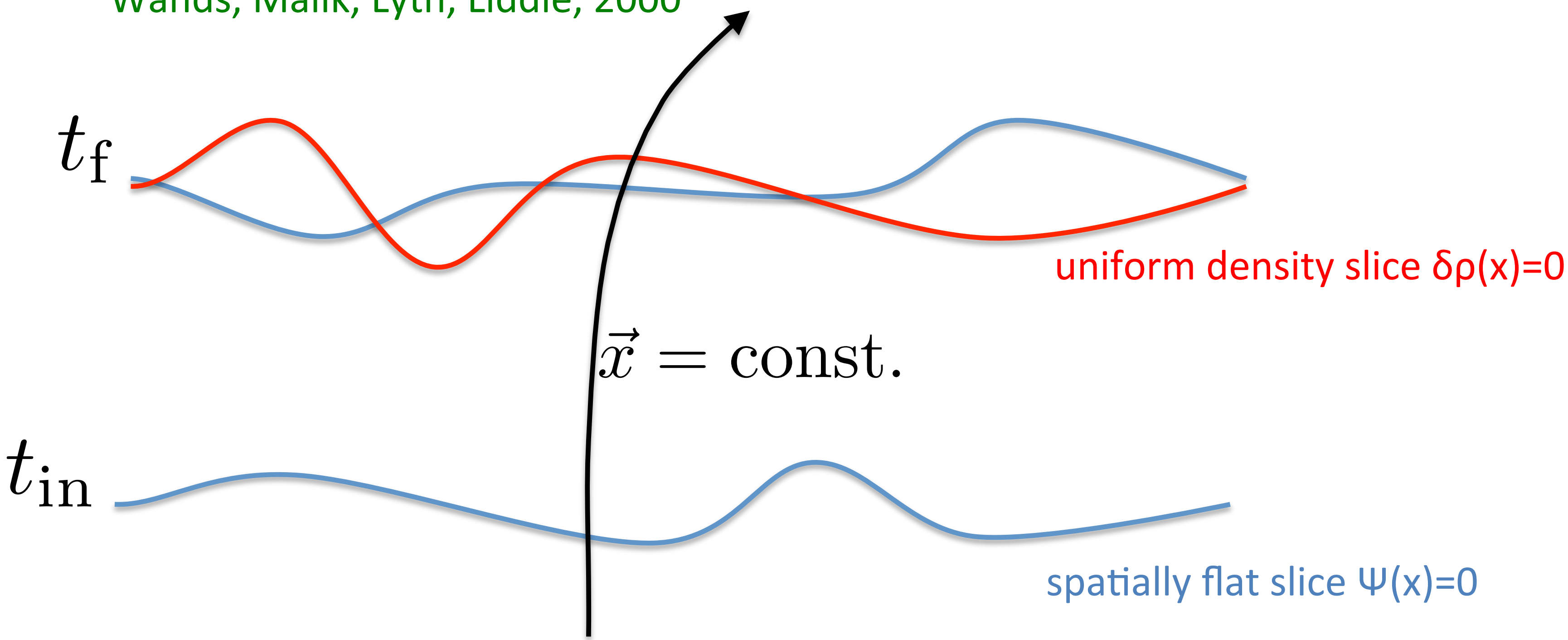
→ Standard result recovered at leading order for $\mathcal{P}_\zeta$

Replica Field Theory Kuhnel and Schwarz, 2008 (test scalar field in de-Sitter)

Stochastic- δ*N* formalism Enqvist, Nurmi, Podolsky, Rigopoulos, 2008
Fujita, Kawasaki, Tada, Takesako, 2013 & 2014

The δ*N* formalism

Starobinsky, 1982 & 1985
Sasaki, Stewart, 1996
Sasaki, Tanaka, 1998
Wands, Malik, Lyth, Liddle, 2000

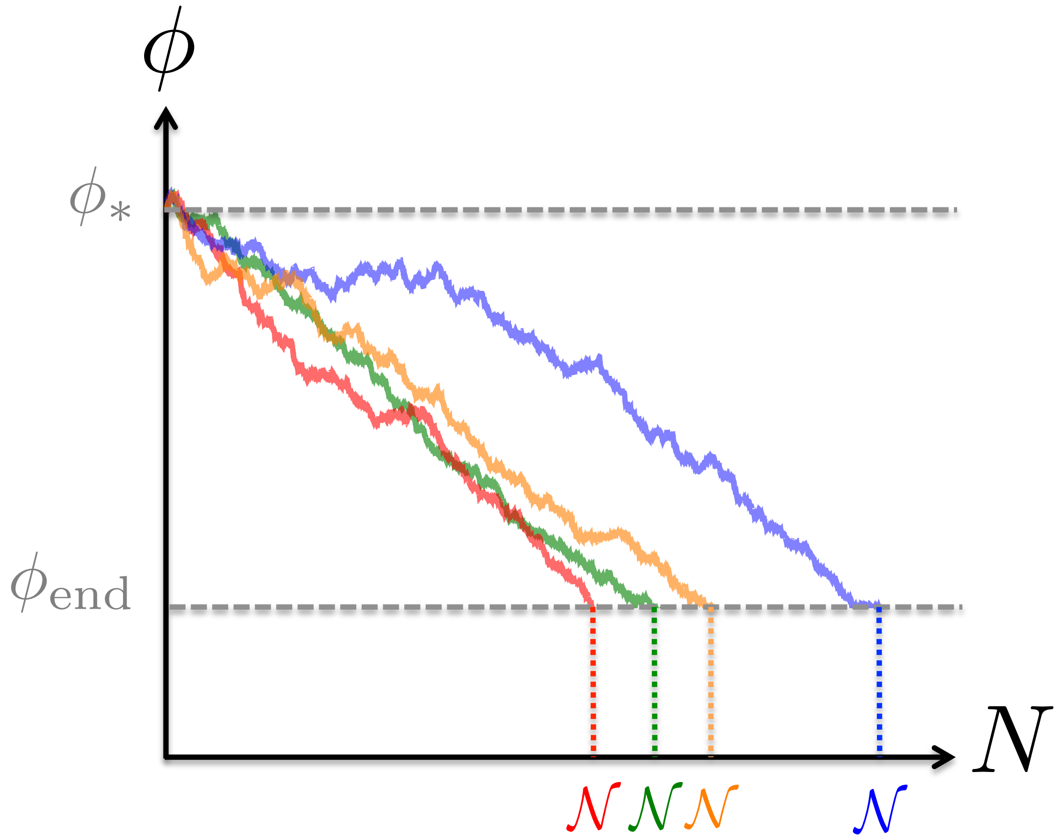


On large scales, the **curvature perturbation** on the uniform density surface is equal to the **perturbation in the number of e-folds** between the uniform density surface and the initial flat slice

$$\zeta(t,x) = N\,(t,x) - N_0(t) \equiv \delta N$$

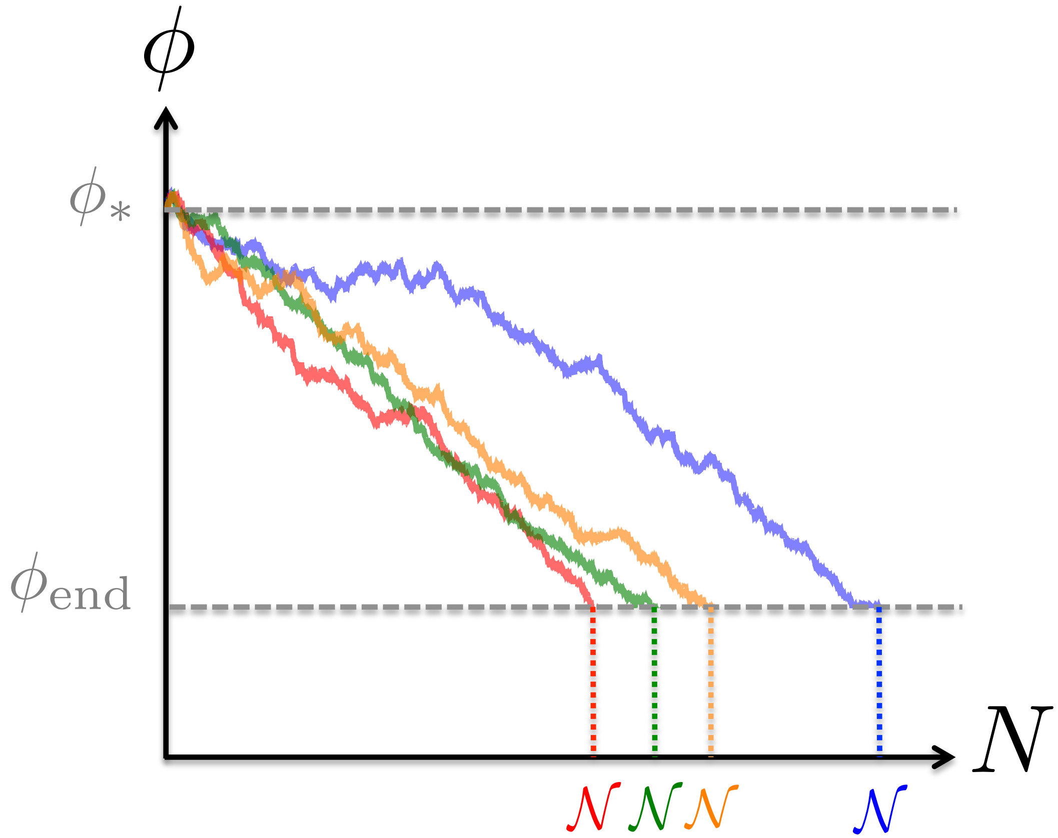
The stochastic-δ*N* formalism

- Location of the observational window: $k \longrightarrow \phi_*(k)$
- Number of e-folds: $\mathcal{N}(\phi_*)$



The stochastic-δN formalism

- Location of the observational window: $k \longrightarrow \phi_*(k)$
- Number of e-folds: $\mathcal{N}\left(\phi_*\right) \longrightarrow \delta\mathcal{N}^2 \equiv \left\langle\mathcal{N}^2\right\rangle - \left\langle\mathcal{N}\right\rangle^2$



The stochastic-δ*N* formalism

- Location of the observational window: $k \longrightarrow \phi_*(k)$
- Number of e-folds: $\mathcal{N}\left(\phi_*\right) \longrightarrow \delta\mathcal{N}^2 \equiv \left\langle\mathcal{N}^2\right\rangle - \left\langle\mathcal{N}\right\rangle^2$
- Integrated Power: $\delta\mathcal{N}^2\left(k\right) = \int_k^{k_\mathrm{end}} \mathcal{P}_{\delta N}\left(k\right)\frac{\mathrm{d}k}{k}$

The stochastic-δ*N* formalism

- Location of the observational window: $k \longrightarrow \phi_*(k)$
- Number of e-folds: $\mathcal{N}\left(\phi_*\right) \longrightarrow \delta\mathcal{N}^2 \equiv \left\langle\mathcal{N}^2\right\rangle - \left\langle\mathcal{N}\right\rangle^2$
- Integrated Power: $$\delta\mathcal{N}^2\left(k\right) = \int_k^{k_\mathrm{end}} \mathcal{P}_{\delta N}\left(k\right)\frac{\mathrm{d}k}{k}$$ $$= \int_{\ln k_\mathrm{end} - \langle\mathcal{N}\rangle(1-\epsilon_{1*}+\cdots)}^{\ln k_\mathrm{end}} \mathcal{P}_{\delta N}\mathrm{d}N$$
- Scalar Power Spectrum: $$\mathcal{P}_\zeta\left(k\right) = \mathcal{P}_{\delta\mathcal{N}}\left(k\right) = \frac{\mathrm{d}\delta\mathcal{N}^2}{\mathrm{d}\left\langle\mathcal{N}\right\rangle}$$ $$= \frac{\mathrm{d}\delta\mathcal{N}^2/\mathrm{d}\phi_*}{\mathrm{d}\left\langle\mathcal{N}\right\rangle/\mathrm{d}\phi_*}$$

→ **Requires to compute** $\left\langle\mathcal{N}\right\rangle\left(\phi_*\right)$ **and** $\delta\mathcal{N}^2\left(\phi_*\right)$

First Passage Time

$$\frac{\partial \phi}{\partial N} + \frac{V'}{3H^2} = \frac{H}{2\pi}\xi \longleftrightarrow \frac{\partial}{\partial N} P(\phi, N) = \frac{\partial}{\partial \phi}\left[\frac{V'}{3H^2} P(\phi, N)\right] + \frac{\partial^2}{\partial \phi^2}\left[\frac{H^2}{8\pi^2} P(\phi, N)\right]$$

$$= -\mathcal{L}_{\mathrm{FP}} \cdot P(\phi, N)$$

Langevin equation

Fokker-Planck equation

First Passage Time: Louis Bachelier, 1900 $\quad \mathcal{L}_{\mathrm{FP}}^{\dagger} \cdot \langle \mathcal{N} \rangle (\phi_*) = 1$

$$\langle \mathcal{N} \rangle'' v - \langle \mathcal{N} \rangle' \frac{v'}{v} = -1 \quad \text{where} \quad v = V/(24\pi^2 M_{\mathrm{Pl}}^4)$$

$$\langle \mathcal{N} \rangle = \int_{\phi_{\mathrm{end}}}^{\phi_*} \frac{\mathrm{d}x}{M_{\mathrm{Pl}}} \int_x^{\bar{\phi}} \frac{\mathrm{d}y}{M_{\mathrm{Pl}}} \frac{1}{v(y)} \exp\left[\frac{1}{v(y)} - \frac{1}{v(x)}\right]$$

How do we recover the **classical result?**

First Passage Time

Saddle Point Approximation

$$\langle\mathcal{N}\rangle = \int_{\phi_{\mathrm{end}}}^{\phi_*}\frac{\mathrm{d}x}{M_{\mathrm{Pl}}}\int_x^{\bar{\phi}}\frac{\mathrm{d}y}{M_{\mathrm{Pl}}}\frac{1}{v\left(y\right)}\exp\left[\frac{1}{v\left(y\right)}-\frac{1}{v\left(x\right)}\right]$$

$$\left|2v-\frac{v''v^2}{v'^2}\right| \ll 1$$

$$\langle\mathcal{N}\rangle \simeq \int_{\phi_{\mathrm{end}}}^{\phi_*}\frac{\mathrm{d}x}{M_{\mathrm{Pl}}^2}\frac{v(x)}{v'(x)}\left[1+v\left(x\right)-\frac{v''\left(x\right)v^2\left(x\right)}{v'^2\left(x\right)}+\cdots\right]$$

Classical result

First order correction

Scalar Power Spectrum

$$\mathcal{P}_\zeta\left(\phi_*\right) = 2\left\{\int_{\phi_*}^{\bar{\phi}} \frac{\mathrm{d}x}{M_{\mathrm{Pl}}} \frac{1}{v\left(x\right)} \exp\left[\frac{1}{v\left(x\right)} - \frac{1}{v\left(\phi_*\right)}\right]\right\}^{-1} \times$$

$$\int_{\phi_*}^{\bar{\phi}} \frac{\mathrm{d}x}{M_{\mathrm{Pl}}} \left\{\int_{x}^{\bar{\phi}} \frac{\mathrm{d}y}{M_{\mathrm{Pl}}} \frac{1}{v\left(y\right)} \exp\left[\frac{1}{v\left(y\right)} - \frac{1}{v\left(x\right)}\right]\right\}^2 \exp\left[\frac{1}{v\left(x\right)} - \frac{1}{v\left(\phi_*\right)}\right]$$

Saddle Point Approximation

$$\left|2v - \frac{v'' v^2}{v'^2}\right| \ll 1$$

$$\mathcal{P}_\zeta\left(\phi_*\right) \simeq \frac{2}{M_{\mathrm{Pl}}^2} \frac{v^3\left(\phi_*\right)}{v'^2\left(\phi_*\right)} \left[1 + 5v\left(\phi_*\right) - 4\frac{v^2\left(\phi_*\right) v''\left(\phi_*\right)}{v'^2\left(\phi_*\right)} + \cdots\right]$$

Classical result

First order correction

First Passage Time

Higher Moments

$$\mathcal{L}_{\mathrm{FP}}^{\dagger} \cdot \left\langle \mathcal{N}^{p} \right\rangle (\phi_*) = p \left\langle \mathcal{N}^{p-1} \right\rangle (\phi_*)$$

$$\delta\mathcal{N}^2 = 2\int_{\phi_*}^{\phi_{\mathrm{end}}} \mathrm{d}x \int_{\phi_\infty}^{x} \mathrm{d}y \langle\mathcal{N}\rangle'^2 (y) \exp\left[\frac{1}{v(y)} - \frac{1}{v(x)}\right]$$

$$\delta\mathcal{N}^3 = 6\int_{\phi_*}^{\phi_{\mathrm{end}}} \mathrm{d}x \int_{\phi_\infty}^{x} \mathrm{d}y \left\langle\mathcal{N}\right\rangle' (y) \delta\mathcal{N}^{2\prime}(y) \exp\left[\frac{1}{v(y)} - \frac{1}{v(x)}\right]$$

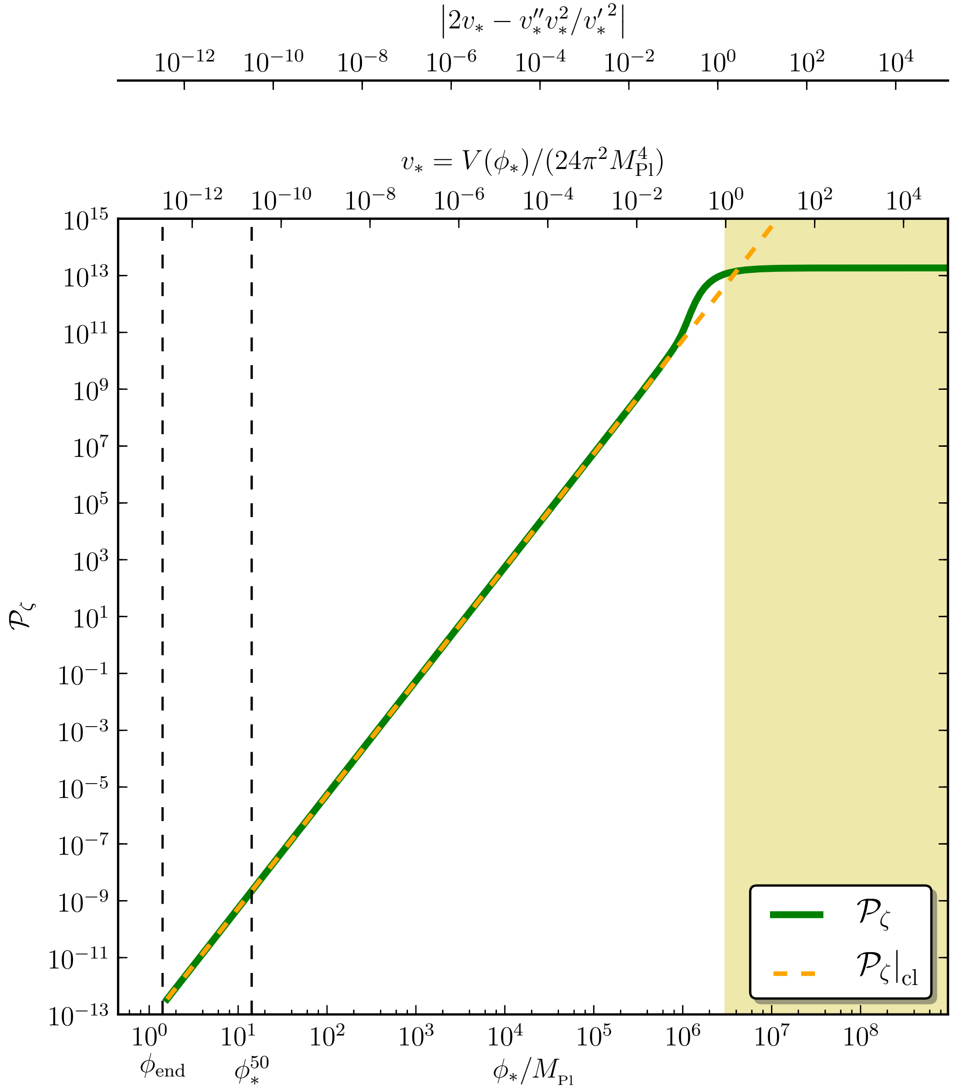
$$\cdots \quad = \quad \cdots$$

$$\delta\mathcal{N}^p(\phi_*) = \int_{\phi_*}^{\phi_{\mathrm{end}}} \mathrm{d}x \int_{\phi_\infty}^{x} \mathrm{d}y \left[2p \left\langle\mathcal{N}\right\rangle' (y)\, \delta\mathcal{N}^{p-1\prime}(y) + p(p-1) \left\langle\mathcal{N}\right\rangle'^2 (y)\, \delta\mathcal{N}^{p-2}(y)\right] \exp\left[\frac{1}{v(y)} - \frac{1}{v(x)}\right]$$

→ **Analytical** expression for **all moments !**

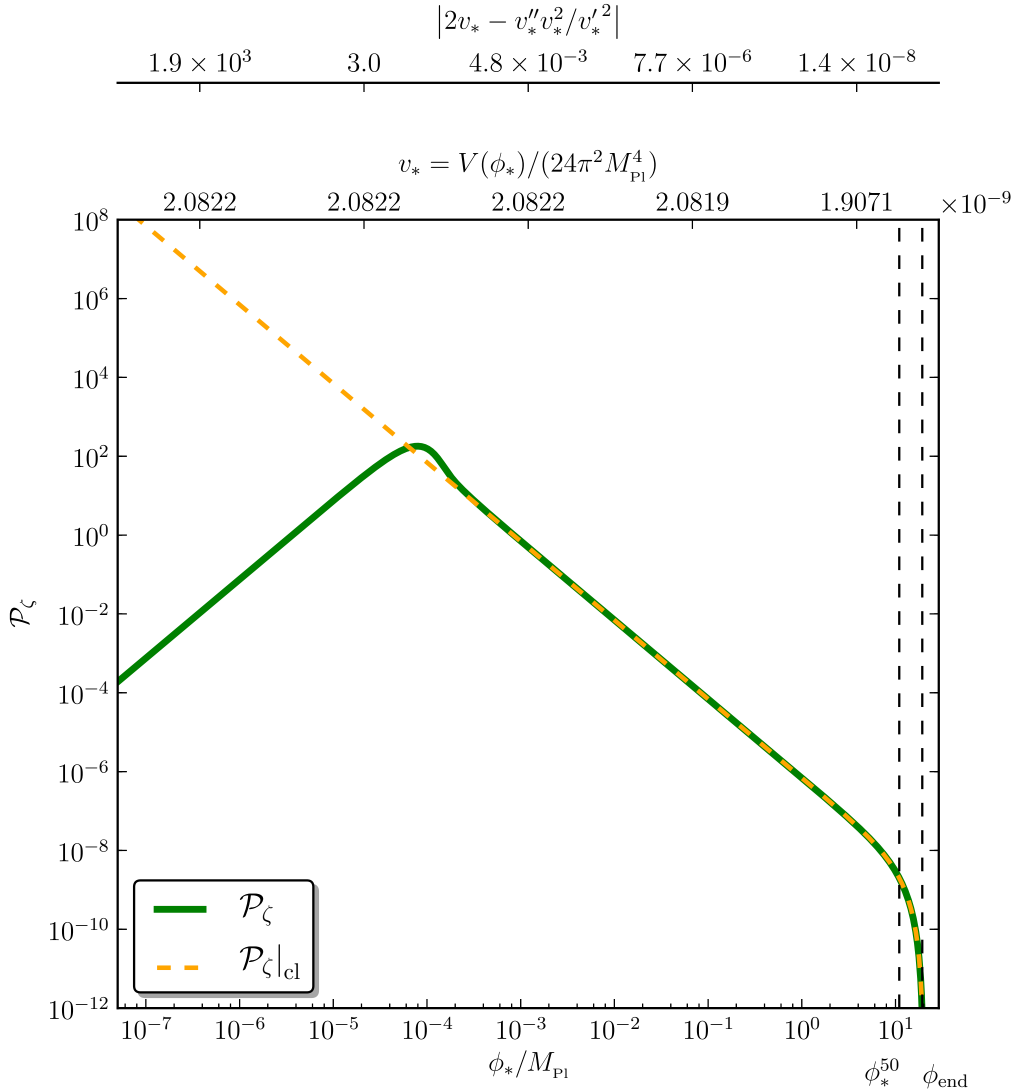
Scalar Power Spectrum

Example 1: Large Field Inflation $V = \frac{m^2}{2}\phi^2$



Scalar Power Spectrum

Example 2: Small Field Inflation $V = M^4 \left[1 - \left(\frac{\phi}{\mu}\right)^2\right]$



Conclusions

Conclusions

- Making use of the ***δN* formalism**, we have derived all the n-point correlation functions of scalar perturbations in **stochastic inflation**

Conclusions

- Making use of the ***δN* formalism**, we have derived all the n-point correlation functions of scalar perturbations in **stochastic inflation**
- The **classical** results arise as a **saddle-point** limit

Conclusions

- Making use of the ***δN* formalism**, we have derived all the n-point correlation functions of scalar perturbations in **stochastic inflation**
- The **classical** results arise as a **saddle-point** limit
- A new **classicality criterion** has been derived: $\left|2v - \frac{v'' v^2}{v'^2}\right| \ll 1$

Conclusions

- Making use of the ***δN* formalism**, we have derived all the n-point correlation functions of scalar perturbations in **stochastic inflation**

- The **classical** results arise as a **saddle-point** limit

- A new **classicality criterion** has been derived: $\left|2v - \frac{v'' v^2}{{v'}^2}\right| \ll 1$

  → **quantum gravity** effect … but **not only**!

Conclusions

- Making use of the ***δN* formalism**, we have derived all the n-point correlation functions of scalar perturbations in **stochastic inflation**

- The **classical** results arise as a **saddle-point** limit

- A new **classicality criterion** has been derived: $\left|2v - \frac{v'' v^2}{{v'}^2}\right| \ll 1$

  → **quantum gravity** effect … but **not only**!

- Can we **see the stochastic effects**? $2v - \frac{v'' v^2}{{v'}^2} = \mathcal{P}_\zeta \left(\epsilon_1 + \frac{\epsilon_2}{4}\right)$

Conclusions

- Making use of the ***δN* formalism**, we have derived all the n-point correlation functions of scalar perturbations in **stochastic inflation**

- The **classical** results arise as a **saddle-point** limit

- A new **classicality criterion** has been derived: $\left|2v - \frac{v'' v^2}{v'^2}\right| \ll 1$

  → **quantum gravity** effect … but **not only**!

- Can we **see the stochastic effects**? $2v - \frac{v'' v^2}{v'^2} = \mathcal{P}_\zeta \left(\epsilon_1 + \frac{\epsilon_2}{4}\right)$

- **Primordial Black Holes** Physics?

Conclusions

- Making use of the ***δN*** **formalism**, we have derived all the n-point correlation functions of scalar perturbations in **stochastic inflation**
- The **classical** results arise as a **saddle-point** limit
- A new **classicality criterion** has been derived: $\left|2v - \frac{v''v^2}{v'^2}\right| \ll 1$

  → **quantum gravity** effect … but **not only**!
- Can we **see the stochastic effects**? $2v - \frac{v''v^2}{v'^2} = \mathcal{P}_\zeta\left(\epsilon_1 + \frac{\epsilon_2}{4}\right)$
- **Primordial Black Holes** Physics?
- What about the **higher moments**?

$$f_{\rm NL} = \frac{5}{24}M_{\rm Pl}^2\left[6\frac{v'^2}{v^2} - 4\frac{v''}{v} + v\left(11\frac{v'^2}{v^2} - 158\frac{v''}{v} - 9\frac{v'''}{v'} + 118\frac{v''^2}{v'^2}\right) + \cdots\right]$$

Conclusions

- Making use of the ***δN* formalism**, we have derived all the n-point correlation functions of scalar perturbations in **stochastic inflation**
- The **classical** results arise as a **saddle-point** limit
- A new **classicality criterion** has been derived: $\left|2v - \frac{v'' v^2}{v'^2}\right| \ll 1$

  → **quantum gravity** effect … but **not only**!
- Can we **see the stochastic effects**? $2v - \frac{v'' v^2}{v'^2} = \mathcal{P}_\zeta \left(\epsilon_1 + \frac{\epsilon_2}{4}\right)$
- **Primordial Black Holes** Physics?
- What about the **higher moments**?

$$f_{\mathrm{NL}} = \frac{5}{24} M_{\mathrm{Pl}}^2 \left[6\frac{v'^2}{v^2} - 4\frac{v''}{v} + v\left(11\frac{v'^2}{v^2} - 158\frac{v''}{v} - 9\frac{v'''}{v'} + 118\frac{v''^2}{v'^2}\right) + \cdots \right]$$

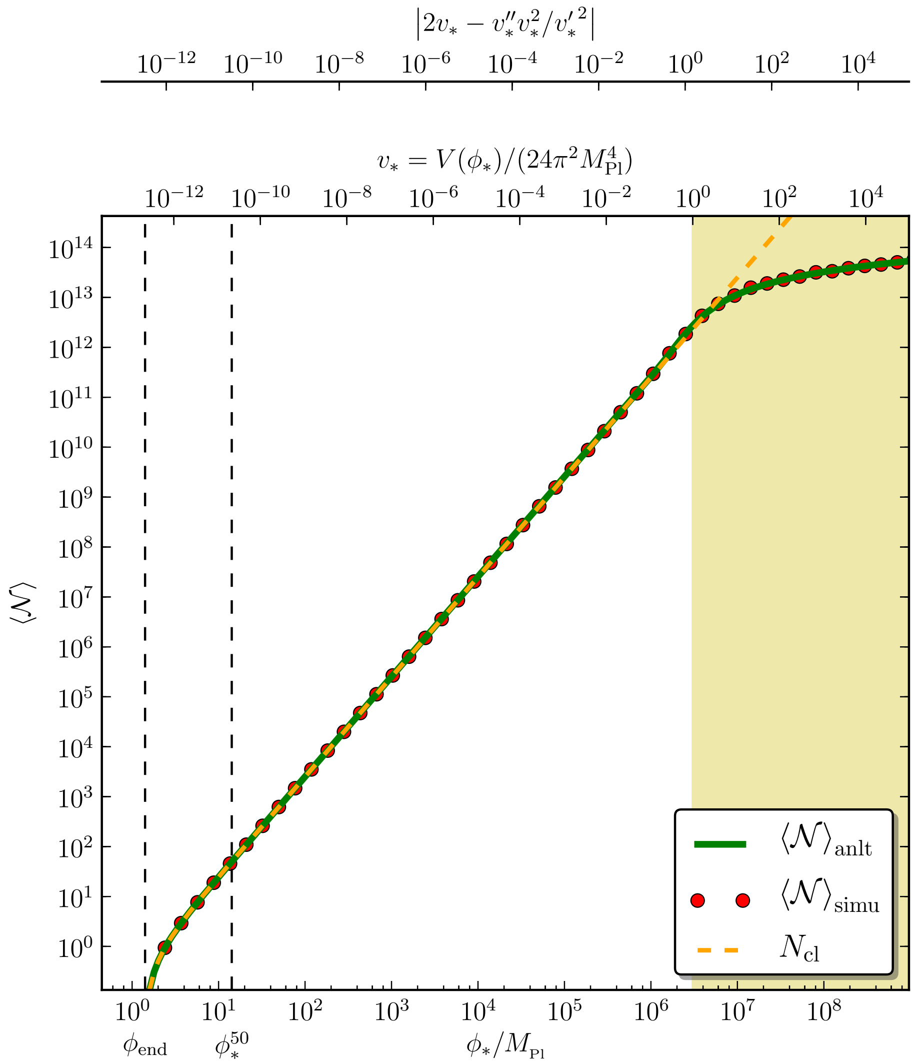
- What about **tensor perturbations**?

Thank you for your attention!

Back Up Slides

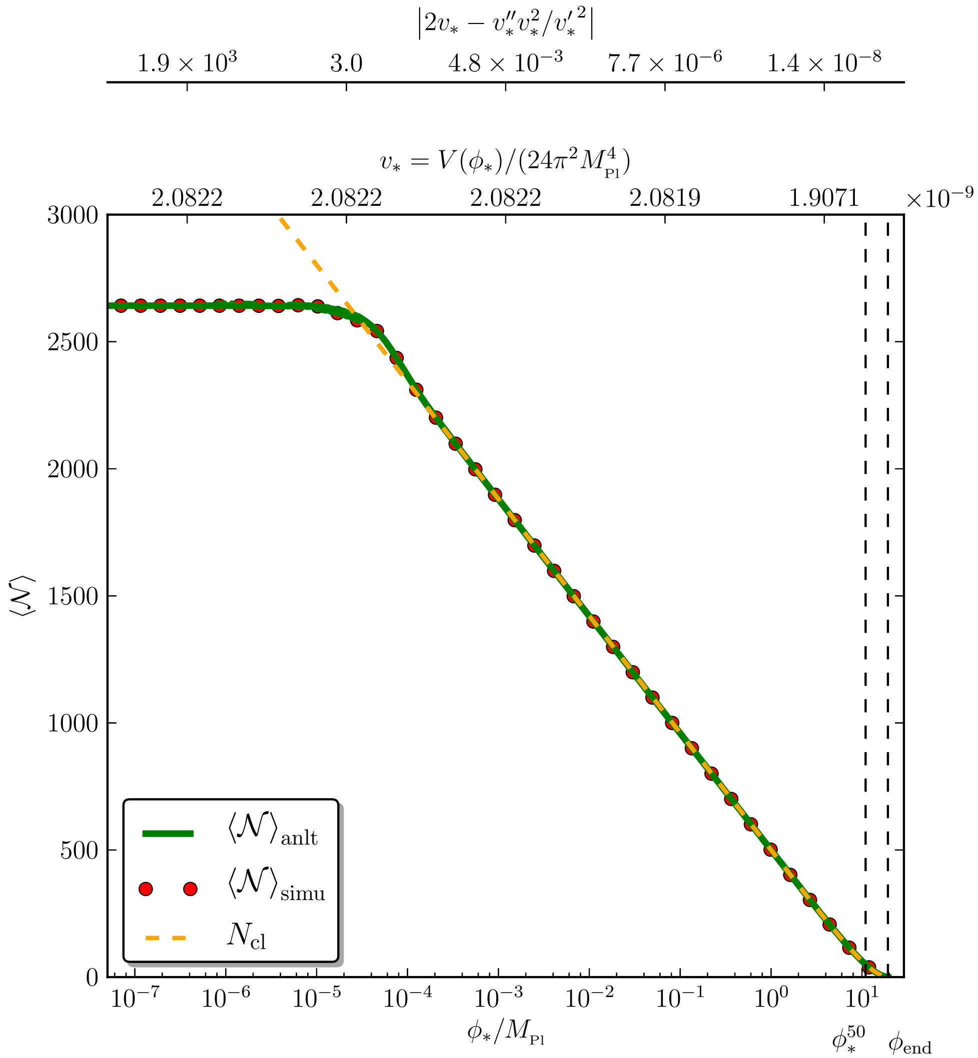
First Passage Time

Example 1: Large Field Inflation $V = \frac{m^2}{2}\phi^2$



First Passage Time

Example 2: Small Field Inflation $V = M^4\left[1-\left(\frac{\phi}{\mu}\right)^2\right]$



The δN formalism

Usual Calculation:

$$\zeta(t,x) = \delta N \simeq \frac{\partial N}{\partial \phi}\delta\phi$$

$$\frac{1}{\sqrt{2\epsilon_1}M_{\rm Pl}}$$

classical trajectory

$$\frac{H}{2\pi}\xi$$

Power Spectrum:

$$\mathcal{P}_\zeta = \frac{1}{2M_{\rm Pl}^2\epsilon_1(k)}\left[\frac{H(k)}{2\pi}\right]^2$$

(standard result)