Resurgence, exact quantization and complex instantons

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An overview of the resurgence program in physics

Main motivations:

- Asymptotic analysis of QFT / QM / string theory / ...
- Understanding of non-BPS saddles / configurations
- Renormalon puzzle in asymptotically free QFTs
- Non-perturbative, continuum definition of QFT
- Real time path integrals $\rightarrow$ non-equilibrium physics
- Analytical continuation of path integrals
- ODEs, PDEs, difference equations, dynamical systems

resurgence: semi-classics $\rightarrow$ "exact semi-classics"

An overview of the resurgence program in physics

Many expansions in physics are divergent-asymptotic:

$$f(\hbar) = \sum_{n=0}^{\infty} c_n \hbar^n \quad , \quad c_n \sim n!$$

QM, QFT, strings, hydrodynamics, QNMs, ...

some examples: (beware! highly incomplete list)

- quartic/cubic oscillator, Mathieu, Zeeman, Stark, ...
- Dyson instability, weak field Euler-Heisenberg, QFT in dS/AdS background, large N, ...
- topological genus expansion ($c_g \sim (2g)!$) [Shenker]
- boost invariant conformal hydrodynamics [Heller,Spalinski; GB, Dunne]

How can we assign a value to an asymptotic series?

Overview: *Borel summation*

Introduce the Borel transform of the series $f(\hbar)$:

$$\mathcal{B}[f](u) = \sum_{n=0}^{\infty} \frac{c_n}{n!} u^n$$

This new series typically has finite radius of convergence.

Borel resummation of the original asymptotic series:

$$\mathcal{S}f(\hbar) = \frac{1}{\hbar} \int_0^{\infty} \mathcal{B}[f](s) e^{-s/\hbar} ds$$

But $\mathcal{B}[f](s)$ in general can have singularities in the s plane. They can be on $\mathbb{R}^+$

How to deal with those singularities?

Overview: *Borel singularities*

We can avoid the singularities on $\mathbb{R}^+$:

Lateral Borel summation:

$$\mathcal{S}_\theta f(\hbar) = \frac{1}{\hbar}\int_0^{e^{i\theta}\infty} \mathcal{B}[f](s) e^{-s/\hbar} ds$$

Go above/below the singularity, $s = s^*$, $\theta = 0^\pm$?

$\Rightarrow$ ambiguous imaginary part: $\propto \pm i\pi e^{-s^*/\hbar}$

$\Rightarrow$ Instability?

Yes (cubic oscl., Stark, $\vec{E}$, ...) No (double well., Zeeman, $\vec{B}$, ...)

idea: non-perturbative physics resolves the ambiguity!

resurgence: relations between perturbative & non-perturbative expansions

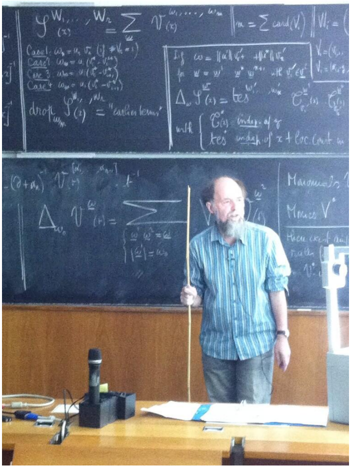
Overview: *Trans-series*

$$f(\hbar) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{k-1} \underbrace{c_{n,k,l}\,\hbar^n}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{\hbar}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{\hbar}\right]\right)^l}_{\text{quasi-zero-modes}}$$

- trans-series: closed under $\int$, ∂, inversion, composition [Écalle, '80] $\left(\sum_n c_n \hbar^n \text{ isn't!}\right)$ Trans-series are ubiquitous.
- trans-monomials, $\hbar^n$, $e^{-1/\hbar}$, $\log(\frac{-1}{\hbar})$, have physical meanings
- respects global symmetries of the original function $\Rightarrow$ analytical continuation
- $\Rightarrow$ exact definition of the function

resurgence: $c_{n,k,l}$s are stringently related

Overview: *Trans-series*



Resurgence and transseries in quantum, gauge and string theories, CERN, 2014 [Photo: J. Edelstein]

"resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities"

J. Écalle

A function may have *different asymptotic expansions* depending on the direction of the expansion in complex plane.

An exponentially small correction might become comparable with the original series as one rotates in complex plane.

This new term "is born" when a Stokes line is crossed.

Borel plane:

Stokes line $\Leftrightarrow$ Line of singularities / branch points

resurgence: keeps track of all the Stokes jumps (bridge equations)

Overview: *a trans-series example*

Stirling expansion for $\psi(x)=\frac{d}{dx}\ln\Gamma(x)$:

$$\psi(1+z)\sim\ln z+\frac{1}{2z}-\frac{1}{12z^2}+\frac{1}{120z^4}-\frac{1}{252z^6}+\dots+\frac{174611}{6600z^{20}}-\dots$$

- functional relation: $\psi(1+z)=\psi(z)+\frac{1}{z}$ ✓
- reflection formula: $\psi(1+z)-\psi(1-z)=\frac{1}{z}-\pi\cot(\pi z)$
- formal series $\Rightarrow$ $\mathrm{Im}\,\psi(1+iy)\sim-\frac{1}{2y}+\frac{\pi}{2}$

  reflection $\Rightarrow$ $\mathrm{Im}\,\psi(1+iy)\sim-\frac{1}{2y}+\frac{\pi}{2}+\pi\sum_{k=1}^{\infty}e^{-2\pi k y}$

  "raw" asymptotics inconsistent with analytic continuation
- resurgence fixes this: series $\to$ trans-series $\Rightarrow$ global properties ✓
- infinite number of exponential terms in trans-series

An overview of the resurgence program in physics

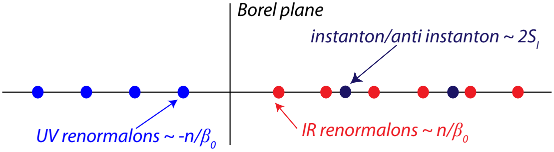
- Écalle's ideas in QM [Pham, Delabaere, Dilinger, Voros, Zinn-Justin]
- Hyperasymptotics [Dingle, Berry, Howls]
- Topological string theory, matrix models, localization [Mariño, Schiappa, Weiss, Aniceto, Edelstein, Couso-Santamaria, Vaz, Vonk]
- Renormalon puzzle in 2d σ models [Cherman, Dobrowski, Dorigoni, Dunne, Ünsal] and 4d gauge theories [Argyres, Ünsal]
- Quantum integrable models, SUSY gauge theories [GB, Dunne; Gorsky, Milekhin; Krefl; Kashani-Poor, Troost]

related:

- Lefschetz thimbles [Pham; Fedoryuk; Howls; Witten et. al.; Tanizaki et. al., ...]
- Lefschetz thimbles $\Leftrightarrow$ Monte Carlo [Scorzato et. al.; Aarts et. al.; GB, Bedaque, Alexandru]
- Thimbles and new saddles in QM [Behtash, Schaefer, Sulejmanpasic, Ünsal]
- More mathematically [Garoufalidis, Costin, Getmanenko, Kontsevich, ...]

Overview: *Resurgence and QFT; renormalon puzzle*

In QFT: extra singularities due to momentum space integrals of Green's functions *"Renormalons"* ['t Hooft]



Claim: semiclassical realization of IR renormalons $\Leftrightarrow$ non-BPS defects with $S \sim \frac{1}{N}$

- semi-class. deformed Yang-Mills *(bions)* [Argyres, Ünsal]
- SUSY gauge theories [Dunne, Shifman, Ünsal]
- $\mathbb{C}P^N$ *(fractional instantons)* [Dunne, Ünsal]
- PCM *(no instantons!)* *(fractional unitons)* [Cherman, Dorigoni, Dunne, Ünsal]
- O(N) *(no instantons!)* $(S = S/\beta_0 = S/(N-2))$ [Dunne, Ünsal]

1D quantum mechanical systems

This talk: 1D quantum mechanical systems

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dz^2} + V(z)\,\psi = u(N,\hbar)\,\psi$$

- One can derive trans-series from first principles
- Prototype for some phenomena in QFTs
- Arise upon compactification of σ models $\mathbb{R}^2 \to \mathbb{R} \times S^1_{L\to 0}$
- Relevant for SUSY gauge theories in D=2,4 [Nekrasov, Shatashvili]
  Quantum integrable systems $\Leftrightarrow$ SUSY gauge theories
- ODE $\Leftrightarrow$ 2D integrable models [Dorey, Tateo; Voros; Bazhanov, Fateev, Lukyanov, Zamolodchikov; ...]

1D quantum mechanical systems

This talk: (for concreteness)

- (mostly) *Mathieu equation* (GB, Dunne; 1501.05671)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dz^2} + cos(z)\,\psi = u(N,\hbar)\,\psi$$

- Most of the conclusions are more general
- Encodes the vacua of $\mathcal{N}=2$, $SU(2)$ theory in its spectrum
  $u \Leftrightarrow \mathrm{tr}\langle\Phi^2\rangle$, moduli space coord.

1D quantum mechanical systems

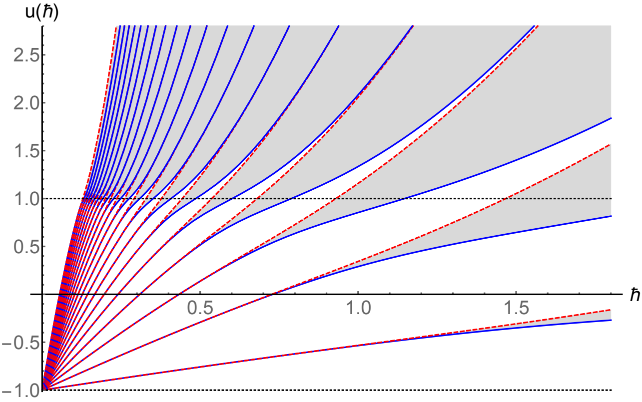
- *Lamé equation*

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dz^2}+\frac{1}{8}\left(M^2-\frac{\hbar^2}{4}\right)\wp\left(\frac{z}{2}+c;\tau\right)\psi=u\,\psi$$

- potential is doubly periodic, tunneling can occur along each period
- additional instantons →"ghost instantons" $S_I < 0$
- $\mathcal{N}=2^*$, $SU(2)$ theory: M=adjoint scalar mass
  $\mathcal{N}=4 \leftarrow 0 \leq M < \infty \rightarrow \mathcal{N}=2$

Mathieu equation

Spectrum of Mathieu equation (with Bloch b.c.)



- $u \sim -1$: small bands, tightly bound states (dyonic)
- $u \sim 1$: gaps $\sim$ bands , crossover region (magnetic)
- $u \gg 1$: small gaps, plane waves (electric)

Spectral regions and expansions

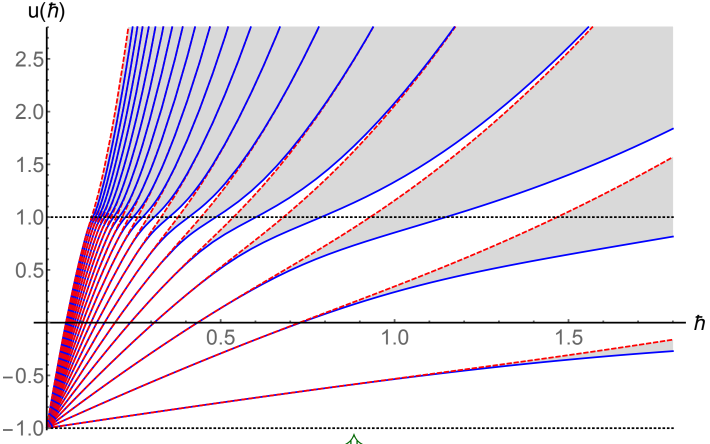
2 expansion parameters: $N, \hbar$. Let $\lambda = N\hbar/2$

- 3 spectral regions and gauge theory interpretations
  - $u \sim -1$ "dyonic" : $\lambda \ll 1$ asymptotic, trans-series
  - $u \sim 1$ "magnetic" : $\lambda \sim 1$ crossover region
  - $u \gg 1$ "electric" : $\lambda \gg 1$ convergent, poles
- Will show both in weak coupling and in strong coupling expansions there are "instanton-like" (semi-classical) contributions to u that are $\mathcal{O}(e^{-N/\lambda})$ and $\mathcal{O}(e^{-2N\log\lambda})$.
  - Instanton/ anti-instanton $\Leftrightarrow$ Borel poles
  - complex instantons$\Leftrightarrow$ poles in convergent expansions for u

A remark: Similar phenomenon $ABJM$ at large N: non-perturbative effects $\leftrightarrow$ complex space-time instantons $\leftrightarrow$ poles in the 't Hooft expansion

[Drukker, Mariño, Putrov; Grassi, Hatsuda, Mariño, ...]

Spectrum of Mathieu equation



$\Uparrow$

Weak coupling expansion: $N\hbar \ll 1$

Weak coupling expansion $(\lambda \ll 1)$: trans-series

localized states at minima, tunneling is exponentially suppressed

$$u(N,\hbar) \sim -1 + \hbar\left[N+\frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right]$$
$$-\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right] - \dots$$
$$+\underbrace{\xi\,(1+\,\mathcal{O}(\hbar))\,\cos\theta}_{1-instanton\,order} + \underbrace{\xi^2\,(f_1(N,\hbar)+\cos 2\theta\, f_2(N,\hbar))}_{2-instanton\,order}$$
$$+\dots$$

- *instanton fugacity:* $\xi = \sqrt{\frac{2}{\pi}}\frac{2^5}{N!}\left(\frac{32}{\hbar}\right)^{N-1/2}\exp\left[-\frac{8}{\hbar}\right]$
- u has a *trans-series* expansion. *trans-monomials:* $\hbar^n$ (perturbative fluctuations), ξ^k (multi instantons), $\log(-1/\hbar)^l$ (quasi zero modes, start at $\mathcal{O}(\xi^2)$: in f_1, f_2)

Resurgence relations

large order growth of perturbative series:

$$c_n(N=0) \sim n! \left(1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \ldots \right)$$

instanton anti-instanton fluctuations: *(leading ambiguity)*

$$\mathcal{I}m\, u(0,\hbar) \;\; \sim \;\; \pi\, e^{-2S_I/\hbar} \left(1 - \frac{5}{2} \cdot \left(\frac{\hbar}{16}\right)^2 - \frac{13}{8} \cdot \left(\frac{\hbar}{16}\right)^4 - \ldots \right)$$

This is just the first step of an infinite tower of perturbative- nonperturbative cancellations!

Graded resurgence triangle [Dunne, Ünsal]

$$
\begin{array}{ccccccc}
 & & & f_{(0,0)} & & & \\
 & & e^{-\frac{S_I}{\hbar}+i\theta} f_{(1,1)} & \updownarrow & e^{-\frac{S_I}{\hbar}-i\theta} f_{(1,-1)} & & \\
 & e^{-\frac{2S_I}{\hbar}+2i\theta} f_{(2,2)} & & e^{-\frac{2S_I}{\hbar}} f_{(2,0)} & & e^{-\frac{2S_I}{\hbar}-2i\theta} f_{(2,-2)} & \\
e^{-\frac{3S_I}{\hbar}+3i\theta} f_{(3,3)} & & e^{-\frac{3S_I}{\hbar}+i\theta} f_{(3,1)} & \updownarrow & e^{-\frac{3S_I}{\hbar}-i\theta} f_{(3,-1)} & & e^{-\frac{3S_I}{\hbar}-3i\theta} f_{(3,-3)} \\
\dots & e^{-\frac{4S_I}{\hbar}+2i\theta} f_{(4,2)} & & e^{-\frac{4S_I}{\hbar}} f_{(4,0)} & & e^{-\frac{4S_I}{\hbar}-2i\theta} f_{(4,-2)} & \dots \\
\iddots & & & \vdots & & & \ddots
\end{array}
$$

Resurgent cancellations happen at every instanton order and are graded by the topological charge.

Exact quantization and trans series

Trans-series can be constructed from the exact quantization relations. Morally: all orders Bohr-Sommerfeld

$$\left(\frac{32}{\hbar}\right)^{-B} \frac{e^{A/2}}{\Gamma\left(\frac{1}{2}-B\right)} + e^{\pm i\pi B}\left(\frac{32}{\hbar}\right)^{-B} \frac{e^{-A/2}}{\Gamma\left(\frac{1}{2}+B\right)} = \sqrt{\frac{2}{\pi}}\cos\theta$$

- Need two functions $A(u,\hbar)$ and $B(u,\hbar)$
- $B(u,\hbar)$: local perturbative series, easy to compute (Rayleigh-Schrödinger)
- $A(u,\hbar)$: instanton fluctuations, quite hard to compute !
- Conjectured by Zinn-Justin & Jentschura ('05)
- Can be derived via *uniform WKB* [Dunne, Ünsal]

Perturbative Non-perturbative connection

$A(u,\hbar)$ and $B(u,\hbar)$ are in fact related!

$$\frac{\partial u(B)}{\partial B} + \frac{\hbar^2}{16}\left(2B + \hbar\frac{\partial A(B)}{\partial \hbar}\right) = 0$$ [Dunne, Ünsal]

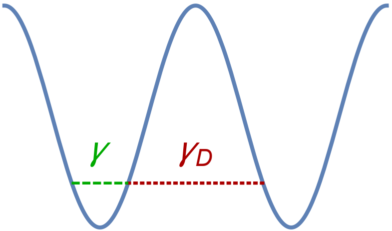
- Knowledge of $B(u,\hbar)$ from ordinary perturbation theory is enough to reconstruct *the whole trans-series!*. Perturbation series encodes *all* the information!
- This is a constructive equation.
  first n terms of p.t. $\Rightarrow$first n terms of instanton fluctuations

A geometric picture

- In this form, the $P \leftrightarrow NP$ relation is somehow mysterious. But there is a simple geometrical interpretation. To see it, we need to go to the Bohr-Sommerfeld picture.
- Set $\hbar = 0$ for now. Classically the (complex) phase space can be identified with the moduli space of complex tori.

$$H = \frac{1}{2}p^2 + \cos z = u \to y^2 = (x^2 - 1)(x - u) : \text{g=1 elliptic curve}$$

A geometric picture



- Semi-classical Bohr-Sommerfeld actions: integrals of abelian differentials over the two independent cycles of torus

$$a_0(u) = \frac{\sqrt{2}}{2\pi}\int_\gamma \sqrt{u - V(z)}\, dz = \frac{\sqrt{2}}{\pi}\int_{\hat{\gamma}} \sqrt{\frac{u-x}{1-x^2}} dx$$

$$a_0^D(u) = \frac{\sqrt{2}}{2\pi}\int_{\gamma_D} \sqrt{u - V(z)}\, dz = \frac{\sqrt{2}}{\pi}\int_{\hat{\gamma}_D} \sqrt{\frac{u-x}{1-x^2}} dx$$

A geometric picture

- leading order BS differentials $\leftrightarrow$ Seiberg-Witten differentials: $a_0 \Leftrightarrow$ scalar v.e.v. (up to monodromy), $M_{n,m} = |na_0 + ma_0^D|$, etc. [Gorsky, Krichever, Marshakhov, Mironov, Morozov]
- $a_0(u, \hbar) = \hbar(N + 1/2)$: determines the location of the band, generates *perturbative* expansion $\leftrightarrow B$
- a_0^D: determines the width of the band $\propto e^{-a_0^D/\hbar} \propto e^{-S_I/\hbar}$, *non-perturbative:* instanton + fluctuations $\leftrightarrow A$
- a_0 and a_0^D are related via *Riemann bilinear identity*

$$a_0 \frac{da_0^D}{du} - a_0^D \frac{da_0}{du} = \frac{2i}{\pi} = \frac{i}{2}\frac{S_I}{T}$$

$T = 2\pi =$ period of the harm. oscll. at the bottom of the well

A geometric picture

- For $\hbar \neq 0$, higher order corrections:

$$a(u,\hbar) = a_0(u) + \hbar^2 a_1(u) + \dots \quad a^D(u,\hbar) = a_0^D(u) + \hbar^2 a_1^D(u) + \dots$$

- The bilinear identity gets modified in a very special way [GB, Dunne]

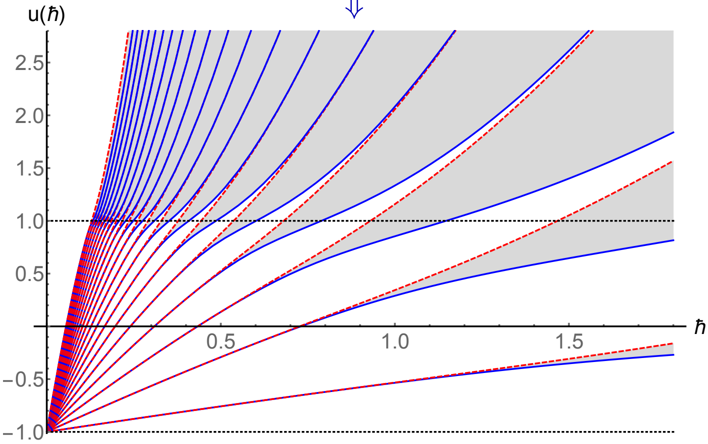
$$\left(a - \hbar \frac{\partial a}{\partial \hbar}\right) \frac{\partial a^D}{\partial u} - \left(a^D - \hbar \frac{\partial a^D}{\partial \hbar}\right) \frac{\partial a}{\partial u} = \frac{2i}{\pi}$$

- "quantum Riemann bilinear identity"
- From this equation one can get the Dunne-Ünsal $P \leftrightarrow NP$ relation. But this equation is valid *everywhere* in the spectrum!
- It can be proven via Matone's relation [Gorsky, Milekhin; GB, Dunne]

Spectrum of Mathieu equation

Strong coupling expansion: $N\hbar \gg 1$

$\Downarrow$



Strong coupling expansion ($\lambda \gg 1$): convergent series

expansion around a free particle on a circle. For $N \gg 1$:

$$u^{cf}(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \dots\right)$$

$$\sim \frac{\hbar^2}{8}\sum_n \frac{P_n(N)}{\prod_{k=1}^n (N^2-k^2)^{2\lfloor \frac{n}{k}\rfloor -1}} Q^{2n} \quad , \quad Q \equiv 4/\hbar^2$$

- u has a convergent expansion in $1/\hbar^4$.
- *Hill's determinant:* $\cos\pi\nu = 1 - 2\Delta(0)\sin^2(\pi\sqrt{2u/\hbar^2})$
- can be derived from continued fractions

$$2Qu - N^2 - \frac{Q^2}{2Qu-(N-2)^2-\frac{Q^2}{2Qu-(N-4)^2-\dots}} = \frac{Q^2}{(N+2)^2-2Qu-\frac{Q^2}{(N+4)^2-2Qu-\dots}}$$

- for given N only good up to the level $Q^{2N} \sim 1/\hbar^{4N}$ due to poles

Strong coupling expansion ($\lambda \gg 1$): convergent series

gauge theory detour [Alday, Gaiotto, Tachikawa; Marshakov et. al.; ...]

$$Z^{inst.}(a;\epsilon_1,\epsilon_2) = \sum_{n=0}^{\infty} \left(\frac{\Lambda^2}{\epsilon_1\epsilon_2}\right)^{2n} Q_\Delta^{-1}([1^n],[1^n]), \qquad Q_\Delta(Y,Y') = \langle\Delta|L_Y\, L_{-Y'}|\Delta\rangle$$

- from AGT: $\Delta = \frac{1}{\epsilon_1\epsilon_2}\left(a^2 - \frac{(\epsilon_1+\epsilon_2)^2}{4}\right)$, $c = 1 - \frac{6(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$
- $\epsilon_2 \to 0$ limit $\Rightarrow$ *twisted superpotential:* [Nekrasov, Shatashvili]

$$\mathcal{W}_{NS}^{inst.}(a;\epsilon_1) \equiv -\frac{\epsilon_1}{4\pi i}\lim_{\epsilon_2\to 0}\epsilon_2 \log\left(Z^{inst.}(a,\epsilon_1,\epsilon_2)\right)$$

- identify $\epsilon_1 = \hbar$, $a = N\hbar/2$

$$\frac{i\pi}{2}\Lambda\frac{\partial \mathcal{W}_{NS}^{inst.}}{\partial\Lambda} = \frac{\hbar^2}{8}\left(\frac{8\Lambda^4}{(N^2-1)\,\hbar^4} + \frac{8\Lambda^8\left(5N^2+7\right)}{(N^2-4)\,(N^2-1)^3\,\hbar^8} + \dots\right)$$

Strong coupling expansion ($\lambda \gg 1$): convergent series

back to QM: level splitting

- In the limit $\lambda \gg 1$ there are small *gaps* in the spectrum
- $u^{cf}(N, \hbar)$ determines the center of the gap
- at large N the gaps are exponentially small
- *gap width:*

$$
\begin{aligned}
\Delta u_N^{\mathrm{gap}} &\sim \frac{\hbar^2}{4} \frac{1_{\text{Text}}}{\left(2^{N-1}(N-1)!\right)^2} \left(\frac{2}{\hbar}\right)^{2N} \left[1 + O\left(\left(\frac{2}{\hbar}\right)^4\right)\right] \\
&\sim \frac{N\,\hbar^2}{2\pi} \left(\frac{e}{N\,\hbar}\right)^{2N} \qquad , \quad N \gg 1
\end{aligned}
$$

Is there a semi-classical interpretation of the exponentially small gaps at strong coupling, similar to the instantons for the case of exponentially small bands at weak coupling ?

Strong coupling expansion: complex instantons

Is there a semi-classical interpretation of the exponentially small gaps at strong coupling, similar to the instantons for the case of exponentially small bands at weak coupling?

YES! complex instantons

- For $u > 1$, the turning points are complex.
- a^D goes around these complex turning points.
- At the limit $\hbar \ll 1$ and $N \gg \hbar^{-1}$ which corresponds to the spectral region $u \gg 1$, semi-classical approximation can be used:

$$\Delta u_N^{\text{gap}} \sim \frac{2}{\pi} \frac{\partial u^{cf}}{\partial N} \, e^{-\frac{2\pi}{\hbar} \mathcal{I}m \, a^D} \sim \frac{N \, \hbar^2}{2\pi} \left(\frac{e}{N \, \hbar} \right)^{2N}$$

$$u \sim \frac{\hbar^2}{8} N^2 + \cdots = \frac{a^2}{2} + \ldots \;, \; \pi \, \mathcal{I}m[a^D] \sim \sqrt{2u} \, (\ln(8u) - 2) + \ldots$$

Strong coupling expansion: complex instantons

A physical analogy:

Schwinger effect in monochromatic electric field $\mathcal{E}\cos(\omega t)$

- Pair production rate behaves differently for different ωs
- Keldysh adiabaticity parameter: $\gamma \equiv \frac{m\,\omega}{\mathcal{E}}$
- $\gamma \ll 1 \leftrightarrow$ constant field, $\gamma \gg 1 \leftrightarrow$ multi-photon limit
- In our analogy: $\hbar \equiv \frac{4\omega^2}{\mathcal{E}} \quad , \quad u \equiv -1 + 2\gamma^2 \quad , \quad N \equiv \frac{m}{\omega}$

$$P_{\mathrm{QED}} = e^{-\frac{m^2\,\pi}{\mathcal{E}}\,g(\gamma)} \sim \begin{cases} e^{-\pi\,\frac{m^2}{\mathcal{E}}} \quad , \quad \gamma \ll 1 \\ \\ e^{-\frac{m^2\,\pi}{\mathcal{E}}\,\frac{4}{\pi\gamma}\,\log(4\gamma)} = \left(\frac{\mathcal{E}}{4m\,\omega}\right)^{4m/\omega} \quad , \quad \gamma \gg 1 \end{cases}$$

- in the worldline formalism:
  $\gamma \ll 1 \leftrightarrow$ real instantons, $\gamma \gg 1 \leftrightarrow$ complex instantons

Fluctuations around complex instantons

- use the "quantum bilinear identity" to relate $u^{cf}(N,\hbar)$ with Δu_N^{gap}, similar to the $P \leftrightarrow NP$ relation in the weak coupling regime

$$u(N,\hbar) \quad \sim \quad \frac{\hbar^2}{8}\sum_{n=1}^{N-1}\frac{P_n(N)}{\prod_{k=1}^{n}(N^2-k^2)^{2\lfloor\frac{n}{k}\rfloor-1}}\left(\frac{4}{\hbar^2}\right)^{2n}$$
$$\pm\frac{1}{\left(2^{N-1}(N-1)!\right)^2}\left(\frac{2}{\hbar}\right)^{2N-1}\sum_{n=1}^{N-1}\frac{R_n(N)}{\prod_{k=1}^{n}(N^2-k^2)^{2\lfloor\frac{n}{k}\rfloor}}\left(\frac{4}{\hbar^2}\right)^{2n}$$

- The level splitting term ("gap width") has the same structure with the leading pertrurbative expansion.
- $P_n(N)$, $R_n(N)$ are related! [GB, Dunne, Ünsal, in prep]
- New results for Mathieu equation!!

Weak and strong coupling expansions

weak coupling expansion, $N\hbar \ll 1$: deep down in the well, tightly bound states, multi instantons and fluctuations around them, trans-series

$$u^{(\pm)}(N,\hbar) \sim \sum_{n=0}^{\infty} c_n(N)\hbar^n \pm \frac{32}{\sqrt{\pi}\,N!}\left(\frac{32}{\hbar}\right)^{N-1/2} e^{-\frac{8}{\hbar}} \sum_{n=0}^{\infty} d_n(N)\hbar^n + \dots$$

- Borel poles at two-instanton location

strong coupling expansion, $N\hbar \gg 1$: far above the barrier, plane waves, degenerate p.t., convergent series

$$u^{(\pm)}(N,\hbar) = \frac{\hbar^2 N^2}{8} \sum_{n=0}^{N-1} \frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8} \frac{\left(\frac{2}{\hbar}\right)^{2N}}{(2^{N-1}(N-1)!)^2} \sum_{n=0}^{N-1} \frac{\beta_n(N)}{\hbar^{4n}} + \dots$$

- Poles at two- complex instanton location

crossover: At the barrier top when $N\hbar \sim 1$ the instantons proliferate. The bands and gaps are of comparable widths.

The crossover region $(\lambda \sim 1)$

- when $N\hbar \sim 8/\pi$ the tightly bound states transform into plane waves.
- Approached from below the barrier, the real instantons proliferate.

$$u_{\text{dyonic}} \sim -1 + \frac{8}{\pi}\left[1 - \frac{1}{16}\frac{8}{\pi} - \frac{1}{2^8}\left(\frac{8}{\pi}\right)^2 - \dots\right] + O(\hbar) = 1 + O(\hbar)$$

- Approached from above the barrier, the complex instantons proliferate.

$$u_{\text{electric}} \sim \frac{1}{2}\left[\left(\frac{4}{\pi}\right)^2 + \frac{1}{2}\left(\frac{\pi}{4}\right)^2 + \dots\right] + O(\hbar) = 1 + O(\hbar)$$

- In the gauge theory side, it is where low energy effective theory is that of weakly coupled monopoles (i.e. $a_0^D \to 0$).

Mathieu spectrum; full picture

weak coupling expansion, $N\hbar \ll 1$: resurgent trans-series:

$$u^{(\pm)}(N,\hbar) \sim \sum_{n=0}^{\infty} c_n(N)\hbar^n \pm \frac{32}{\sqrt{\pi}\,N!}\left(\frac{32}{\hbar}\right)^{N-1/2} e^{-\frac{8}{\hbar}} \sum_{n=0}^{\infty} d_n(N)\hbar^n + \dots$$

- asymptotic, Borel poles at two-instanton location

strong coupling expansion, $N\hbar \gg 1$: convergent series

$$u^{(\pm)}(N,\hbar) = \frac{\hbar^2 N^2}{8} \sum_{n=0}^{N-1} \frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8} \frac{\left(\frac{2}{\hbar}\right)^{2N}}{(2^{N-1}(N-1)!)^2} \sum_{n=0}^{N-1} \frac{\beta_n(N)}{\hbar^{4n}} + \dots$$

- Poles at two- complex instanton location

crossover, $N\hbar \sim \frac{8}{\pi}$ proliferation of instantons

$$u_N^{(\pm)} \sim 1 - \frac{c_N}{\left(N \pm \frac{1}{4}\right)^2}\left(Q - \frac{\pi^2}{16}\left(N \pm \frac{1}{4}\right)^2\right) \quad , \quad c_N \sim O(1)$$

Lamé equation

Lamé equation and ghost instantons

(GB, Dunne, Ünsal; 1308.1108)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dz^2}+\frac{1}{8}\left(M^2-\frac{\hbar^2}{4}\right)\wp\left(\frac{z}{2}+c;\tau\right)\psi=u\,\psi$$

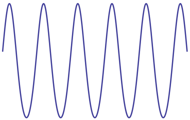
Weierstrass elliptic function: Doubly periodic, meromorphic

$$\begin{aligned}\wp(z;\omega_1,\omega_2) &= \frac{1}{z^2}+\sum_{\lambda\in\Lambda}\left(\frac{1}{(z+\lambda)^2}-\frac{1}{\lambda^2}\right)\\ \Lambda &= \{n\omega_1+m\omega_2|(n,m)\in\mathbb{Z}^2/(0,0)\}\end{aligned}$$

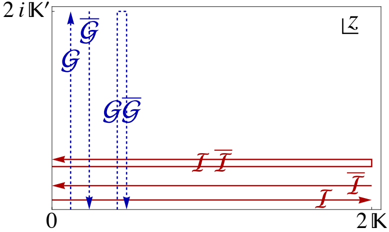
(for gauge theory: M=adjoint scalar mass ,
$\mathcal{N}=4 \leftarrow 0\leq M<\infty \rightarrow \mathcal{N}=2$, u $\leftrightarrow$moduli space coordinate,)

Lamé equation and ghost instantons

- The potential is *doubly periodic.*
- Set the periods to be along real and imaginary axes



- Tunneling along real and imaginary axes:
  Real and ghost instantons



actions:

$$\frac{S_{\mathcal{I}}(m)}{\hbar} = \frac{2\sin^{-1}(\sqrt{m})}{\hbar\sqrt{mm'}} \geq \frac{2}{\hbar}$$

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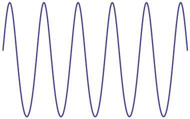

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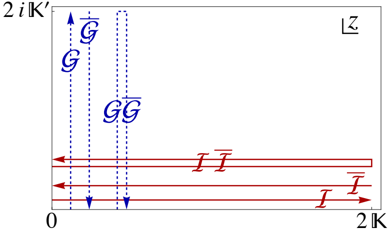

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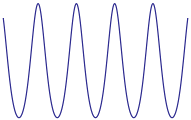

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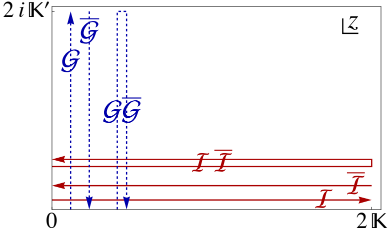

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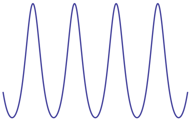

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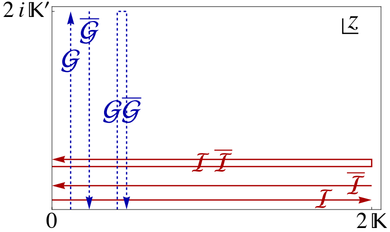

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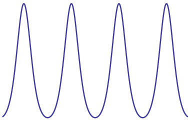

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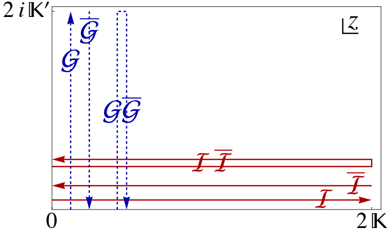

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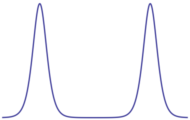

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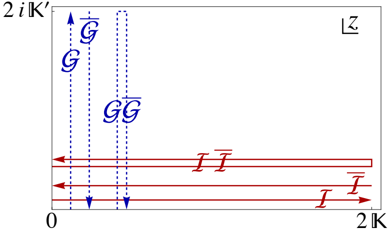

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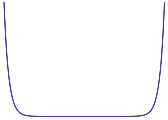

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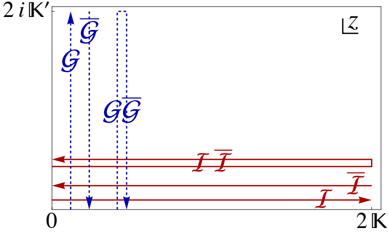

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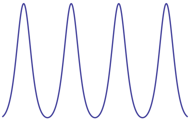

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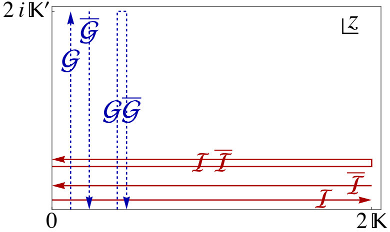

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Lamé equation and ghost instantons

The ghost instantons have negative actions. We do not include them in the path integral.

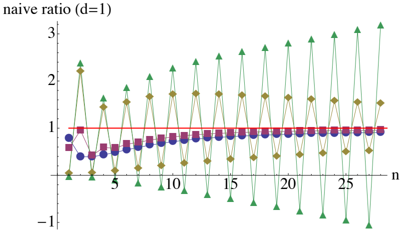
$$\sum_k G_k(\hbar)\, e^{+\frac{1}{\hbar}|S_k|}$$

Nevertheless they do have influence on physical quantities:

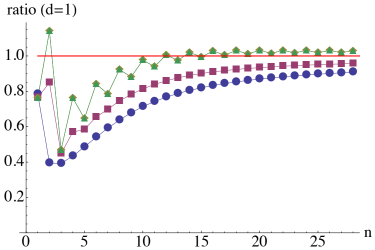
- They contribute to large order perturbation series.
- They can be important for extending the theory to complex/negative couplings.
- They can play a role in quantum phase transitions.

Lamé equation and ghost instantons

The large order growth of perturbation theory



without ghost instantons

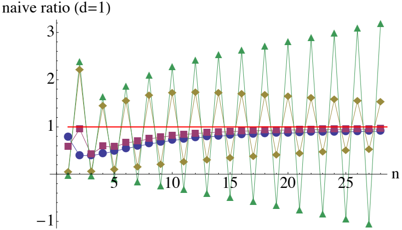


with ghost instantons

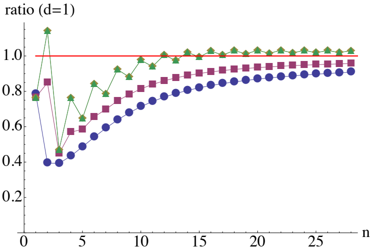
$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$

Lamé equation and ghost instantons

The large order growth of perturbation theory



without ghost instantons



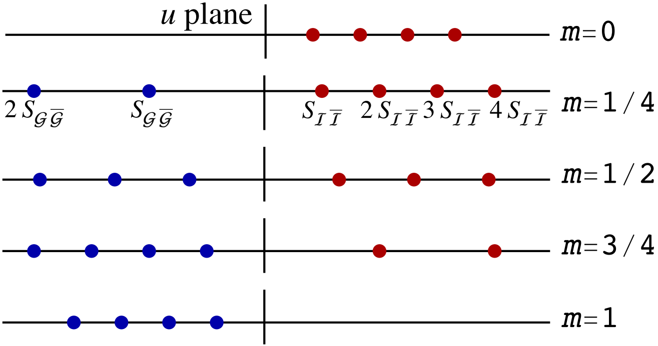
with ghost instantons

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$

Notice the leading singularity is at $[\mathcal{I}\bar{\mathcal{I}}]$ or $[\mathcal{G}\bar{\mathcal{G}}]$

Lamé equation and ghost instantons

Borel plane:



- The distance of the singularity to origin determines the how dominant its contribution to the large order growth is.

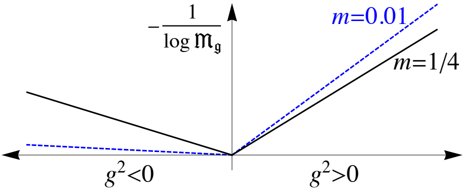
Quantum phase transition:

self-duality for the Hamiltonian

$$H_m = -\frac{d^2}{dz^2} + \frac{1}{\hbar} V(z|m) \xrightarrow{\hbar \to -\hbar} H_{m'} = -\frac{d^2}{dz^2} + \frac{1}{\hbar} V(z|m')$$

m : parameter that controls the period

mass gap/band-width $\mathfrak{M}_\hbar \sim e^{-\frac{1}{m'\hbar}} \xrightarrow{\hbar \to -\hbar} e^{-\frac{1}{m\hbar}}$



non-analytic at origin

Lamé equation and ghost instantons

lesson: take all the saddles seriously, even though they seem unphysical

Resurgence and Hydrodynamics [Heller, Spalinski; GB, Dunne [1509.05046]]

- resurgence: generic feature of differential equations
- boost invariant conformal hydrodynamics
- second-order hydrodynamics: $T^{\mu\nu} = \mathcal{E}\, u^\mu\, u^\nu + T^{\mu\nu}_\perp$

$$\begin{aligned}
\tau \frac{d\mathcal{E}}{d\tau} &= -\frac{4}{3}\mathcal{E} + \Phi \\
\tau_\Pi \frac{d\Phi}{d\tau} &= \frac{4}{3}\frac{\eta}{\tau} - \Phi - \frac{4}{3}\frac{\tau_\Pi}{\tau}\Phi - \frac{1}{2}\frac{\lambda_1}{\eta^2}\Phi^2
\end{aligned}$$

[Baier, Romatschke, Son, Starinets, Stephanov]

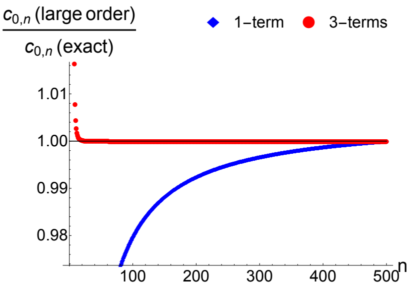
asymptotic hydro expansion: $\mathcal{E} \sim \frac{1}{\tau^{4/3}}\left(1 - \frac{2\eta_0}{\tau^{2/3}} + \dots\right)$

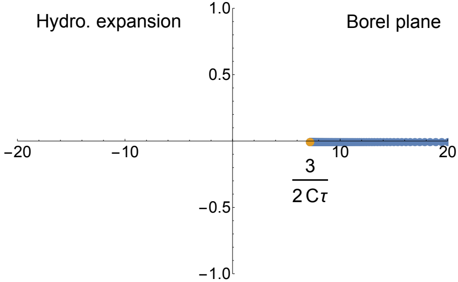
- formal series $\rightarrow$ trans-series

$$\mathcal{E} \sim \mathcal{E}_{\rm pert} + e^{-S\tau^{2/3}} \times (\text{fluc}) + e^{-2S\tau^{2/3}} \times (\text{fluc}) + \dots$$

- non-hydro modes clearly visible in the asymptotic hydro series study large-order behavior:

$$c_{0,k} \sim S_1 \frac{\Gamma(k+\beta)}{2\pi i\, S^{k+\beta}} \left(c_{1,0} + \frac{S\, c_{1,1}}{k+\beta-1} + \frac{S^2\, c_{1,2}}{(k+\beta-1)(k+\beta-2)} + \dots \right)$$





- resurgent large-order behavior and Borel structure verified to 4-instanton level
- $\Rightarrow$ trans-series for metric coefficients in AdS ?

Conclusions

- Resurgence provides a concrete relation between perturbative and non-perturbative expansions in a vast class of physical problems
- Trans-series contains all the information to construct the underlying function and its global properties
- Different limits of the problem might have different expansions, similar perturbative-non perturbative relations exist
- In large N matrix models, string theory and QFTs there are very similar phenomena described here
- In QM how general is this construction? geometric construction? relation to complex integrable models?
- Formulation in terms of Picard-Lefschetz theory