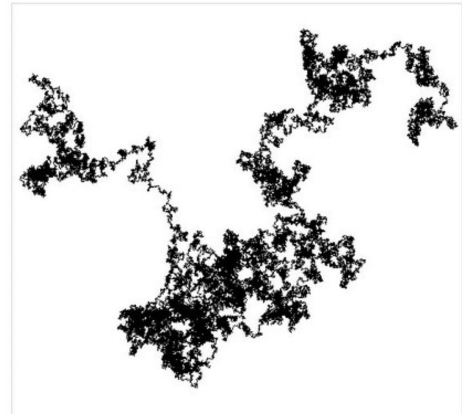


Brownian Motion in AdS/CFT

IPMU, Kashiwa, 2 Nov 2009

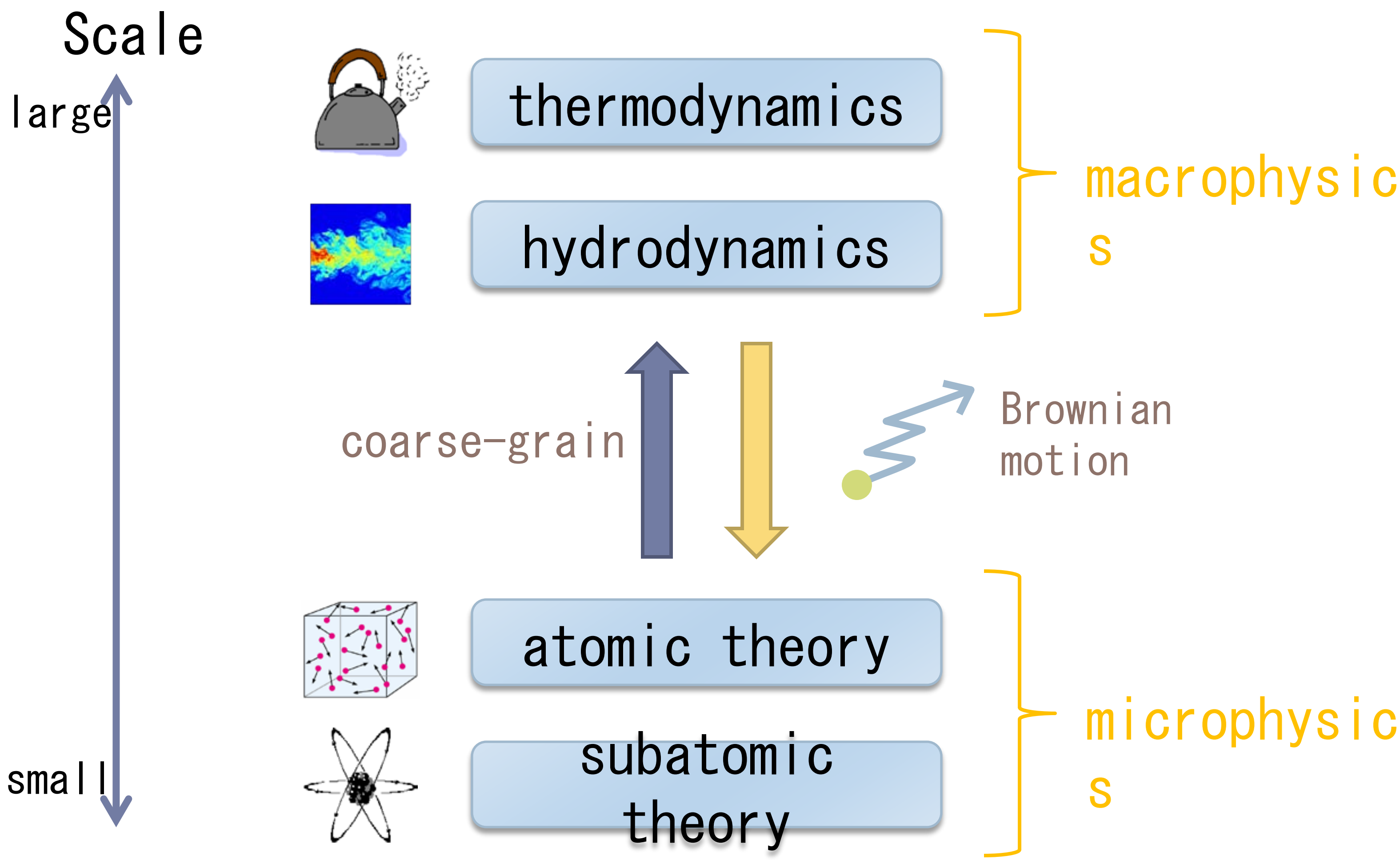
Masaki Shigemori (Amsterdam)

This talk is based on:

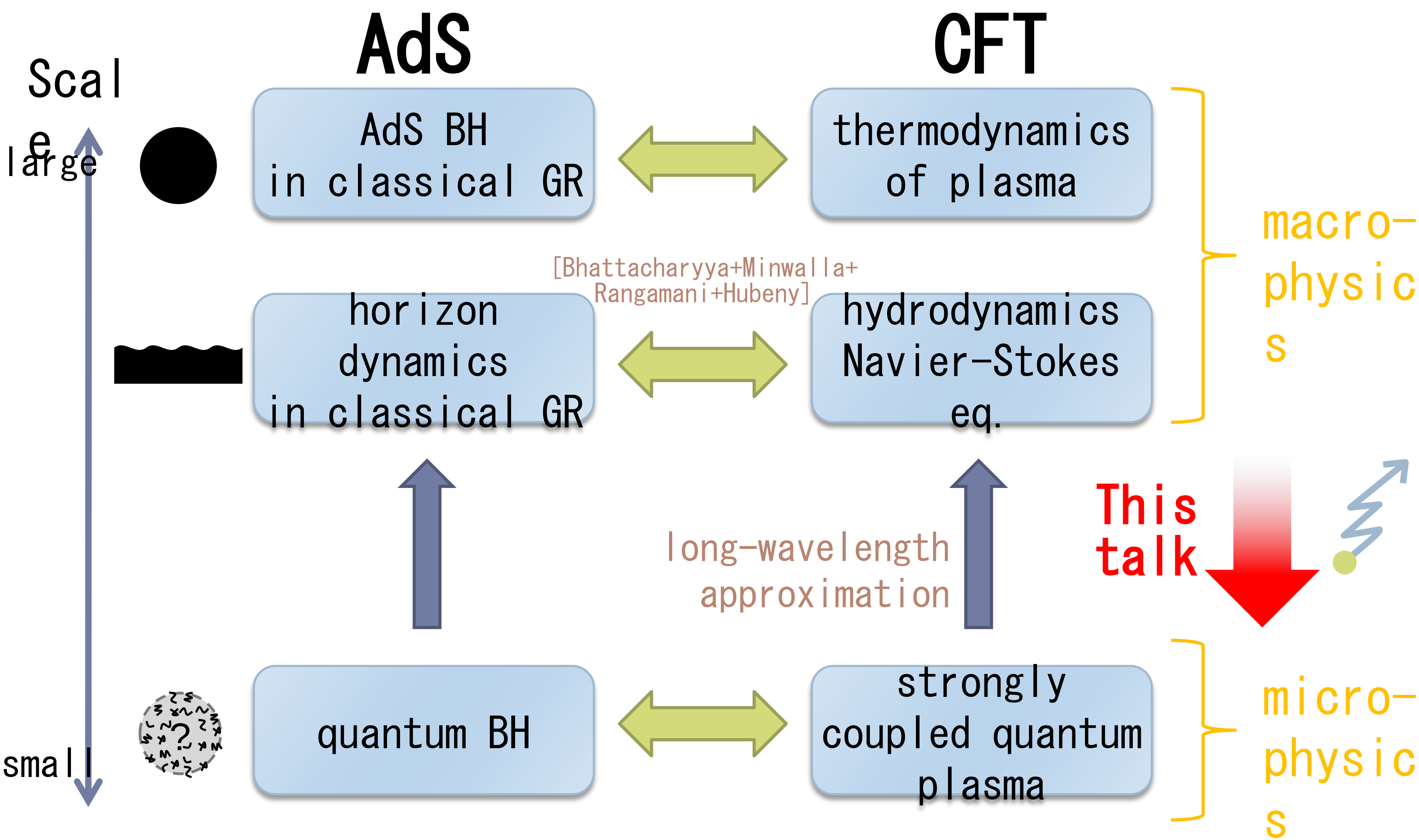
- J. de Boer, V. Hubeny, M. Rangamani, M.S., "Brownian motion in AdS/CFT," arXiv:0812.5112.
- A. Atmaja, J. de Boer, M.S., B. van Rees,
  in preparation.

Introduction / Motivation

Hierarchy of scales in physics



Hierarchy in AdS/CFT

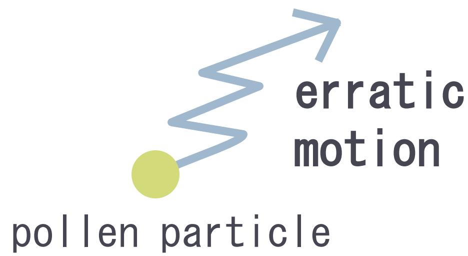


Brownian motion

— Historically, a crucial step toward microphysics of nature

Robert Brown (1773-1858)

- 1827 Brown



- Due to collisions with fluid particles
- Allowed to determine Avogadro #: $N_A = 6 \times 10^{23} < \infty$
- Ubiquitous
- Langevin eq. (friction + random force)

Brownian motion in AdS/CFT

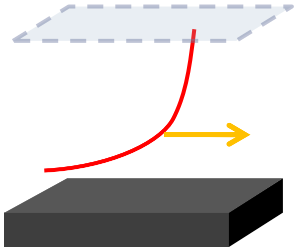
→ Do the same in AdS/CFT!

- Brownian motion of an external quark in CFT plasma
- Langevin dynamics from bulk viewpoint?
- Fluctuation-dissipation theorem
- Read off nature of constituents of strongly coupled plasma
- Relation to RHIC physics?

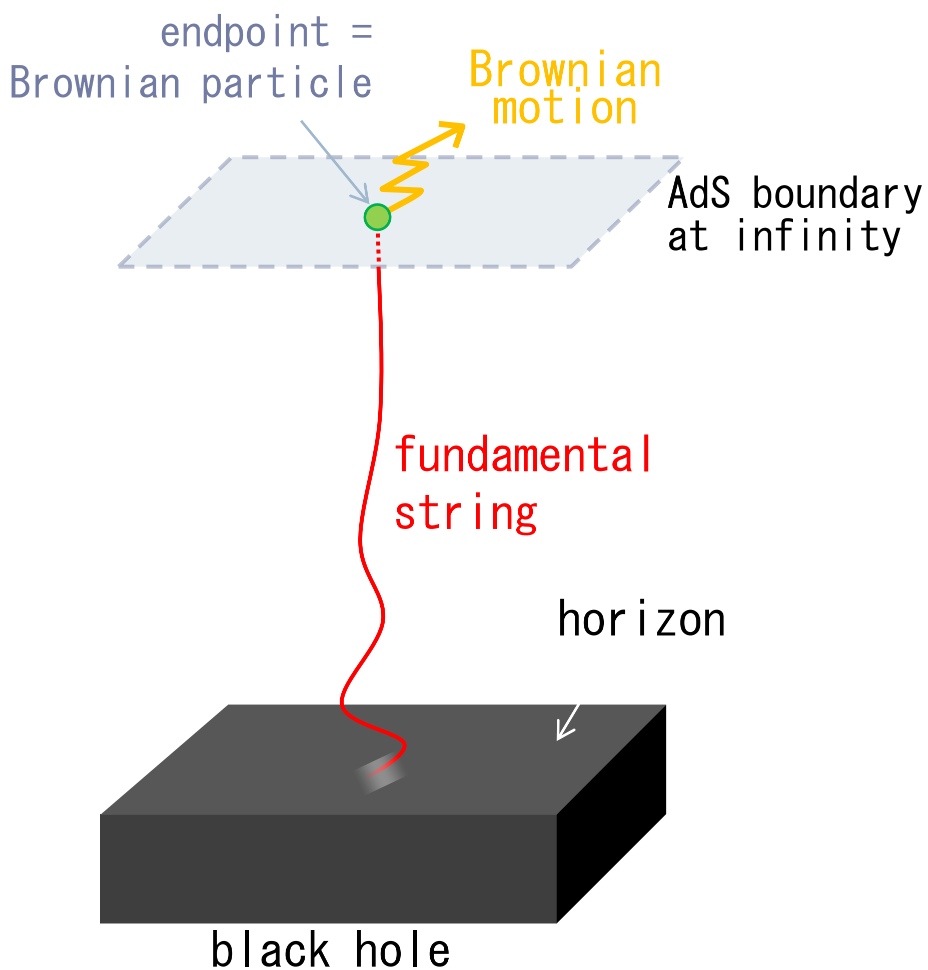
Related work:

Drag force: Herzog+Karch+Kovtun+Kozcaz+Yaffe, Gubser, Casalderrey-Solana+Teaney

Transverse momentum broadening: Gubser, Casalderrey-Solana+Teaney



Preview: BM in AdS/CFT



Outline

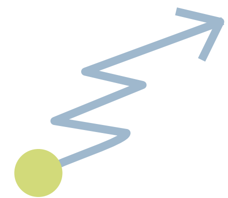
- Intro/motivation
- Boundary BM
- Bulk BM
- Time scales
- BM on stretched horizon


Paul Langevin (1872–1946)

Boundary BM

– Langevin dynamics

Simplest Langevin eq.



$x, p = m\dot{x}$

$$\dot{p}(t) = -\gamma_0 p(t) + R(t)$$

(instantaneous) friction — random force

$$\langle R(t) \rangle = 0, \qquad \langle R(t) R(t') \rangle = \kappa_0 \delta(t - t')$$

white noise

Simplest Langevin eq.

- Displacement:

$$\langle s^2(t)\rangle \equiv \langle [x(t) - x(0)]^2 \rangle$$

$$\approx \begin{cases} \dfrac{T}{m} t^2 & (t \ll t_{relax}) \\ 2Dt & (t \gg t_{relax}) \end{cases}$$

ballistic regime (init. velocity $\sim\sqrt{T/m}$)

diffusive regime (random walk)

- Diffusion constant: $D \equiv \dfrac{T}{\gamma_0 m}$ (S-E relation)
- Relaxation time: $t_{\mathrm{relax}} = \dfrac{1}{\gamma_0}$ FD theorem → $\gamma_0 = \dfrac{\kappa_0}{2mT}$

Generalized Langevin equation

$$\dot{p}(t) = -\int_{-\infty}^{t} dt' \underbrace{\gamma(t-t')p(t')}_{\text{delayed friction}} + \underbrace{R(t)}_{\text{random force}} + \underbrace{F(t)}_{\text{external force}}$$

$$\langle R(t) \rangle = 0, \qquad \langle R(t)R(t') \rangle = \kappa(t-t')$$

- Qualitatively similar to simple LE
  - ballistic regime
  - diffusive regime

Time scales

- Relaxation time $t_{\text{relax}} \equiv \frac{1}{\gamma_0}, \quad \gamma_0 = \int_0^\infty dt\, \gamma(t)$
- Collision duration time t_{coll}

$$\langle R(t)R(0)\rangle \sim e^{-t/t_{\text{coll}}}$$

→ time elapsed in a single collision

- Mean-free-path time t_{mfp} → time between collisions

Typically

$$t_{\text{relax}} \gg t_{\text{mfp}} \gg t_{\text{coll}}$$

but not necessarily so for strongly coupled plasma

How to determine γ, κ

$$p(\omega) = \frac{R(\omega) + F(\omega)}{\gamma[\omega] - i\omega} \equiv \underbrace{\mu(\omega)}_{\text{admittance}}[R(\omega) + F(\omega)]$$

1. Forced motion $F(t) = F_0 e^{-i\omega t}$

   $$\langle p(t) \rangle = \mu(\omega) F_0 e^{-i\omega t}$$ → read off μ

2. No external force $F, = 0$

   $$\underbrace{\langle pp \rangle}_{\text{measure}} = \underbrace{|\mu|^2}_{\text{known}} \underbrace{\langle RR \rangle}$$ → read off κ

Bulk BM

Bulk setup

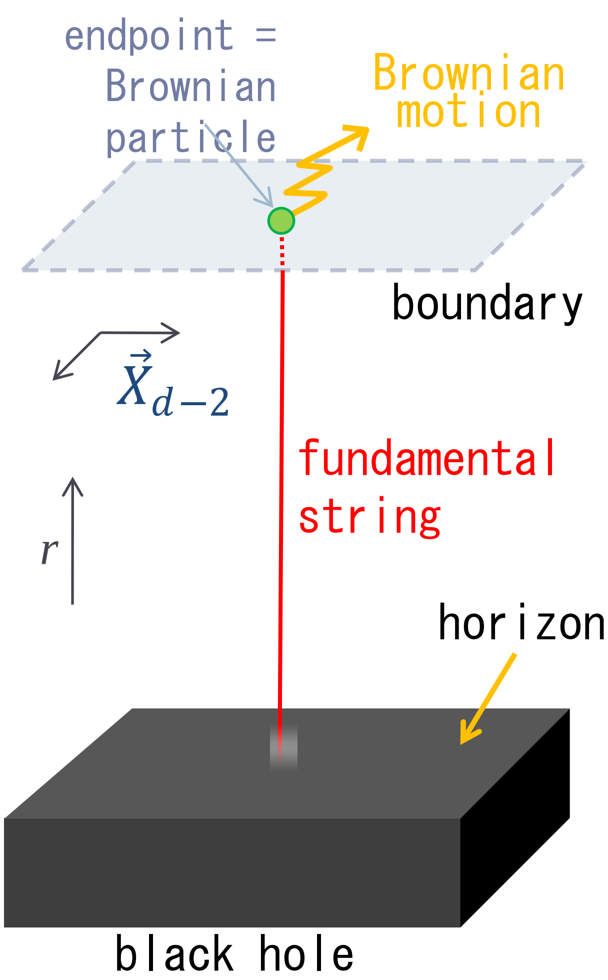
- AdS$_d$ Schwarzschild BH (planar)

$$ds_d^2 = \frac{r^2}{l^2}\left(-h(r)dt^2 + d\vec{X}_{d-2}^2\right) + \frac{l^2 dr^2}{r^2 h(r)}$$

$$h(r) = 1 - \left(\frac{r_H}{r}\right)^{d-1}$$

$$T = \frac{1}{\beta} = \frac{(d-1)r_H}{4\pi l^2}$$

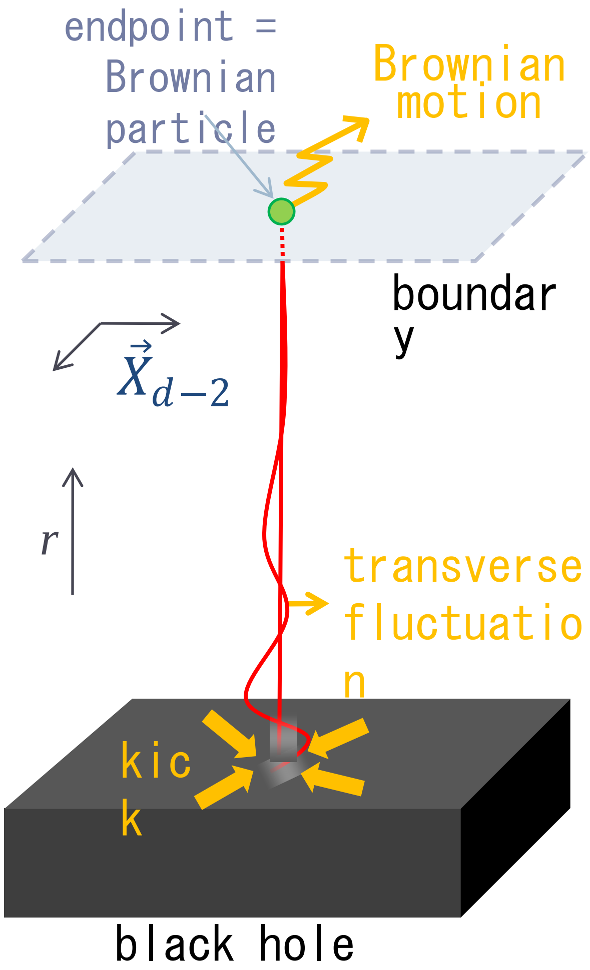
l: AdS radius



Physics of BM in AdS/CFT

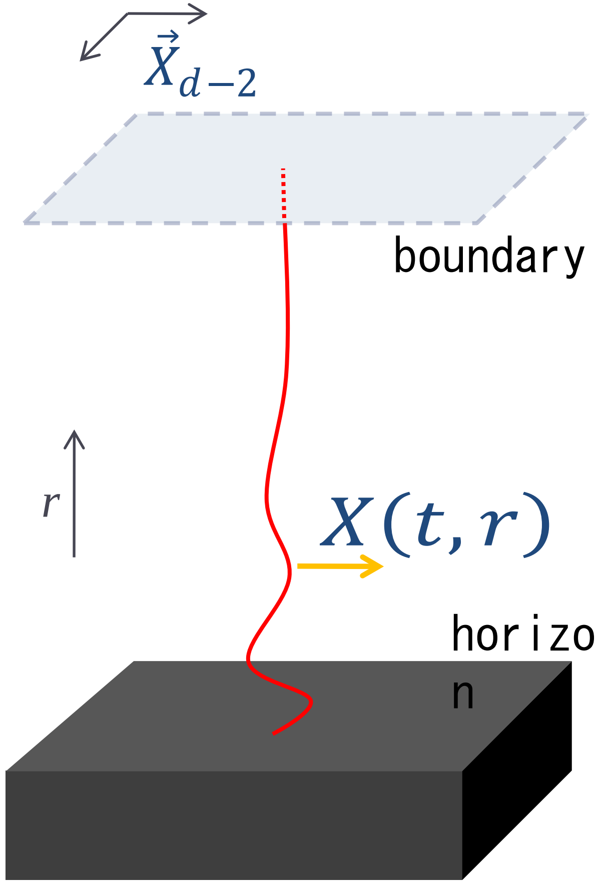
- Horizon kicks endpoint on horizon
(= Hawking radiation)
- Fluctuation propagates to AdS boundary
- Endpoint on boundary (= Brownian particle) exhibits BM

Whole process is dual to quark hit by plasma particles



Assumptions

- Probe approximation
- Small g_s
  - No interaction with bulk
  - Only interaction is at horizon
- Small fluctuation
  - Expand Nambu-Goto action to quadratic order
  - Transverse positions are similar to Klein-Gordon scalar



Transverse fluctuations

- Quadratic action

$$S_{\mathrm{NG}} = \mathrm{const} + S_2 + S_4 + \cdots$$

$$S_2 = -\frac{1}{4\pi\alpha'}\int dt\, dr \left[\frac{\dot{X}^2}{h(r)} - \frac{r^4 h(r)}{l^4} X'^2\right]$$

- Mode expansion

$$X(t,r) = \int_0^\infty d\omega \left(f_\omega(r) e^{-i\omega t} a_\omega + \mathrm{h.c.}\right)$$

$$\left[\omega^2 + \frac{h(r)}{l^4}\partial_r\left(r^4 h(r)\partial_r\right)\right] f_\omega(r) = 0$$

d=3: can be solved exactly
d>3: can be solved in low frequency limit

Bulk-boundary dictionary

Near horizon:

outgoing mode — ingoing mode

$$X(t,r) \sim \int_0^{\infty} \frac{d\omega}{\sqrt{2\omega}} [(e^{-i\omega(t-r_*)} + e^{i\theta_\omega} e^{-i\omega(t+r_*)}) a_\omega + \mathrm{h.c.}]$$

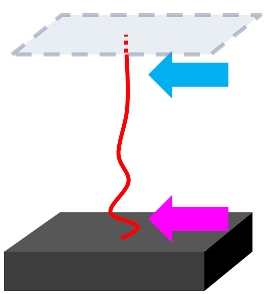
phase shift

r_* : tortoise coordinate

Near boundary:

$$X(t,r_c) \equiv x(t) = \int_0^{\infty} d\omega \, [f_\omega(r_c) e^{-i\omega t} a_\omega + \mathrm{h.c.}]$$

r_c : cutoff



$$\langle x(t_1) x(t_2) \cdots \rangle \leftrightarrow \langle a_{\omega_1} a^\dagger_{\omega_2} \cdots \rangle$$

observe BM in gauge theory ⟷ correlator of radiation modes

Can learn about quantum gravity in principle!

Semiclassical analysis

- Semiclassically, NH modes are thermally excited:

$$\langle a_\omega a_\omega^\dagger \rangle \propto \frac{1}{e^{\beta\omega} - 1}$$

➡ Can use dictionary to compute $x(t), s^2(t)$ (bulk → boundary)

- AdS$_3$

$$s^2(t) \equiv \langle :[x(t) - x(0)]^2: \rangle \approx \begin{cases} \frac{T}{m} t^2 & (t \ll t_{\text{relax}}) \\ \frac{\alpha'}{\pi l^2 T} t & (t \gg t_{\text{relax}}) \end{cases}$$

: ballistic

: diffusive

diffusive

➡ Does exhibit Brownian motion

Semiclassical analysis

- Momentum distribution

Probability distribution for $p = m\dot{x}$

$$f(p) \propto \exp(-\beta E_p)\,, \qquad E_p = \frac{p^2}{2m}$$

→ Maxwell-Boltzmann

- Diffusion constant

$$D = \frac{(d-1)^2\alpha'}{8\pi l^2 T}$$

→ Agrees with drag force computation

[Herzog+Karch+Kovtun+Kozcaz+Yaffe]
[Liu+Rajagopal+Wiedemann]
[Gubser] [Casalderrey-Solana+Teaney]

Forced motion

- Want to determine $\mu(\omega), \kappa(\omega)$
  — follow the two-step procedure

Turn on electric field $E(t)=E_0 e^{-i\omega t}$
on "flavor D-brane" and measure response
$\langle p(t) \rangle$

Forced motion: results (AdS_3)

- Admittance

$$\mu(\omega) = \frac{1}{\gamma[\omega] - i\omega} = \frac{\alpha' \beta^2 m}{2\pi} \frac{1 - i\omega/\pi\alpha' m}{1 - \alpha' m \beta^2 \omega/2\pi}$$

- Random force correlator

$$\kappa(\omega) = \frac{4\pi}{\alpha' \beta^3} \frac{1 - (\beta\omega/2\pi)^2}{1 - (\omega/2\pi\alpha' m)^2}$$

⇨ $t_{\rm coll} \sim \frac{1}{T}$ (no λ)

- FD theorem

$$2m\,\mathrm{Re}(\gamma[\omega]) = \beta\kappa(\omega)$$

⇨ satisfied

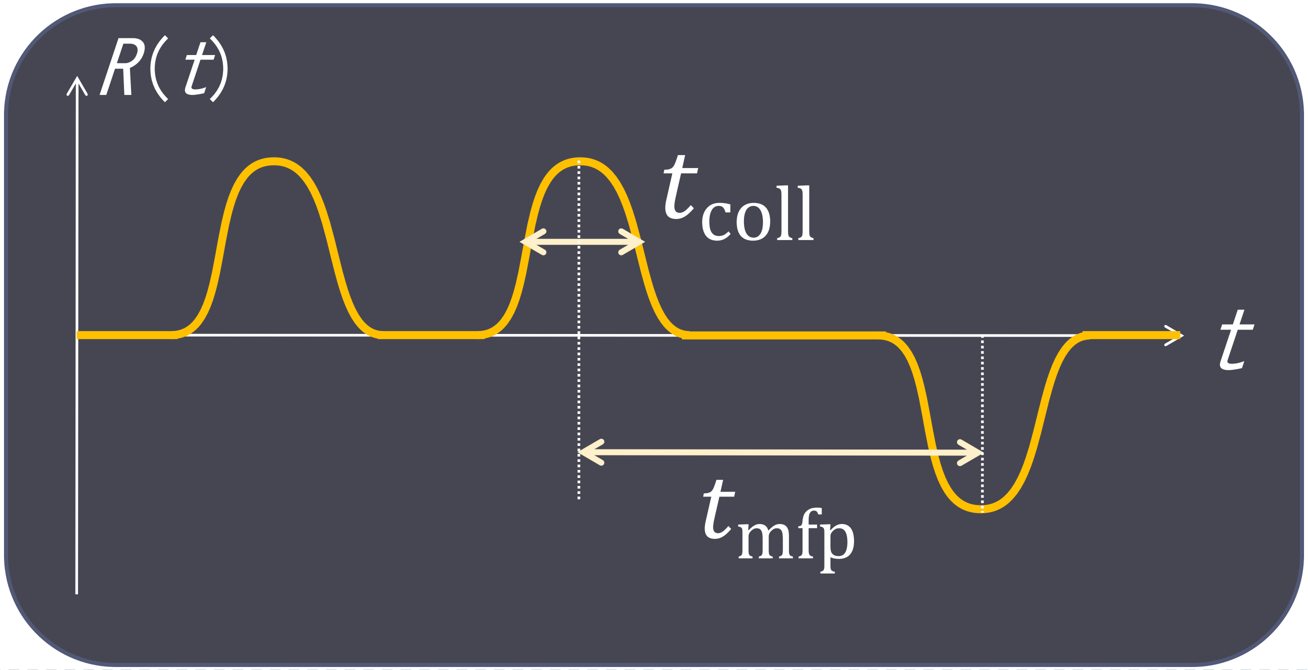
can be proven generally

Time scales

Time scales

t_{relax} t_{mfp} t_{coll}

information about
plasma constituents



Time scales from R-correlators

A toy model:

- $R(t)$: consists of many pulses randomly distributed

$$R(t)=\sum_{i=1}^{k}\epsilon_i f(t-t_i)$$

$f(t)$: shape of a single pulse

$\epsilon_i=\pm 1$: random sign

- Distribution of pulses = Poisson distribution

μ : number of pulses per unit time $\sim 1/t_{\mathrm{mfp}}$

Time scales from R-correlators

- 2-pt func $\langle R(t)R(0)\rangle \to t_{\mathrm{coll}}$
- Low-freq. 4-pt func $\langle \tilde{R}(\omega_1)\tilde{R}(\omega_2)\tilde{R}(\omega_3)\tilde{R}(\omega_4)\rangle \to t_{\mathrm{mfp}}$

$$\langle \tilde{R}(\omega_1)\tilde{R}(\omega_2)\rangle \approx 2\pi\mu\delta(\omega_1+\omega_2)\tilde{f}(0)^2$$

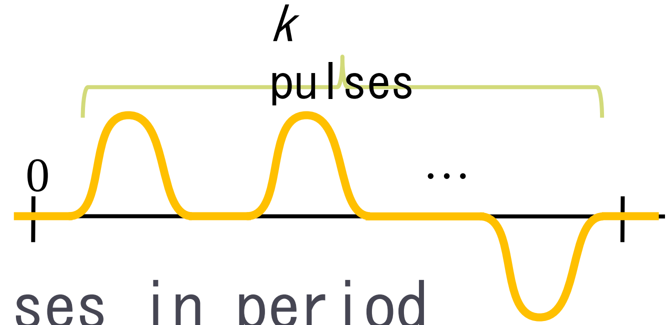
$$\langle \tilde{R}(\omega_1)\tilde{R}(\omega_2)\tilde{R}(\omega_3)\tilde{R}(\omega_4)\rangle_{\mathrm{conn}}$$
$$\approx 2\pi\mu\delta(\omega_1+\omega_2+\omega_3+\omega_4)\tilde{f}(0)^4$$

tilde = Fourier transform

⇒ Can determine μ, thus t_{mfp}

Sketch of derivation (1/2)

$$R(t) = \sum_{i=1}^{k} \epsilon_i f(t - t_i)$$



Probability that there are k pulses in period $[0,\tau]$:

$$P_k(\tau) = e^{-\mu\tau} \frac{(\mu\tau)^k}{k!}$$ (Poisson dist.)

2-pt func:

$$\langle R(t)R(t')\rangle = \sum_{k=1}^{\infty} P_k(\tau) \sum_{i,j=1}^{k} \langle \epsilon_i \epsilon_j f(t - t_i) f(t - t_j)\rangle_k$$

$$\epsilon_i = \pm 1 : \text{ random signs} \rightarrow \langle \epsilon_i \epsilon_j \rangle = \delta_{ij}$$

$$\langle f(t - t_i) f(t' - t_i)\rangle_k = \frac{k}{\tau} \int_0^{\tau} du\, f(t - u) f(t' - u)$$

Sketch of derivation (2/2)

$$\Rightarrow \quad \langle R(t)R(t')\rangle = \mu \int_{-\infty}^{\infty} du\, f(t-u) f(t'-u)$$

$$\langle \tilde{R}(\omega)\tilde{R}(\omega')\rangle = 2\pi\mu\delta(\omega+\omega')\tilde{f}(\omega)\tilde{f}(\omega')$$

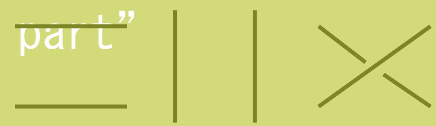
Similarly, for 4-pt func,

$$\langle \tilde{R}(\omega)\tilde{R}(\omega')\tilde{R}(\omega'')\tilde{R}(\omega''')\rangle$$

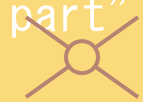
$$= \langle \tilde{R}(\omega)\tilde{R}(\omega')\rangle \langle \tilde{R}(\omega'')\tilde{R}(\omega''')\rangle + (2 \text{ more terms})$$

$$+2\pi\mu\delta(\omega+\omega'+\omega''+\omega''')\tilde{f}(\omega)\tilde{f}(\omega')\tilde{f}(\omega'')\tilde{f}(\omega''')$$

"disconnected part"



"connected part"



$$\Rightarrow \quad \langle \tilde{R}(\omega_1)\tilde{R}(\omega_2)\rangle \approx 2\pi\mu\delta(\omega_1+\omega_2)\tilde{f}(0)^2$$

$$\langle \tilde{R}(\omega_1)\tilde{R}(\omega_2)\tilde{R}(\omega_3)\tilde{R}(\omega_4)\rangle_{\rm conn} \approx 2\pi\mu\delta(\omega_1+\omega_2+\omega_3+\omega_4)\tilde{f}(0)^4$$

⟨RRRR⟩ from bulk BM

- Can compute t_{mfp} from connected to 4-pt func.
- Expansion of NG action to higher order:

$$S_{\mathrm{NG}} = \mathrm{const} + S_2 + S_4 + \cdots$$

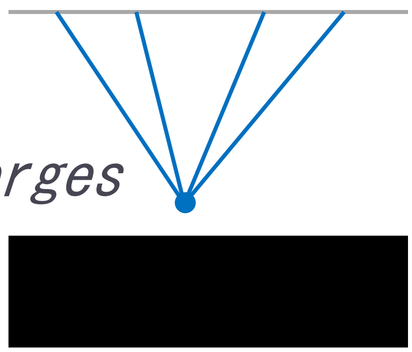
$$S_4 = \frac{1}{16\pi\alpha'}\int dt\,dr\left[\frac{\dot{X}^2}{h(r)} - \frac{r^4 h(r)}{l^4}X'^2\right]^2$$

4-point vertex

⇒ Can compute $\langle RRRR\rangle_{\mathrm{conn}}$ and thus t_{mfp}

⟨RRRR⟩ from bulk BM

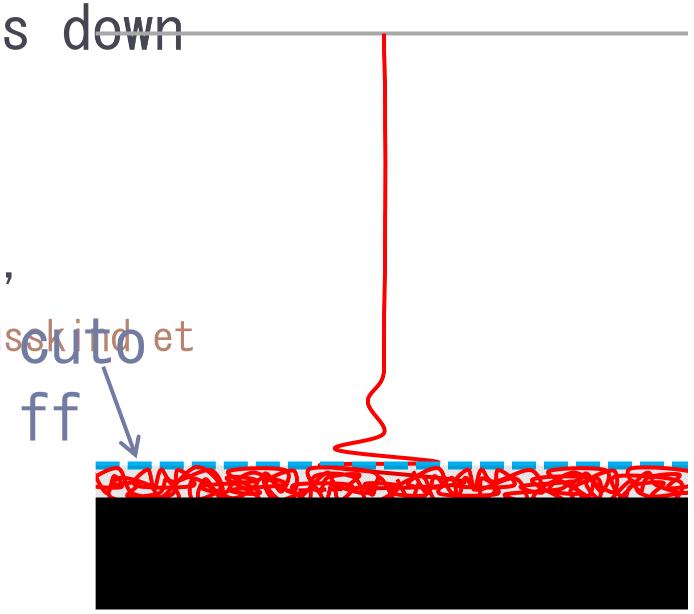
- Holographic renormalization [Skenderis]
  - Similar to KG scalar, but not quite
- Lorentzian AdS/CFT [Skenderis + van Rees]
- IR divergence
  - Near the horizon, bulk integral *diverges*



IR divergence

Reason:

- Near the horizon, local temperature is high
- String fluctuates wildly
- Expansion of NG action breaks down



Interpretation:

- String covers entire horizon, becoming a "part" of BH [Susskind et al.]
- Background gets renormalized

Remedy:

- Introduce an IR cut off where expansion breaks down

Times scales from AdS/CFT (weak)

Resulting timescales:

$$t_{\text{relax}} \sim \frac{m}{\sqrt{\lambda}\,T^2} \qquad t_{\text{coll}} \sim \frac{1}{T} \qquad t_{\text{mfp}} \sim \frac{1}{T \log \lambda} \qquad \lambda \equiv \frac{l^4}{\alpha'^2}$$

- weak coupling $\lambda \ll 1$

⇒ $t_{\text{relax}} \gg t_{\text{mfp}} \gg t_{\text{coll}}$

⇒ conventional kinetic theory is good

Times scales from AdS/CFT (strong)

Resulting timescales:

$$t_{\mathrm{relax}} \sim \frac{m}{\sqrt{\lambda}\,T^2} \qquad t_{\mathrm{coll}} \sim \frac{1}{T} \qquad t_{\mathrm{mfp}} \sim \frac{1}{T \log \lambda} \qquad \lambda \equiv \frac{l^4}{\alpha'^2}$$

- strong coupling $\lambda \gg 1$

⟹ $t_{\mathrm{mfp}} \ll t_{\mathrm{coll}}$

⟹ Multiple collisions occur simultaneously.

Cf. “fast scrambler”

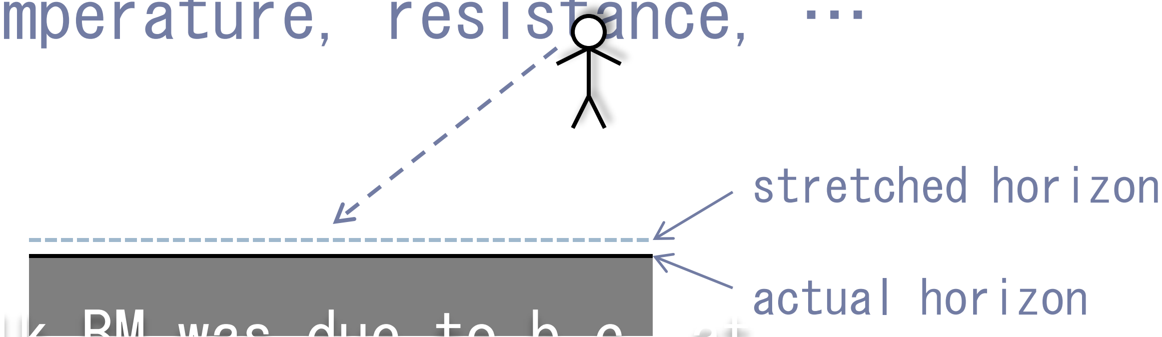
[Hayden+Preskill]
[Sekino+Susskind]

BM on stretched horizon

Membrane paradigm?

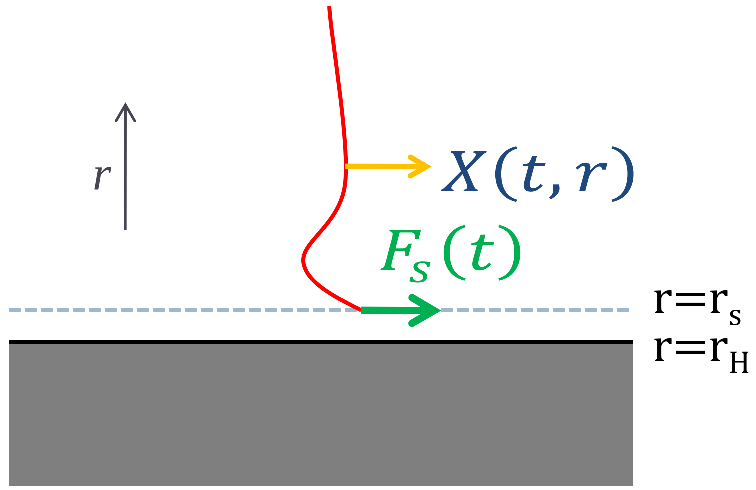
- Attribute physical properties to "stretched horizon"

  E.g. temperature, resistance, …



Q. Bulk BM was due to b.c. at horizon
(ingoing: thermal, outgoing: any).
Can we reproduce this by interaction
between string and "membrane" ?

Langevin eq. at stretched horizon



- EOM for endpoint: $-\#\partial_r X(t,r_s) = F_s(t)$
- Postulate:

$$F_s(t) = -\int_{-\infty}^{t} dt' \gamma_s(t-t')\partial_t X(t',r_s) + R_s(t)$$

$$\langle R_s(t)\rangle = 0, \qquad \langle R_s(t)R_s(t')\rangle = \kappa_s(t-t')$$

Langevin eq. at stretched horizon

$$X(t, r \sim r_H) = \int \frac{d\omega}{\sqrt{2\omega}} [\underbrace{a_\omega^{(+)} e^{-i\omega(t-r_*)}}_{\text{Ingoing (thermal)}} + \underbrace{a_\omega^{(-)} e^{-i\omega(t+r_*)}}_{\text{outgoing (any)}} + \text{h.c.}]$$

Plug into EOM

- Can satisfy EOM if

$$\gamma_s(t) \propto \delta(t), \qquad R_s(\omega) \propto \sqrt{\omega}\, a_\omega^{(+)}$$

Friction precisely cancels outgoing modes

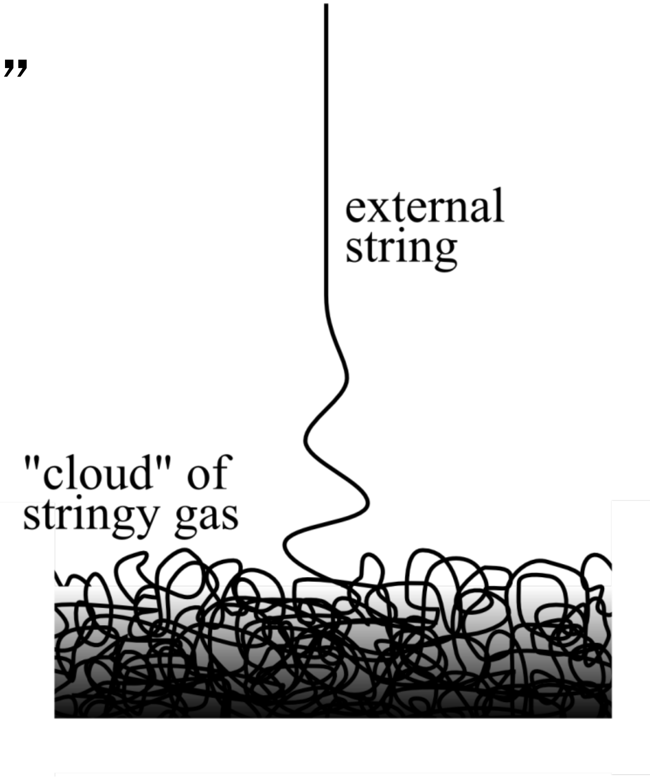
Random force excites ingoing modes thermally

- Correlation function:

$$\langle R_s(\omega)^\dagger R_s(\omega) \rangle \propto \langle a_\omega^{(+)\dagger} a_\omega^{(+)} \rangle \propto \frac{\omega}{e^{\beta\omega} - 1}$$

Granular structure on stretched horizon

- BH covered by "stringy gas"
  [Susskind et al.]
  - Frictional / random forces can be due to this gas
  - Can we use Brownian string to probe this?



Granular structure on stretched horizon

- AdS_d BH

$$ds_d^2 = \frac{r^2}{l^2}\left(-h(r)dt^2 + d\vec{X}_{d-2}^2\right) + \frac{l^2 dr^2}{r^2 h(r)}$$

$$h(r) = 1 - \left(\frac{r_H}{r}\right)^{d-1} \qquad T = \frac{1}{\beta} = \frac{(d-1)r_H}{4\pi l^2}$$

- One quasiparticle / string bit per Planck area

⇩

$$\Delta X \sim \frac{l}{r_H}\ell_P$$

⇩ qp' s are moving at speed of light

$$\Delta t \sim \frac{\Delta X}{\sqrt{\epsilon}} \sim \frac{l\ell_P}{\sqrt{\epsilon}\, r_H}$$

(stretched horizon at $r_s = (1+2\epsilon)r_H$)

- Proper distance from actual horizon: $L \sim \sqrt{\epsilon}\, l$

Granular structure on stretched horizon

- String endpoint collides with a quasiparticle once in time

$$\Delta t \sim \frac{\ell_P}{TL}$$

- Cf. Mean-free-path time read off R_s-correlator:

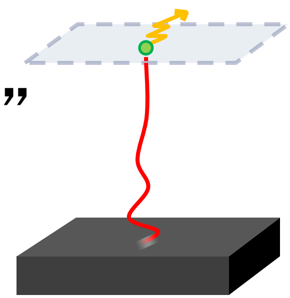
$$t_{\mathrm{mfp}} \sim \frac{1}{T}$$

- Scattering probability for string endpoint:

$$\sigma = \frac{\Delta t}{t_{\mathrm{mfp}}} \sim \frac{\ell_P}{L} \sim \begin{cases} 1 & (L \sim \ell_{\mathrm{P}}) \\ g_{\mathrm{s}}^{\#} & (L \sim \ell_{\mathrm{s}}) \end{cases}$$

Conclusions

Conclusions



- Boundary BM ↔ bulk "Brownian string"

  Can study QG in principle

- Semiclassically, can reproduce Langevin dyn. from bulk

  random force ↔ Hawking rad. ("kick" by horizon)

  friction ↔ absorption

- Time scales in strong coupling QGP: t_{relax}, t_{coll}, t_{mfp}
- FD theorem

Conclusions

- BM on stretched horizon
- Analogue of Avogadro # $N_A = 6x10^{23} < \infty$?
  - Boundary: $E/T \sim \mathcal{O}(N^2) < \infty$
  - Bulk: $M/T \sim \mathcal{O}(G_N^{-1}) < \infty$

→ energy of a Hawking quantum is tiny
as compared to BH mass, but finite

Thanks!