Shaving the Black Hole

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Outline

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Clots of gravity

One of the most exciting predictions of Einstein's General Relativity(GR) is that there exist Black Holes: objects whose gravitational fields are so strong that no body or signal can break free and escape.

- Occupy special position in observational astrophysics, theoretical efforts at unification of forces etc.
- Reveal profound relationships between gravitation, quantum theory and thermodynamics.
- Many fundamental ideas like Holographic principle, string dualities etc are related to the study of black holes.
- Black Holes provide a very useful context where quantum gravitational effects are calculable and highly precise tests are possible. They lead to non-trivial tests of nonperturbative consistency of string theory as a theory of quantum gravity.

Black Holes are the extreme examples of the dynamical nature of space-time(expressed by metric tensor) in General Relativity.

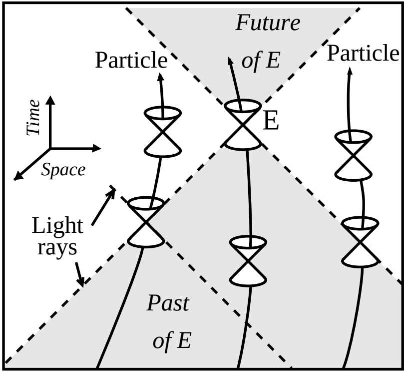


Figure 2

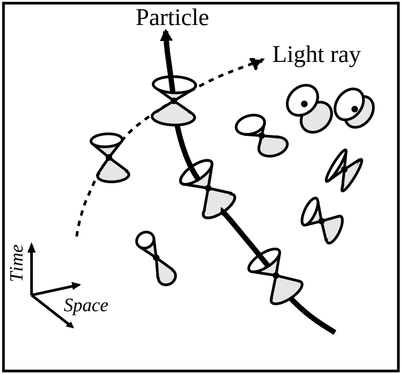


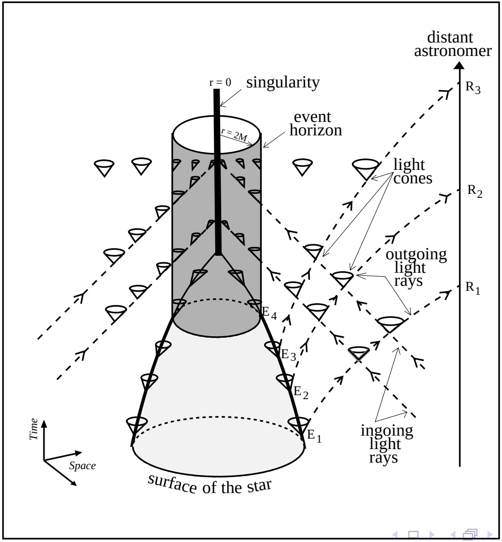
Figure 3

Properties

Light Imprisoned

- Black Holes can occur as end products of complete gravitational collapse. Theoretically, they are solutions to equations of general relativity.
- Black Holes have a singularity, covered by an imaginary surface, Event Horizon which serves as a causal boundary.
- Schwarzschild Black Holes : $ds^2 = (1 - \frac{r_h}{r})dt^2 + (1 - \frac{r_h}{r})^{-1}dr^2 + r^2 d\Omega_2^2$

- Spherically symmetric,non-rotating. Radius of event horizon $r_h = 2GM/c^2$. For an object to be black hole, $r_h \gg \lambda_c$ where λ_c is the compton wavelength.
- Surface gravity κ is the force required by a faraway observer to hold a unit mass at the horizon. For Schwarzschild BH, $\kappa = \frac{c^4}{4GM}$.


distant
astronomer
r = 0
singularity
event
horizon
r = 2M
light
cones
outgoing
light
rays
ingoing
light
rays
R_3
R_2
R_1
E_4
E_3
E_2
E_1
Time
Space
surface of the star

Black Hole thermodynamics

Classical black holes satisfy several theorems which are tantalizingly like laws of thermodynamics

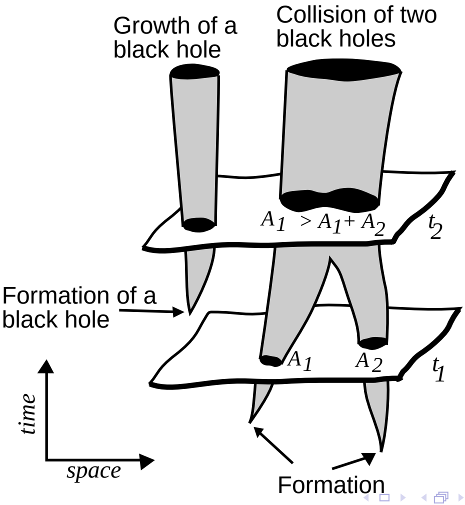
Laws of Thermodynamics

- Zeroth Law: T constant throughout body in thermal equilibrium
- First Law: $dE = TdS + workterms$
- Second Law: Change in Entropy $\delta S \geq 0$ in any process.
- Third Law: Impossible to achieve $T = 0$ in physical processes

Laws of Black Hole Mechanics

- Zeroth Law:Surface gravity κ is constant over the horizon of a stationary black hole.
- First Law: $dM = \frac{\kappa}{8\pi G} dA + \omega_h dJ + \Phi_e dQ$
- Second Law: Change in the area of the event horizon $\delta A \geq 0$ always increases in any classical process.
- Third Law:It is impossible to achieve $\kappa = 0$ in finite number of steps.

Area Law


Growth of a black hole
Collision of two black holes
$A_1 > A_1 + A_2$
t_2
Formation of a black hole
A_1
A_2
t_1
time
space
Formation

Semi-classical Black Holes

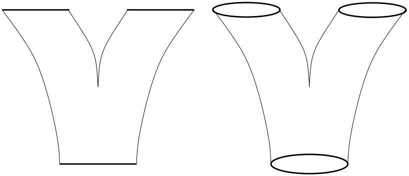
Black Holes and Second Law of Thermodynamics

One can violate second law of thermodynamics in observable universe by throwing stuff into black holes.

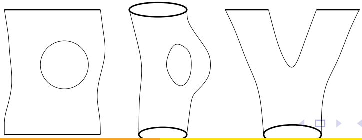
- Based on analogy of black holes with laws of thermodynamics, Beckenstein proposed to save second law by assigning black hole an entropy proportional to area.
- Hawking coupled quantum matter to a classical black hole and showed that they emit black body radiation at a temperature $T = \frac{h\kappa}{2\pi c}$. For Schwarzschild BH, $T \approx 6 \times 10^{-8}(M_{sun}/M)K$
- Given the black hole temperature, first law of BH mechanics assigns an entropy $S_{BH} = \frac{Ac^3}{4hG}$ to black hole. $S_{total} = S_{matter} + S_{BH}$ obeys the second law.
- In conventional statistical mechanics, entropy of a system has a microscopic explanation. $S = \ln(d_{micro})$. Here d_{micro} is the number of (quantum) microstates available to the system for a given set of macroscopic charges like energy, total electric charge etc. Huge entropy of the black hole implies that it should have large number of microstates. For $M = M_{sun}$, no. of d.o.f is $\sim 10^{10^{78}}$!

Outline

- String theory posits that the fundamental degrees of freedom are string-like extended objects(closed and open) instead of point like elementary particles Different elementary particles arise as different oscillation modes of string. Graviton arises as one such oscillation.



- Quantization leads to extra spatial dimensions(10 dimensions when quantization is done in flat space).
- Newton's constant $G \sim g_s^2 l_s^8$ in 10 dimensions where g_s and l_s are string coupling and string length respectively. Basic string interaction is splitting and joining of strings and is controlled by g_s (perturbative description possible when $g_s \ll 1$).



String Theory and Supersymmetry

- String theory has supersymmetry and in (extended) supersymmetric theories, massive states satisfy BPS bound $M \geq Q$.
- States saturating BPS bound belong to short multiplet, preserve some supersymmetries and are stable. BPS relation doesn't change as moduli (like string coupling) are varied and the dimension of the multiplet(hence the counting of BPS states) also doesn't change.
- As long as joining of multiplets doesn't happen, one can follow BPS states from weak to strong coupling.
- String theory contains other extended objects of various dimensionalities also. Important ones are D-branes which are non-perturbative objects with $M \sim \frac{1}{g_s}$. At weak coupling they are described by surfaces on which open strings can end.

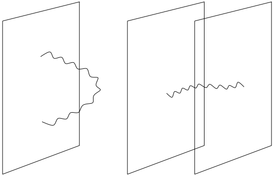


Fig 3a Fig 3b

Breakthrough

Breakthrough came in 1996 when Strominger and Vafa considered a charged black hole made of D-branes.

- Charged Black Holes are astrophysically not very viable but are theoretically interesting since they satisfy a bound $M \geq Q$.
- Black Holes with zero temperature are called extremal and are often, but not always, supersymmetric. BPS blackholes are stable, even quantum-mechanically. Since they cease to Hawking radiate, the notion of degeneracy is also better defined.
- In string theory, black holes made of strings and D-branes are strong coupling analogs of BPS states. One can calculate their entropy by adiabatically varying the coupling
- Start with an extremal black hole and calcualte it's entropy $S = \frac{A}{4G}$. Then imagine reducing the coupling g_s to a regime where one obtains a weakly coupled system of strings and branes and one can count the number of BPS states.
- For a wide variety of extremal and near-extremal black holes in various dimensions,we get a perfect match between the macroscopic and microscopic entropy calculations. Not only the entropies but rate of radiation and slight deviations from thermal spectrum also match.

Moduli

- Black holes appear when string theory is compactified to lower dimensions and when branes are wrapped on non-trivial cycles of the compact manifold (so that final configuration is point like).
- In string theory, parameters of a consistent string background are determined by the vacuum expectation of scalar fields.
- These parameters (called moduli) include various coupling constants,shape and size of compactification manifold, expectation values of Wilson lines of gauge fields around non-trivial cycles etc.
- These moduli appear as part of the black hole solutions, which turn out to exist for generic values of asymptotic values of these moduli.
- If black hole entropy depended on these continuous parameters(moduli) then it would be problematic for a microscopic interpretation of entropy as logarithm of the no. of microstates.

Attractors

- It turns out that the entropy of an extremal black hole is determined by the behavior of the solution at the horizon of the black hole and not at infinity.
- Moduli field vary with the radius in such a way that their values at the horizon are completely determined by the discrete quantities like charges, regardless of their values at infinity.
- Hence, the differential equations for radial dependence of moduli have attractor solution..
- Existence of an attractor is necessary for a microscopic description of black hole entropy to be possible. Since string coupling is one of the moduli, this guarantees that entropy wouldn't depend on it and hence can be matched between strong and weak coupling regimes.

Outline

Agenda

- In string theory, Beckenstein-Hawking formula is only an approximate formula (valid when string theory reduces to classical general relativity).
- It works well when charges carried by black hole are large and hence the curvature at the horizon is small. On the microscopic side also, the large charge limit allows us to use simple approximation techniques to calculate the degeneracy.

Can we do better? We need to have methods to calculate both macroscopic (black hole) entropy and microscopic (statistical) degeneracy beyond the leading order.

Helicity Trace

In string theoretic approach to black holes, one typically computes an index, rather than degeneracy on the microscopic side. For dyonic black holes that we consider, helicity trace is the index. It is protected in the sense that it does not change continuously as we vary the asymptotic moduli.

- For a black hole that breaks $4n$ supersymmetries, we define

$$B_{2n} = \frac{1}{(2n)!} Tr(-1)^{2h}(2h)^{2n} \tag{3.1}$$

  where h is the helicity(eigenvalue of the diagonal generator of the little group) and the trace is over bosonic and fermionic states of the representation (fixed charge).
- For each pair of broken supersymmetries, we have a pair of fermion zero modes which give a bose-fermi pair of degenerate states after quantization. Witten index $Tr(-1)^{2h}$ on such a pair of states will vanish unless we prevent it by inserting factors of $2h$.
- In order that a given state gives a non-vanishing contribution to this index, the number of supersymmetries broken by the state must be less than or equal to $4n$.
- In the classical limit, index matches with the Beckenstein-Hawking formula for the degeneracy.

- In full quantum string theory, black hole entropy formula is modified due to stringy and quantum corrections.
- Typically in string theory, a black hole is characterized by multiple charges. Quantum corrections and stringy corrections are characterized by different combination of charges. Depending on the value of charges, either stringy corrections or quantum corrections or both may be important.
- General Relativity is governed by an action which involves derivatives of the relevant field-metric tensor

$$I = \frac{1}{16\pi G}\int \sqrt{g} R_{abcd} g^{ac} g^{bd} \tag{3.2}$$

- String Theory predicts extra terms (stringy corrections) involving higer derivatives in the action,apart from the two derivative action.

$$I = \frac{1}{16\pi G}\int \sqrt{g}\left(R_{abcd} g^{ac} g^{bd} + c_1 (R_{abcd} g^{ac} g^{bd})^2 + c_2 R_{abcd} R^{abcd} + ..\right) \tag{3.3}$$

- If we adjust the charges so that we can ignore quantum corrections then we have an exact formula, due to Wald, which gives entropy

$$S_{BH} = 2\pi \int_{S^2} \epsilon_{\mu\nu}\epsilon_{\rho\lambda} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\lambda}} d^2\Omega \tag{3.4}$$

This can be compared with microscopic results and it agrees in all cases studied so far.

Microscopic Approach

- Count the states in the quantum hilbert space of string theory with the same charges and mass as the black holes. In general, quite complicated, but sometimes we can do exact counting.
- Degeneracy of states in string theory is,in many cases, given by fourier coefficients of modular forms. For states with just electric charge Q (these are half BPS in $\mathcal{N} = 4$ theories), the degeneracy $\Omega(Q) = c(Q^2/2)$ where

$$Z(q) = \frac{1}{q \prod_n (1 - q^n)^{24}} = \sum_N c(N) q^N \tag{3.5}$$

$$\Omega(Q) = \oint d\sigma \frac{e^{-\pi i Q^2 \sigma}}{\eta^{24}(\sigma)} \tag{3.6}$$

  where $\eta(\sigma)$ is the Dedekind eta-function and $q = e^{2\pi i \sigma}$.
- For 4D dyonic black holes(with both electric and magnetic charges) in $\mathcal{N} = 4$ theories, degeneracy given in terms of fourier coefficients of Siegel Modular form

$$Z(q,p,y) = \frac{1}{\Phi(q,p,y)} = \sum c(N,M,L) q^N p^M y^L \tag{3.7}$$

$$\Omega(Q,P) = c(Q^2/2, P^2/2, Q \cdot P) \tag{3.8}$$

Matching

So the general approach for dyonic black hole entropy calculations in string theory is following:

- Calculate the entropy from the effective action including subleading corrections. Classical corrections to $A/4G$ are given by Wald's formula. For quantum corrections, Ashoke Sen's Quantum entropy function is one proposal.
- Compute the degeneracy from the asymptotic expansion of the fourier coefficients of Siegel Modular form for large charges.
- Compare these two completely different computations and see if they match. For $N = 4$ strings, we find matching with great precision to several subleading orders. Even many exponentially suppressed terms can be matched.

Outline

Expectations

Best hope to have an exact relation of the form $d_{macro} = d_{micro}$ is for extremal black holes. In many cases, d_{micro} (rather an index) is exactly known. The macroscopic entropy of an extremal black hole is determined completely by its near horizon geometry.

Arguments

- The Beckenstein-Hawking entropy of a BH is proportional to the area of its event horizon. Wald's modification to include higher-derivative corrections still uses only horizon data (classical lagrangian density in the near-horizon region). Quantum Entropy function also is on near-horizon geometry only.
- For extremal black holes, an infinite throat separates the horizon from the rest of the black hole spacetime.

Conclusion: Two different black holes with identical near-horizon geometries should have identical macroscopic and microscopic degeneracies.

A trivial counter-example

Due to attractor mechanism, near-horizon geometry is independent of asymptotic values of moduli. But microscopic degeneracy jumps across the walls of marginal stability as we vary the asymptotic moduli.

- Resolution is provided by the existence of multicentered black holes with same set of charges as single-centered black holes.
- As we cross a wall of marginal stability, some of these multicentered black holes cease to exist and hence cause a jump in the degeneracy.

Reformulation : String Theory in the near-horizon geometry captures information about microscopic degeneracy of single centered black holes only.

A non-trivial Counter-example

The postulate that two BHs with identical near horizon geometry will have identical degeneracies seems to be violated by the following example:

(1) *BMPV Black Hole*

- Microscopic description - A D1-D5 system of IIB on $K3 \times S^1 \times 5D$ flat spacetime, carrying momentum along S^1 and equal angular momentum in two planes transverse to the D5-brane.
- Macroscopic description - A 5D rotating black hole.

VERSUS

(2) *Four-dimensional Rotating Black Hole*

- Microscopic description - BMPV moving in a Kaluza-Klein monopole background whose microscopic degenaracies are different from just BMPV due to modes on taub-nut and relative motion between BMPV and taub-nut.
- Macroscopic description - A 4D black hole with same near horizon geometry as BMPV.

Puzzle and Resolution

Two black holes have same near horizon geometries but different microscopic degeneracy in subleading order.

Proposed Resolution

- Microscopic degeneracy computation captures contribution from d.o.f. living outside the horizon as well. These outside d.o.f are called "hair".
- For 2 BHs with identical near horizon geometry, we should get identical degeneracy after we remove these "hair d.o.f.". For single-centered black holes, $d_{micro} = d_{hair} * d_{horizon} = d_{macro}$
- For supersymmetric black holes, hair can identified as classical, supersymmetric, normalizable deformations of the black hole solution with support outside the horizon. One can then do geometric quanitzation over this space of solutions to get d_{hair}.

Outline

Microscopic Components

We describe a **quarter BPS** state with 12 out of 16 SUSYs broken. The microscopic description involves Q_5 D5-branes wrapped on $K3 \times S^1$ and Q_1 D1-branes wrapped on S^1 carrying $-n$ units of momentum along S^1 (with $n > 0$) and J units of angular momentum.

- We take $Q_5 = 1$ without any loss of generality as the result depends through the combination $Q_5(Q_1 - Q_5)$.
- We denote the $SO(4)$ rotation group of $5D$ space-time by $SU(2)_L \times SU(2)_R$.
- We identify the angular momentum J with twice the diagonal generator of $SU(2)_L$.
- We denote by h the eigenvalue of the diagonal generator of $SU(2)_R$.
- We choose the convention that left-chiral spinors of $SO(1,1)$ carry $(J = 0, 2h = \pm 1)$ and right-chiral spinors of $SO(1,1)$ carry $(J = \pm 1, h = 0)$.

Partition Function

We define partition function of 5D black hole with quantum numbers (n, Q_1, J)in terms of helicity trace $d_{5D}(n, Q_1, J) = Tr\left((-1)^{2h+J}(2h)^2\right)/2!$ as

$$Z_{5D}(\rho, \sigma, v) = \sum_{n, Q_1, J} d_{5D}(n, Q_1, J) \exp\left[2\pi i\{(Q_1 - 1)\sigma + (n-1)\rho + vJ\}\right]$$

- The -1 in $Q_1 - 1$ is because $D5$ brane on K3 carries -1 unit of $D1$ charge.
- The -1 in $n - 1$ is because charge at infinity and horizon differ due to chern-simons coupling in the action.

Explicit calculation gives

$$\begin{aligned} Z_{5D}(\rho, \sigma, v) \quad = \quad & e^{-2\pi i\rho - 2\pi i\sigma} \prod_{\substack{k,l,j\in \mathbf{Z} \\ k\geq 1, l\geq 0}} \left(1 - e^{2\pi i(\sigma k + \rho l + vj)}\right)^{-c(4lk-j^2)} \\ & \times \left\{ \prod_{l\geq 1} (1 - e^{2\pi i(l\rho + v)})^{-2} (1 - e^{2\pi i(l\rho - v)})^{-2} (1 - e^{2\pi i l\rho})^4 \right\} \\ & \times (-1)\left(e^{\pi i v} - e^{-\pi i v}\right)^2 \end{aligned} \qquad (5.9)$$

First line above is due to relative motion of the D1-D5 system (elliptic genus of symmetric products of K3's) while rest is the contribution of centre of mass modes of D1-D5 system.

C.O.M contribution

Ground state of D1-D5 system breaks four translational symmetries and 8 susys. This gives 4 goldstone bosons and 8 goldstinos. BPS condition freezes all right moving excitations except the zero modes. Centre of mass contribution then consists of following :

1. 4 left moving bosons carrying $J = \pm 1$.
2. 4 left-moving fermion zero modes ($J = 0, 2h = \pm 1$) (soak up $(2\,h)^2/2!$ in helicity trace) and 4 right-moving ($J = \pm 1, 2h = 0$) fermion zero-modes from the SUSYs broken by $D1 - D5$ ground state.
3. After soaking the zero modes, helicity trace reduces to Witten index for 4 left-moving fermionic fields

Classical Solution

Since the microscopic counting does not distinguish between hair and horizon d.o.f. we need the macroscopic description.

Metric

The solution has following 6D metric :

$$\begin{aligned} dS^2 &= \left(1+\frac{r_0}{r}\right)^{-1}\Big[-dt^2+(dx^5)^2+\frac{r_0}{r}(dt+dx^5)^2+ \\ &\quad +\frac{\widetilde{J}}{4r}\,(dt+dx^5)\,(dx^4+\cos\theta\, d\phi)\Big] \\ &\quad +\left(1+\frac{r_0}{r}\right)\, ds^2_{flat} \end{aligned}$$

Here, we set all scalar fields to fixed values with their asymptotic values equal to attractor values.

$$e^{\Phi}=\lambda \quad , \quad r_0=\frac{\lambda(Q_1-Q_5)}{4V}=\frac{\lambda Q_5}{4}=\frac{\lambda^2|n|}{4R_5^2 V}\ , \ \tilde{J}=\frac{J\lambda^2}{2R_5 V}$$

Hair Degrees of Freedom

ZERO MODES

Since Black hole breaks translational symmetry and 12 out of 16 SUSY's, we expect following zero modes on black hole world-volume.

- 4 bosonic zero modes from 4 broken translational symmetries.
- 4 left-chiral fermion zero modes (which soak up $(2\,h)^2/2!$ in the helicity trace) and 8 right-chiral fermionic zero modes which, when quantized, give a factor $(\exp \pi i v - \exp -\pi i v)^4$.

LEFT-MOVING FIELDS

Given a zero mode, we explore if it is possible to lift it to a full fledged field in $(1+1)$ dimensions spanned by the coordinates (t, x^5).

- g_{++} and g_{+i} components of the metric which violate $(1+1)$ dimensional Lorentz invariance don't give rise to g^{++} or g^{+i} component and the left-moving fields φ, for which $\partial_- \varphi = 0$ do not couple to g^{--} or g^{-i}, continue to describe solutions to linearized e.o.m. around the black hole background.

So black-hole world-volume has 4 left-moving bosonic fields and fermion fields.

Partition Function for Hair d.o.f.

Bosonic modes turn out to be singular at the horizon. So we don't count those. The total contribution to the partition function from the degrees of freedom living outside the horizon is thus :

$$\begin{aligned} Z^{hair}_{5D}(\rho,\sigma,v) &= (e^{\pi i v} - e^{-\pi i v})^4 \\ &\times \prod_{l\geq 1}(1-e^{2\pi i l\rho})^4 \end{aligned}$$

so that we have the relation

$$\begin{aligned} Z^{hor}_{5D}(\rho,\sigma,v) &= Z_{5D}/Z^{hair}_{5D} \\ &= -e^{-2\pi i\rho-2\pi i\sigma}\,(e^{\pi i v}-e^{-\pi i v})^{-2} \prod_{\substack{k,l,j\in\mathbf{Z}\\ k\geq 1, l\geq 0}} \left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)} \\ & \left\{\prod_{l\geq 1}(1-e^{2\pi i(l\rho+v)})^{-2}\,(1-e^{2\pi i(l\rho-v)})^{-2}\right\}. \qquad (5.10) \end{aligned}$$

Outline

Microscopic Components

Partition Function

- We define the partition function of the 4D BH with quantum numbers (n, Q_1, J) in terms of the *sixth* helicity trace $d_{4D}(n, Q_1, J) = Tr((-1)^{2h+J}(2h)^6)/6!$ as :

$$Z_{4D}(\rho, \sigma, v) = \sum_{n, Q_1, J} d_{4D}(n-1, Q_1, J) \exp\left[2\pi i\{(Q_1 - 1)\sigma + (n-1)\rho + vJ\}\right]$$

- We have $n-1$ in the argument of d_{4D} because charge measured at horizon and infinity agree for 4D black hole.

Explicit computation gives

$$Z_{4D}(\rho, \sigma, v) = -e^{-2\pi i\rho - 2\pi i\sigma - 2\pi i v} \prod_{\substack{k,l,j \in \mathbb{Z} \\ k,l \geq 0, j<0 \text{ for } k=l=0}} \left(1 - e^{2\pi i(\sigma k + \rho l + vj)}\right)^{-c(4lk - j^2)}$$

The $e^{-2\pi i\rho}$ factor is due to ground state of the Taub-Nut carrying -1 unit of momentum along S^1 (due to higher derivative terms)

4D BH Solution

- The 4D BH is obtained by placing the 5D BMPV BH at the center of Taub-NUT space. The metric is given by

$$\begin{aligned} dS^2 \quad &= \quad \left(1+\frac{r_0}{r}\right)^{-1}\left[-dt^2+(dx^5)^2+\frac{r_0}{r}(dt+dx^5)^2\right. \\ &\left. +\frac{\widetilde{J}}{4}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)(dx^4+\cos\theta\, d\phi)(dt+dx^5)\right]+\left(1+\frac{r_0}{r}\right)\, ds_{TN}^2 \end{aligned}$$

- It has the same near horizon geometry (metric) as the BMPV BH.
- Black hole admits a normalizable two form, inherited from Taub-Nut

$$\omega=-\frac{r}{4r+R_4^2}\sin\theta d\theta\wedge d\phi+\frac{R_4^2}{(4r+R_4^2)^2}dr\wedge(dx^4+\cos\theta d\phi)$$

Hair Degrees of Freedom

ZERO MODES

- 3 bosonic zero modes from 3 broken translational symmetries.
- 4 left-chiral and 8 right-chiral fermionic zero modes from 12 broken spacetime SUSYs which soak up $(2\,h)^6/6!$ in the helicity trace.

Black hole solution admits a normalizable closed two-form. Any 2-form along this gives rise to a scalar mode.

a. 1 scalar from NS-NS 2-form field
b. 1 scalar from R-R 2-form field
c. 22 scalars from 19 left-chiral and 3 right chiral 2-form fields from the 4-form field with self-dual field strength reduced on the 22 internal cycles of $K3$.

By arguments similar to 5D black hole case, right movers are frozen but left-movers are full fields on worldvolume.

Hair 4D

- $3 + 21 = 24$ left-moving scalars
- 4 left-moving fermionic fields

Partition Function for Hair d.o.f.

Four fermionic modes cancel the contribution from four of the bosonic modes. So only 20 bosons contribute.

Hair Partition function

The total contribution to the partition function from the degrees of freedom living outside the horizon is :

$$Z_{4D}^{hair}(\rho, \sigma, v) = \prod_{l=1}^{\infty} \left(1 - e^{2\pi i l \rho}\right)^{-20}$$

Using the relation

$$Z_{4D} = Z_{4D}^{hor} \times Z_{4D}^{hair}$$

Partition Function for Horizon d.o.f.

The partition function obtained for the horizon d.o.f. of the 4D BH is :

$$
\begin{aligned}
Z_{4D}^{hor}(\rho,\sigma,v) &= Z_{4D}/Z_{4D}^{hair} \\
&= -e^{-2\pi i\rho-2\pi i\sigma}\left(e^{\pi iv}-e^{-\pi iv}\right)^{-2}\prod_{\substack{k,l,j\in\mathbf{Z}\\ k\geq 1,l\geq 0}}\left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)} \\
&\quad \left\{\prod_{l\geq 1}\left(1-e^{2\pi i(l\rho+v)}\right)^{-2}\left(1-e^{2\pi i(l\rho-v)}\right)^{-2}\right\}
\end{aligned}
$$

where is the last step we have used $c(-1)=2$, $c(0)=20$.

which is the SAME as that obtained for the BMPV BH.

Outline

Transverse Oscillations of BMPV

Garfinkle-Vachaspati technique

Given a space-time with metric G_{MN} satisfying the supergravity equations and a null, killing and hypersurface orthogonal vector field k_M, *i.e.*, satisfying the following properties

$$k^M k_M = 0, \ k_{M;N} + k_{N;M} = 0, \ k_{M;N} = \frac{1}{2}(k_M A_{,N} - k_N A_{,M})$$

for some scalar function A, one can construct a new exact solution of the equations of motion by defining

$$G'_{MN} = G_{MN} + e^{-A} T k_M k_N$$

where the function T satisfies

$$\nabla^2 T = 0 \ , \ k^M \partial_M T = 0$$

The new metric G'_{MN} describes a gravitational wave on the background of the original metric provided the matter fields, if any, satisfy some conditions.

Applying the transformations to BMPV metric, we get

$$ds^2 = \psi^{-1}(r)\left\{dudv + (\psi - 1 + T(v,\vec{w}))dv^2 + \chi_i(r)dvdw_i\right\} + \psi(r)dw_i dw_i$$

where $r = (w_i w_i)/4$ and $T(v,\vec{w})$ satisfies the flat four dimensional Laplace equation:

$$\partial_{w^i}\partial_{w^i} T(v,\vec{w}) = 0$$

Matter fields remain unchanged. Demanding asymptotic flatness and regularity, we get

transverse oscillations metric

$$ds^2 = \psi^{-1}(r)\left[dudv + \left\{\psi - 1 + \vec{f}(v)\cdot\vec{w}\right\}dv^2 + \chi_i(r)dvdw_i\right] + \psi(r)dw_i dw_i$$

This doesn't look asymptotically flat but can be made by coordinate transformation.

Fermionic Deformations

Gravitino

Since the black hole solution breaks 12 of the 16 space-time supersymmetries, there are 12 fermionic zero modes.

- Four of these lift to full left moving fields on the two dimensional world volume of black hole.

The linearized equation of motion of Ψ_M^α and $\chi^{\alpha r}$ in the background where all the scalars are constants and $\chi^{\alpha r}$ are set to zero are

$$\Gamma^{MNP} D_N \Psi_P^\alpha - \bar{H}^{kMNP} \Gamma_N \widehat{\Gamma}^k_{\alpha\beta} \Psi_P^\beta = 0,$$
$$H^{sMNP} \Gamma_{MN} \Psi_P^\alpha = 0$$

Since background is self-dual, $H^{sMNP} = 0$, second equation is automatically solved.

To solve, gravitino equation, we make the ansatz

- $$\Psi_M^\alpha = 0 \quad \text{for} \quad M \neq v$$
- Gauge condition
$$\Gamma^M \Psi_M^\alpha = 0 \quad \rightarrow \quad \Gamma^v \Psi_v^\alpha = 0$$

We also have projection condition

$$\partial_u \Psi_v^\alpha = 0 \quad , \quad \widetilde{\Gamma}^0 \, \widetilde{\Gamma}^1 \, \Psi_v^\alpha = \Psi_v^\alpha$$

Gravitino Solution

Solving the gravitino equation,we get following solutions

$$\begin{aligned} \Psi_v &= \psi^{-3/2}\, \eta(v,\theta,\phi) \quad \text{for} \quad \widehat{\Gamma}^1 \, \eta = -\eta \,, \\ \Psi_v &= \psi^{-1/2}\, \eta(v,\theta,\phi) \quad \text{for} \quad \widehat{\Gamma}^1 \eta = \eta \end{aligned}$$

where $\eta(v,\theta,\phi)$ is an SO(5,1) spinor and also an $SO(5)$ spinor. Only the first solution preserves supersymmetry.

Bosonic deformations representing transverse oscillations

Fermionic deformations are exactly same as for BMPV in flat space. So we only discuss bosonic deformations. Deformations describing the oscillation of the black hole in the three transverse non-compact direction.

Garfinkle-Vachaspati in 4D

$$ds^2 = \psi^{-1}(r)\left[du\,dv + \left(\psi(r) - 1 + \widetilde{T}(v,\vec{y},x^4)\right)dv^2 - 2\widetilde{\zeta}\,dv\right] + \psi(r)\,ds^2_{TN}$$

Here

$$\widetilde{T}(v,\vec{y},x^4) \equiv \widetilde{T}(v,\vec{y}) = \vec{g}(v)\cdot\vec{y}\,, \qquad \int_0^{2\pi R_5} g_i(v)dv = 0$$

where $(g_1(v), g_2(v), g_3(v))$ are three arbitrary functions.

Oscillations of the 2-form fields

Taub-NUT space has a self-dual harmonic form ω_{TN} given by

$$\omega_{TN} = -\frac{r}{4r + R_4^2} \sin\theta d\theta \wedge d\phi + \frac{R_4^2}{(4r + R_4^2)^2} dr \wedge (dx^4 + \cos\theta d\phi)$$

Metric Perturbation due to 2-form

We now switch on a deformation of the form

$$\delta(ds^2) = \psi^{-1}(r) \left(\widetilde{S}(v, \vec{y}, x^4) \right) dv^2, \qquad \delta H^s = h^s(v)\ dv \wedge \omega_{TN}$$

where $h^s(v)$ are arbitrary functions Following solution describes a normalizable deformation of the metric, outside the horizon:

$$\widetilde{S}(v, \vec{y}, x^4) = \frac{C(v) r}{2R_4^2 (4r + R_4^2)}$$

Killing spinors and Fermion-Zero Modes

Besides the hair modes described above, both black holes carry twelve fermionic zero modes associated with the broken supersymmetry generators.

Fermion Zero modes in SUGRA

- We take a local supersymmetry transformation whose parameter approaches a constant spinor other than the Killing spinor at infinity and vanishes at the horizon, and apply it to the original black hole solution to generate a fermionic zero mode.
- Since there are 12 independent supersymmetry transformations whose parameters do not approach a Killing spinor at infinity, this generates 12 fermion zero modes.

The Killing spinor equation in the BMPV black hole and BMPV black hole in the Taub-NUT space, obtained by setting $\delta\Psi_M^\alpha = 0$, is

$$D_M \epsilon - \frac{1}{4}\bar{H}^i_{MNP}\Gamma^{NP}\widehat{\Gamma}^i \epsilon = 0$$

where $\bar{H}^i_{MNP}$ for $1 \leq i \leq 5$ are self-dual field strengths of 2-form fields in six dimensions.

Killing spinors

We find the following solutions:

$$\epsilon = \psi(r)^{-1/2}\, e^{i\phi/2}\, \epsilon^1_\theta, \qquad \epsilon = \psi(r)^{-1/2}\, e^{-i\phi/2}\, \epsilon^2_\theta$$

where $\epsilon^1_\theta = (\cos\theta/2\ ,\ \ \sin\theta/2)^T$ and $\epsilon^2_\theta = (\cos\theta/2\ ,\ \ -\sin\theta/2)^T$. Killing spinors for both flat space and TN are same in our conventions.

TN Killing spinors

- The Taub-NUT space has $SU(2)$ holonomy, which by convention is identified with $SU(2)_L$ subgroup of its $SO(4)$ tangent space symmetry.
- Fermions in the Taub-NUT space transform as $(2,1)+(1,2)$ under $SO(4) = SU(2)_L \times SU(2)_R$. Thus half of the fermions are neutral under $SU(2)_L$ and hence behave as free fermions as far as the Taub-NUT space is concerned.
- In our conventions, $SU(2)_L$ singlets are left moving with respect to $SO(1,1)$ and $SU(2)_R$ singlets are right moving. Since Killing spinors corresponds to unbroken supersymmetry, which in our convention are left moving spinors of $SO(1,1)$, they are singlet of the Taub-NUT holonomy group $SU(2)_L$.
- As a result the Killing spinors are unaffected when we replace flat space by the Taub-NUT space.

Outline

Non-Singular coordinates for the Black Hole

We have studied BMPV and 4d black holes in Schwarzschild-type or isotropic coordinates. In these coordinates, there is a coordinate singularity at the horizon , at $r \to 0$ and $u, v \to \infty$. Are the hair modes regular at the horizon?

The original BMPV metric may be expressed as

$$ds^2 = \psi^{-1}(dudv + Kdv^2) + \psi\left(r^{-1}dr^2 + 4\,r\,d\Omega_3^2\right)\,, \quad d\Omega_3^2 \equiv \frac{1}{4}\left((dx^4 + \cos\theta d\phi)^2 + d\Omega_2^2\right)\,, \tag{8.11}$$

where,

$$\psi = 1 + \frac{r_0}{r}, \qquad K = \psi - 1\,. \tag{8.12}$$

We will now do the following coordinate transformation:

$$V = -\sqrt{r_0}\exp(-\frac{v}{\sqrt{r_0}}), \qquad W = \frac{1}{R}\exp(\frac{v}{2\sqrt{r_0}}), \qquad U = u + \frac{R^2}{2\sqrt{r_0}} + 2v\,, \tag{8.13}$$

$$R \equiv 2\sqrt{r_0\left(1 + \frac{r_0}{r}\right)}\,. \tag{8.14}$$

Note that the region outside the horizon has $V < 0$.

In these new coordinates the metric becomes

$$\begin{aligned}\frac{ds^2}{4r_0} &= W^2 dUdV + dV^2 r_0 W^4 Z^{-3}(24 + 128\sqrt{r_0}VW^2 + 192 r_0 V^2 W^4) \\ &\quad - dVdW 4\sqrt{r_0} W Z^{-3}(3 + 12\sqrt{r_0}VW^2 + 16 r_0 V^2 W^4) + W^{-2}Z^{-3}dW^2 + Z^{-1}d\Omega_3^2\end{aligned}$$

Here $Z = 1 + 4\sqrt{r_0}VW^2$. It is now easy to see that the metric is regular at the future horizon $V = 0$. In fact the metric components are polynomials in V and therefore they are analytic functions of V. Thus all derivatives of the metric components, and hence the Riemann tensor, remain finite at the horizon for finite W.

Regularity Analysis

Steps

- Usually to show regularity, one calculates the curvature invariants. But for our case, curvature invariants before and after adding deformations are all same!! Still there is possibility of null singularities.
- A way to characterize such singularities is to calculate Riemann tensor in a frame which is parallely propagated along a geodesic. This quantity occurs in geodesic deviation equation.
- For complicated metrics like ours, finding geodesics is difficult. Thankfully, for checking curvature singularities, it suffices to use a frame continuous across the horizon and calculate Riemann tensor in that frame.
- For all the modes, we can find non-singular coordinates such that modes are continuous at the horizon.
- When we calculate Riemann tensor, we find that transverse oscillations of 5d BMPV black hole lead to divergence in some componenets of Riemann tensor. It's counterpart in 4d, the relative motion between BMPV and taub-nut is also singular.

Outline

Conclusion

- Removing the divergent modes from 4d and 5d black hole hair counting, we get the exact matching of partition function associated with horizon d.o.f
- Future Directions: Hair modes in other systems? Connection with fuzzballs? Geometric Quantization of these modes ?

To conclude : The equality of Z_{4D}^{hor} and Z_{5D}^{hor} shows that the microscopic and macroscopic degeneracies associated with the near horizon degrees of freedom of the 4D and the 5D black holes are identical.