Localization on twisted spheres and supersymmetric GLSMs

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Supersymmetric gauge theories in two dimensions

Two-dimensional supersymmetric gauge theories—a.k.a. GLSM—are an interesting playground for the quantum field theorist.

- They exhibit many of the qualitative behaviors of their higher-dimensional cousins.
- Supersymmetry allows us to perform exact computations.
- They provide useful UV completions of non-linear σ-models, including conformal ones, and of other interesting 2d SCFTs.
- Consequently, they are useful tools for string theory and enumerative geometry:
  - $\mathcal{N} = (2, 2)$ susy: IIB string theory compactifications.
  - $\mathcal{N} = (0, 2)$ susy: heterotic compactifications.

GLSM Observables

Consider a GLSM with at least one $U(1)$ factor. We have the complexified FI parameter

$$\tau = \frac{\theta}{2\pi} + i\xi$$

which is classically marginal in 2d.

Schematically, expectation values of appropriately supersymmetric local operators $\mathcal{O}$ have the expansion

$$\langle \mathcal{O} \rangle \sim \sum_k q^k Z_k(\mathcal{O}) \; , \qquad\qquad q = e^{2\pi i \tau} \; .$$

The 2d instantons are *gauge vortices*.

GLSM supersymmetric observables

We consider half-BPS local operators.

In the $\mathcal{N} = (2, 2)$ case, we have two choices (up to charge conjugation):

- $[\tilde{Q}_-, \mathcal{O}] = [\tilde{Q}_+, \mathcal{O}] = 0$, chiral ring.
- $[Q_-, \mathcal{O}] = [\tilde{Q}_+, \mathcal{O}] = 0$, twisted chiral ring.

The so-called "twisted" theories [Witten, 1988] efficiently isolate these subsectors: B- and A-twist, respectively. We will focus on the latter.

In the $(0, 2)$ case, half-BPS operators commute with a single supercharge and there is no chiral ring, in general. However, some interesting models share properties with the $(2, 2)$ case. We will discuss them in the second part of the talk.

$S^2_{\epsilon_\Omega}$ correlators for $(2,2)$ theories

We will consider correlations of twisted chiral ring operators on the Ω-deformed sphere,

$$\langle \mathcal{O} \rangle_{S^2_\Omega} \; .$$

This Ω-background constitutes a one-parameter deformation of the A-twist at genus zero.

We will derive a formula for GLSM supersymmetric observables on S^2_Ω of the schematic form:

$$\langle \mathcal{O} \rangle = \sum_k q^k \oint_{\mathcal{C}} d^r\sigma \, Z_k^{\text{1-loop}}(\sigma)\, \mathcal{O}(\sigma) \; ,$$

valid for any standard GLSM. This results simplifies previous computations [Morrison, Plesser, 1994; Szenes, Vergne, 2003] and generalizes them to non-Abelian theories.

Some further motivations

In field theory:

- These 2d $\mathcal{N} = (2,2)$ theories appear on the worldvolume of *surface operators* in 4d $\mathcal{N} = 2$ theories.
- Our 2d setup can also be uplifted to 4d $\mathcal{N} = 1$ on $S^2 \times T^2$.
  [C.C., Shamir, 2013, Benini, Zaffaroni, 2015, Gadde, Razamat, Willett, 2015]

In string theory or "quantum geometry":

- Think in terms of a target space X_d with $\xi \sim \mathrm{vol}(X_d)$. New localization results can give new tools for enumerative geometry.
  [Jockers, Kumar, Lapan, Morrison, Romo, 2012]
- The $(0,2)$ results are relevant for heterotic string compactifications.

Outline

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- Curved-space supersymmetry in 2d
- $(2,2)$ GLSM and supersymmetric observables
- Localization on the Coulomb branch
- Examples and applications
- Generalization to (some) $(0,2)$ theories with a Coulomb branch
- Conclusion

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Curved-space $(2,2)$ supersymmetry

The first step is to define the theory of interest in *curved space*, while preserving some supersymmetry. A systematic way to do this is by coupling to background supergravity. [Festuccia, Seiberg, 2011]

Assumption: The theory possesses a vector-like R-symmetry, $R_V = R$.

In that case, we have:

$$j_\mu^{(R)} \,, \qquad S_\mu \,, \qquad T_{\mu\nu} \,, \qquad j_\mu^{Z} \,, \qquad j_\mu^{\tilde{Z}}$$

$$A_\mu^{(R)} \,, \qquad \Psi_\mu \,, \qquad g_{\mu\nu} \,, \qquad C_\mu \,, \qquad \tilde{C}_\mu$$

A supersymmetric background corresponds to a non-trivial solution of the generalized Killing spinor equations, $\delta_\zeta \Psi_\mu = 0$.

Supersymmetric backgrounds in 2d

The allowed supersymmetric background are easily classified.

[C.C., Cremonesi, 2014]

For Σ a closed orientable Riemann surface of genus g:

- If $g > 1$, we need to identify $A_\mu^{(R)} = \pm\frac{1}{2}\omega_\mu$. Witten's A-twist.
- If $g = 1$, this is flat space.
- If $g = 0$, we have two possibilities, depending on

$$\frac{1}{2\pi}\int_\Sigma dA^R = 0, \pm 1$$

Supersymmetric backgrounds on S^2

On the sphere, we can have:

$$\frac{1}{2\pi}\int_{S^2} dA^R = 0 \,, \qquad \frac{1}{2\pi}\int_{S^2} dC = \frac{1}{2\pi}\int_{S^2} d\tilde{C} = 1$$

This was studied in detail in [Doroud, Le Floch, Gomis, Lee, 2012; Benini, Cremonesi, 2012]. In this case, the R-charge can be arbitrary but the real part of the central charge, $Z + \tilde{Z}$, is constrained by Dirac quantization.

The second possibility is

$$\frac{1}{2\pi}\int_{S^2} dA^R = 1 \,, \qquad \frac{1}{2\pi}\int_{S^2} dC = \frac{1}{2\pi}\int_{S^2} d\tilde{C} = 0$$

This is the case of interest to us. Note that the R-charges must be integers, while $Z, \tilde{Z}$ can be arbitrary.

Equivariant A-twist, a.k.a. Ω-deformation

Consider this latter case. We preserve two supercharges if the metric on S^2 has a $U(1)$ isometry with Killing vector V^μ. This gives a one-parameter deformation of the A-twist:

$$\mathcal{Q}^2 = 0 \,, \qquad \tilde{\mathcal{Q}}^2 = 0 \,, \qquad \{\mathcal{Q}, \tilde{\mathcal{Q}}\} = Z + \epsilon_\Omega \mathcal{L}_V \,.$$

The supergravity background reads:

$$ds^2 = \sqrt{g}(|z|^2) dz d\bar{z} \,, \qquad A_\mu^{(R)} = \frac{1}{2}\omega_\mu \,, \qquad C_\mu = \frac{1}{2}\epsilon_\Omega V_\mu \,, \qquad \tilde{C}_\mu = 0 \,.$$

Using the general results of [C.C., Cremonesi, 2014], we can write down any supersymmetric Lagrangian we want.

GLSMs: Lightning review

Let us consider 2d $\mathcal{N}=(2,2)$ supersymmetric GLSM on this S^2_Ω.

We have the following field content:

- Vector multiplets $\mathcal{V}_a$ for a gauge group G, with Lie algebra $\mathfrak{g}$.
- Chiral multiplets Φ_i in representations $\mathfrak{R}_i$ of $\mathfrak{g}$.

We also have interactions dictated by:

- A superpotential $W(\Phi)$
- A twisted superpotential $\hat{W}(\sigma)$, where $\sigma \subset \mathcal{V}$.

Assumption: The classical twisted superpotential is linear in σ:

$$\hat{W} = \tau^I \operatorname{Tr}_I(\sigma) \ .$$

That is, we turn on one FI parameter for each $U(1)_I$ factor in G.

The FI term often runs at one-loop:

$$\tau(\mu) = \tau(\mu_0) - \frac{b_0}{2\pi i} \log\left(\frac{\mu}{\mu_0}\right) ,$$

If $b_0 = 0$, we expect an SCFT in infrared.

This $\hat{W}$ preserves a $U(1)_A$ axial R-symmetry, broken to $\mathbb{Z}_{2b_0}$ by an anomaly if $b_0 \neq 0$.

Examples with $G = U(1)$

Example 1: $\mathbb{C}P^{n-1}$ model. With n chirals with $Q_i = 1$, $r_i = 0$.
τ runs at one-loop ($b_0 = n$), and there is a dynamical scale:

$$\Lambda = \mu q^{\frac{1}{n}} \ .$$

For $\xi \gg 0$, target space is $\mathbb{C}P^{n-1}$.

Example 2: The quintic model. 5 chirals x_i with $Q_i = 1$, $r_i = 0$, and one chiral p with $Q_p = -5$, $r_p = 2$, with a superpotential

$$W = pF(x_i)$$

F is homogeneous of degree 5.

$b_0 = 0$. For $\xi \gg 0$: quintic CY_3 in $\mathbb{C}P^4$.

Non-Abelian examples

Example 3: Grassmanian models. Consider a $U(N_c)$ vector multiplet with N_f chirals in the fundamental.

This non-Abelian GLSM flows to the NLσM on the Grassmanian $Gr(N_c, N_f)$.

The Grassmanian duality

$$Gr(N_c, N_f) \cong Gr(N_f - N_c, N_f)$$

corresponds to a Seiberg-like duality of the GLSMs.

We can also study new classes of CY manifolds inside Grassmanians (and generalizations thereof). [Hori, Tong, 2006; Jockers, Kumar, Morrison, Lapan, Romo, 2012]

Example 4: The Rødland CY_3 model. Consider $G = U(2)$ with 7 chirals Φ_i in the fundamental with $r_i = 0$ and 7 chirals P_α in the $\det^{-1}$ rep. with $r_\alpha = 2$. We have the baryons

$$B_{ij} = \epsilon_{a_1 a_2} \Phi_i^{a_1} \Phi_j^{a_2} \ ,$$

charged under the diagonal $U(1) \subset U(2)$. Let $G^\alpha(B)$ be polynomials of degree one in B_{ij}. We have a superpotential

$$W = \sum_{\alpha=1}^{7} P_\alpha G^\alpha(B)$$

The target space for $\xi \gg 0$ is a complete intersection in the Grassmanian $G(2,7)$ known as the Rødland CY_3.

Supersymmetric observables

When $\epsilon_\Omega = 0$, the only local operators (built from elementary fields) which are Q-closed and not Q-exact are

$$\mathcal{O}(\sigma) ,$$

the gauge-invariant polynomials in σ. Supersymmetry also ensures that the theory is topological. In particular:

$$\partial_\mu \langle \mathcal{O}_x \cdots \rangle = \langle \{Q, \cdots\} \rangle = 0 .$$

When $\epsilon_\Omega \neq 0$, instead:

$$[Q, \sigma] \sim \epsilon_\Omega \, V^\mu \Lambda_\mu .$$

Thus σ is only Q-closed at the fixed points of V.

Supersymmetric observables

We can insert $\mathcal{O}(\sigma)$ at the north or south poles of S^2_Ω:

$$\langle \mathcal{O}_N(\sigma)\mathcal{O}_S(\sigma)\rangle$$

This is what we shall compute explicitly, as a function of q and ϵ_Ω.

Note: One can write down a supersymmetric local term:

$$S = \int d^2x (F(\omega)R + \cdots) \sim F(\omega)$$

Thus, correlators $\langle \mathcal{O} \rangle$ are only defined up to an overall holomorphic function.

Localizations

Localization principle: For any $\mathcal{O}$ which is Q-closed,

$$\langle \mathcal{O} \rangle = \langle \, e^{t\, S_{\rm loc}}\, \mathcal{O} \rangle \qquad \text{if} \quad S_{\rm loc} = \{Q, \Psi_{\rm loc}\}\ .$$

Therefore, we can take $t \to \infty$ and *localize* the path integral on the saddle point configurations of $S_{\rm loc}$. The question is how to choose $S_{\rm loc}$.

We can consider two distinct localizations:

- "Higgs branch" localization: Sum over vortices.

  [Morrison, Plesser, 1994]

- "Coulomb branch" localization: Contour integral.

We will discuss the latter. The contour picks 'poles' on the Coulomb branch corresponding to the vortices.

“Coulomb branch” localization

Choose:

$$\mathscr{L}_{\text{loc}} = \mathscr{L}_{\text{YM}} \;.$$

Note: We also localize the matter sector with its standard kinetic term.

The saddles are on the Coulomb branch:

$$\sigma = \text{diag}(\sigma_a) \;, \qquad G \to H = \prod_{a=1}^{\text{rank}(G)} U(1)_a$$

There is a family of gauge field saddles for each allowed (GNO) flux:

$$k = (k_a) \in \Gamma_{\mathbf{G}^\vee}$$

When $\epsilon_\Omega \neq 0$, there is a non-trivial profile for σ

$$\sigma = \sigma(|z|^2) ,$$

related to the gauge flux by supersymmetry:

$$f_{1\bar{1}} = -\frac{1}{\epsilon_\Omega \sqrt{g}} \partial_{|z|^2} \sigma .$$

The important feature is that:

$$\sigma_N = \hat{\sigma} - \epsilon_\Omega \frac{k}{2} , \qquad \sigma_S = \hat{\sigma} + \epsilon_\Omega \frac{k}{2} ,$$

with $\hat{\sigma}$ a constant mode, over which we need to integrate.

In this localization scheme, we also have gaugino zero modes, $\lambda, \tilde{\lambda} = \text{constant}$.

The path integral reduces to a supersymmetric ordinary integral:

$$\langle \mathcal{O}_{N,S}(\sigma) \rangle \sim \sum_k \int d\lambda d\tilde{\lambda} \int dD \int d^2\hat{\sigma}\, \mathcal{Z}_k(\hat{\sigma}, \hat{\tilde{\sigma}}, \lambda, \tilde{\lambda}, D)\, \mathcal{O}_{N,S}(\sigma_{N,S})$$

We refrained from integrating over the constant mode of the auxiliary field D in the vector multiplet.

We have

$$\mathcal{Z}_k = e^{-S_{\rm cl}}\, \mathcal{Z}_k^{1-\rm loop} \,.$$

The one-loop term results from integrating out the chiral multiplets and the W-bosons. It can be computed explicitly by standard techniques.

The integration over the gaugino zero-modes can be performed implicitly by using the residual supersymmetry of $\mathcal{Z}_k$. We have

$$\delta\sigma = 0\,, \qquad \delta\tilde{\sigma} = \tilde{\lambda}\,, \qquad \delta\tilde{\lambda} = 0\,, \qquad \delta\lambda = D\,, \qquad \delta D = 0\,.$$

and therefore

$$\delta\mathcal{Z}_k = \left(\tilde{\lambda}\partial_{\tilde{\sigma}} + D\partial_\lambda\right)\mathcal{Z}_k = 0 \quad \Rightarrow \quad D\,\partial_\lambda\partial_{\tilde{\lambda}}\mathcal{Z}_k\Big|_{\lambda=\tilde{\lambda}=0} = \partial_{\tilde{\sigma}}\mathcal{Z}_k\Big|_{\lambda=\tilde{\lambda}=0}$$

This crucial step leads to a contour integral on the σ-plane:

$$\int d^2\lambda d^2\sigma\,\mathcal{Z} \sim \int d^2\sigma\frac{1}{D}\partial_{\tilde{\sigma}}\mathcal{Z} \sim \oint d\sigma\frac{1}{D}\mathcal{Z}\,.$$

This is like in case of the flavored elliptic genus. [Benini, Eager, Hori, Tachikawa, 2013]

The Coulomb branch formula

The remaining steps are similar to previous works [Benini, Eager, Hori, Tachikawa, 2013; Hori, Kim, Yi, 2014]. We find:

$$\langle \mathcal{O}_{N,S}(\sigma) \rangle = \frac{1}{|W|} \sum_k \oint_{\text{JK}} \prod_{a=1}^{\text{rank}(G)} \left[d\hat{\sigma}_a \, q_a^{k_a} \right] Z_k^{1-\text{loop}}(\hat{\sigma}) \, \mathcal{O}_{N,S}\left(\hat{\sigma} \mp \frac{1}{2} \epsilon_\Omega k \right)$$

- $|W|$ denotes the order of the Weyl group.
- The contour is determined by a Jeffrey-Kirwan residue.
- The result depends on the FI parameters explicitly and through the definition of the contour.
- The sum is over all fluxes k's. However, only some chambers in $\{k_a\}$ effectively contribute residues.

The Coulomb branch formula

$$\langle \mathcal{O}_{N,S}(\sigma) \rangle = \frac{1}{|W|} \sum_k \oint_{\text{JK}} \prod_{a=1}^{\text{rank}(G)} \left[d\hat{\sigma}_a \, q_a^{k_a} \right] Z_k^{1-\text{loop}}(\hat{\sigma}) \, \mathcal{O}_{N,S}\left(\hat{\sigma} \mp \frac{1}{2} \epsilon_\Omega k \right)$$

- The distinct q_a's are a formal device. We have as many actual q's as the number of $U(1)$ factors in $\mathfrak{g}$.
  For instance, for $G = U(N)$ we have $q_a = q$ for $a = 1, \cdots, N$.
- The one-loop term reads

$$Z_k^{1-\text{loop}}(\hat{\sigma}) = \prod_{\alpha \in \mathfrak{g}} Z_k^W(\alpha(\hat{\sigma})) \prod_{\rho \in \mathfrak{R}} Z_k^\Phi(\rho(\hat{\sigma}))$$

  from the W-bosons and chiral multiplets.

The Coulomb branch formula

$$\langle \mathcal{O}_{N,S}(\sigma) \rangle = \frac{1}{|W|} \sum_k \oint_{\text{JK}} \prod_{a=1}^{\text{rank}(G)} \left[d\hat{\sigma}_a \, q_a^{k_a} \right] Z_k^{1-\text{loop}}(\hat{\sigma}) \, \mathcal{O}_{N,S}\left(\hat{\sigma} \mp \frac{1}{2} \epsilon_\Omega k \right)$$

- For chiral multiplet of $U(1)$ charge Q and R-charge r, we have

$$Z_k^{\Phi}(\hat{\sigma}) = {\epsilon_\Omega}^{Qk+1-r} \frac{\Gamma\left(Q\frac{\hat{\sigma}}{\epsilon_\Omega} - Q\frac{k}{2} + \frac{r}{2} \right)}{\Gamma\left(Q\frac{\hat{\sigma}}{\epsilon_\Omega} + Q\frac{k}{2} - \frac{r}{2} + 1 \right)} = \frac{{\epsilon_\Omega}^{Qk+1-r}}{\left(Q\frac{\hat{\sigma}}{\epsilon_\Omega} - Q\frac{k}{2} + \frac{r}{2} \right)_{Qk-r+1}} \, .$$

- The W-boson W^α contributes exactly like a chiral of R-charge $r = 2$ and gauge charges α.
- Twisted masses m_i for global symmetries can be introduced in the obvious way.

A-model Coulomb branch formula ($\epsilon_\Omega = 0$)

For $\epsilon_\Omega = 0$, the Coulomb branch formula simplifies to:

$$\langle \mathcal{O}(\sigma) \rangle_0 = \frac{1}{|W|} \sum_k \oint_{\text{JK}} \prod_{a=1}^{\text{rank}(G)} \left[d\hat{\sigma}_a \, q_a^{k_a} \right] Z_k^{1-\text{loop}}(\hat{\sigma}) \, \mathcal{O}\left(\hat{\sigma}\right)$$

with

$$Z_k^{1-\text{loop}}(\hat{\sigma}) = (-1)^{\sum_{\alpha>0}(\alpha(k)+1)} \prod_{\alpha>0} \alpha(\hat{\sigma})^2 \prod_i \prod_{\rho_i \in \mathfrak{R}_i} \rho_i(\hat{\sigma})^{r_i - 1 - \rho_i(k)}$$

In the Abelian case, this is a known mathematical result by [Szenes, Vergne, 2003] about volumes of vortex moduli spaces. Our physical derivation generalizes it to non-Abelian GLSMs.

A-model Coulomb branch formula ($\epsilon_\Omega = 0$)

In favorable cases, one can do the sum over fluxes explicitly:

$$\langle \mathcal{O}(\sigma) \rangle_0 = \frac{1}{|W|} \oint_{\text{JK}} \prod_{a=1}^{\text{rank}(G)} \left[d\hat{\sigma}_a \, \frac{1}{1 - e^{2\pi i \partial_{\sigma_a} \hat{W}_{\text{eff}}}} \right] Z_0^{1-\text{loop}}(\hat{\sigma}) \, \mathcal{O}(\hat{\sigma})$$

Here $\hat{W}_{\text{eff}}$ is the one-loop effective twisted superpotential. Finally, if the critical locus

$$e^{2\pi i \partial_{\sigma_a} \hat{W}_{\text{eff}}} = 1, \qquad \sigma_a \neq \sigma_b \ (\text{if } a \neq b)$$

consists of isolated points (such as typically happens for massive theories), we can write the contour integral as

$$\langle \mathcal{O}(\sigma) \rangle_0 = \sum_{\hat{\sigma}^* | d\hat{W} = 0} \frac{Z_0^{1-\text{loop}}(\hat{\sigma}^*) \, \mathcal{O}(\hat{\sigma}^*)}{H(\hat{\sigma}^*)} \, , \qquad H = \det \partial_{\sigma_a} \partial_{\sigma_b} \hat{W}$$

This same formula appeared in [Nekrasov, Shatashvili, 2014] and also in [Melnikov, Plesser, 2005].

$U(1)$ examples

Example 1. In the $\mathbb{C}P^{n-1}$ model, we have

$$\langle \mathcal{O}_{N,S}(\sigma)\rangle = \sum_{k=0}^{\infty} q^k \oint d\hat{\sigma} \prod_{p=0}^{k} \prod_{i=1}^{n} \frac{1}{\hat{\sigma} - m_i - k/2 + p}\, \mathcal{O}\left(\hat{\sigma} \mp \frac{k}{2}\right)$$

with m_i the twisted masses coupling to the $SU(n)$ flavor symmetry.

In the A-model limit and with $m_i = 0$, this simplifies to

$$\langle \mathcal{O}(\sigma)\rangle_{\epsilon_\Omega=0} = \oint d\hat{\sigma} \left(\frac{1}{1 - q\hat{\sigma}^{-n}}\right) \frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^n} = \oint d\hat{\sigma} \frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^n - q}$$

This reproduces known results.

Example 2. For the quintic model, we have

$$\langle \mathcal{O}_N(\sigma) \rangle = \frac{1}{{\epsilon_\Omega}^3} \sum_{k=0}^{\infty} q^k \oint ds \frac{\prod_{l=0}^{5k}(-5s-l)}{\prod_{p=0}^{k}(s+p)^5} \mathcal{O}(\epsilon_\Omega s)$$

In the A-model limit, we obtain

$$\langle \mathcal{O}(\sigma) \rangle_{\epsilon_\Omega = 0} = \sum_{k=0}^{\infty} (-5^5 q)^k \oint d\hat{\sigma} \frac{5\hat{\sigma}\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^5} = \frac{5}{1+5^5 q} \oint d\hat{\sigma} \frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^4}$$

For any ϵ_Ω, we find $\langle \sigma^n \rangle = 0$ if $n = 0, 1, 2$, and

$$\langle \sigma^3 \rangle = \frac{5}{1+5^5 q}\,, \qquad \langle \sigma^4 \rangle = 10\epsilon_\Omega \frac{5^5 q}{(1+5^5 q)^2}\,, \cdots$$

in perfect agreement with [Morrison, Plesser, 1994].

Non-Abelian examples

For simplicitly, let us focus on $\epsilon_\Omega = 0$, the A-model.

Example 3. For the Grassmanian model, the residue formula gives

$$\langle \mathcal{O} \rangle_0 = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}} q^{\mathbf{k}} \mathcal{Z}_k(\mathcal{O}) \; ,$$

with

$$\mathcal{Z}_{\mathbf{k}} = \frac{1}{N_c!} \sum_{k_a | \sum_a k_a = \mathbf{k}} \frac{(-1)^{2\rho_W(k)}}{(2\pi i)^{N_c}} \oint d^{N_c}\sigma \frac{\prod_{a \neq b}^{N_c} (\sigma_a - \sigma_b)}{\prod_{a=1}^{N_c} \prod_{i=1}^{N_f} (\sigma_a - m_i)^{1+k_a}} \mathcal{O}(\sigma) \; .$$

Here m_i are twisted masses, corresponding to a $SU(N_f)$-equivariant deformation of $Gr(N_c, N_f)$.

For $m_i = 0$, the numbers $\mathcal{Z}_{\mathbf{k}}$ are the $g = 0$ Gromov-Witten invariants.

Example 3, continued. This simplifies explicit formulas found in the math literature. For instance, one finds [C.C., N. Mekareeya, work in progress]

$$\langle u_1(\sigma)^p\rangle_0 = \delta_{p,(N_f-N_c)N_c+\mathbf{k}N_f}\; q^{\mathbf{k}} \deg(K^{\mathrm{k}}_{N_f-N_c,N_c})$$

with $\deg(K^{\mathrm{k}}_{N_f-N_c,N_c})$ given by [Ravi, Rosenthal, Wang, 1996]

$$(-1)^{k(N_c+1)+\frac{1}{2}N_c(N_c-1)}[N_c(N_f-N_c+\mathbf{k}N_f)]! \sum_{k_a \,|\, \sum_a k_a=\mathbf{k}} \sum_{\sigma\in S_{N_c}} \prod_{j=1}^{N_c} \frac{1}{(N_f-2N_c-1+j+\sigma(j)+k_jN_f)!}\,,$$

Example: for $N_c=2, N_f=5$, we have the non-vanishing correlators:

$$\langle u_1^6\rangle_0 = 5\,, \quad \langle u_1^{11}\rangle_0 = 55\,q\,, \quad \langle u_1^{16}\rangle_0 = 610\,q^2\,, \quad \langle u_1^{21}\rangle_0 = 6765\,q^3\,, \quad \cdots$$

This generalizes to the computation of GW invariants of non-CY target space, and is thus complementary of the techniques of [Jockers, Kumar, Lapan, Morrison, Romo, 2012] valid for conformal models.

Example 4. For the Rødland CY_3 model, our formula reads

$$\frac{1}{2}\sum_{k_1,k_2=0}^{\infty} q^{k_1+k_2}\oint_{(\hat\sigma_a=0)} d\hat\sigma_1 d\hat\sigma_2 (\hat\sigma_1-\hat\sigma_2)^2 \frac{(-\hat\sigma_1-\hat\sigma_2)^{7(1+k_1+k_2)}}{\hat\sigma_1^{7(1+k_1)}\hat\sigma_2^{7(1+k_2)}}\mathcal{O}(\hat\sigma)\ .$$

The observables are polynomials in the gauge invariants

$$u_1(\sigma)=\mathrm{Tr}(\sigma)=\sigma_1+\sigma_2\ ,\qquad u_2(\sigma)=\mathrm{Tr}(\sigma^2)=\sigma_1^2+\sigma_2^2\ .$$

The only non-vanishing correlators are given by:

$$\begin{aligned}\langle u_1(\sigma)^3\rangle &= \frac{42-14q}{1-57q-289q^2+q^3}\ ,\\ \langle u_2(\sigma)u_1(\sigma)\rangle &= \frac{14+126q}{1-57q-289q^2+q^3}\ .\end{aligned}$$

Note:

- The Yukawa $\langle u_1(\sigma)^3 \rangle$ was computed by mirror symmetry in [Batyrev *et al.*, 1998]. The second correlator is a new result.
- More generally, the correlators

$$\langle u_n(\sigma) \cdots \rangle \,, \qquad n > 1 \,,$$

  in any non-Abelian GLSM are new results which could not be obtained by previous methods (to the best of my knowledge).
- Many more examples can be considered. In particular, one can study the $PAX/PAXY$ models of [Jockers, Kumar, Morrison, Lapan, Romo, 2012] for determinantal CY varieties.

$\mathcal{N} = (0,2)$ observables

A priori, the above would not generalize to $(0,2)$ theories, which only have two right-moving supercharges:

$$\{Q_+, \tilde{Q}_+\} = -4P_{\bar{z}} \ .$$

Half-BPS operators are $\tilde{Q}_+$-closed, and generally do not form a ring but a chiral algebra:

$$\mathcal{O}_a(z)\mathcal{O}_b(0) \sim \sum_c \frac{f_{abc}}{z^{s_a+s_b-s_c}} \mathcal{O}_c(z) \ ,$$

In some favorable cases with an extra $U(1)_L$ symmetry, there exists a subset of the $\mathcal{O}_a$, of spin $s = 0$, with trivial OPE. These pseudo-chiral rings are known as “topological heterotic rings".

[Adams, Distler, Ernebjerg, 2006]

Theories with a $(2,2)$ locus and $A/2$-twist

I will focus on $(0,2)$ supersymmetric GLSMs with a $(2,2)$ locus. Schematically, they are determined by the following $(0,2)$ matter content:

- A vector multiplet $\mathcal{V}$ and a chiral Σ in the adjoint of the gauge group G, with $\mathfrak{g} = \mathrm{Lie}(G)$.
- Pairs of chiral and Fermi multiplets Φ_i and Λ_i, in representations $\mathfrak{R}_i$ of $\mathfrak{g}$.

The interactions are encoded in two sets of holomorphic functions of the chiral multiplets:

$$\mathcal{E}_i(\Sigma, \Phi) = \Sigma E_i(\Phi) \ , \qquad\qquad J_i = J_i(\Phi)$$

By assumption, we preserve an additional $U(1)_L$ symmetry classically, which leads to $\mathcal{E}_i$ linear in Σ

We also turn on an FI term τ^I for each $U(1)_I$ in G.

Theories with a $(2,2)$ locus and $A/2$-twist

We assign the R-charges:

$$R_{A/2}[\Sigma] = 0 \;, \qquad R_{A/2}[\Phi_i] = r_i \;, \qquad R_{A/2}[\Lambda_i] = r_i - 1 \;,$$

which is always anomaly-free.

We can define the theory on S^2 (with any metric) by a so-called half-twist:

$$S = S_0 + \frac{1}{2} R_{A/2} \;,$$

preserving one supercharge $\tilde{\mathcal{Q}} \sim \tilde{Q}_+$. The R-charges r_i must be integers (typically, $r_i = 0$ or 2). “Pseudo-topological.”

Incidentally, half-twisting is the only way to preserve supersymmetry on the sphere, unlike for $(2,2)$ GLSM.

The Coulomb branch of theories with a $(2,2)$ locus

If we have a generic $\mathcal{E}_i$ potentials, there is a Coulomb branch spanned by the scalar σ in Σ:

$$\sigma = \mathrm{diag}(\sigma_a) \ .$$

The matter fields obtain a mass

$$M_{ij} = \partial_j \mathcal{E}_i \Big|_{\phi=0} = \sigma_a \, \partial_j E_i^a \Big|_{\phi=0} \ .$$

By gauge invariance, M_{ij} is block-diagonal, with each block spanned by fields with the same gauge charges. We denote these blocks by M_γ. (On the $(2,2)$ locus, $M_{ij} = \delta_{ij} Q_i(\sigma)$.)

Let us introduce the notation

$$P_\gamma(\sigma) = \det M_\gamma \ \in \mathbb{C}[\sigma_1, \cdots, \sigma_r] \ , \qquad\qquad (r = \mathrm{rank}(G))$$

which is a homogeneous polynomial of degree $n_\gamma \geq 1$ in σ.

A residue formula for $A/2$-model correlators on S^2

All the fields are massive on the Coulomb branch, and the localization argument can be carried out similarly to the $(2,2)$ case, allowing us to compute the $A/2$-twisted correlators on S^2 with an half-twist:

$$\langle \mathcal{O}(\sigma)\rangle_{A/2} = \sum_k \frac{1}{|W|} \sum_k \oint_{\text{JKG}} \prod_{a=1}^{\text{rank}(G)} \left[d\sigma_a\, q_a^{k_a} \right] Z_k^{1-\text{loop}}(\sigma)\, \mathcal{O}(\sigma) \ ,$$

with

$$Z_k^{1-\text{loop}}(\sigma) = (-1)^{\sum_{\alpha>0}(\alpha(k)+1)} \prod_{\alpha>0} \alpha(\sigma)^2 \prod_\gamma \prod_{\rho_\gamma \in \mathfrak{R}_\gamma} \left(\det M_{(\gamma,\rho_\gamma)}\right)^{r_\gamma - 1 - \rho_\gamma(k)} \ .$$

Here we have a new residue prescription generalizing the Jeffrey-Kirwan residue relevant on the $(2,2)$ locus.

In the Abelian case, this reproduces previous results of [McOrist, Melnikov, 2007].

The Jeffrey-Kirwan-Grothendieck residue

In the $(2,2)$ case, the Jeffrey-Kirwan residue determined a way to pick a middle-dimensional contour in

$$\mathbb{C}^r - \cup_{i\in I} H_i \ , \qquad I = \{i_1, \cdots, i_s\} \ (s \geq r) \ , \qquad H_i = \{\sigma_a \,|\, Q_i(\sigma) = 0\} \ ,$$

when the integrand has poles on H_i only.

For generic $(0,2)$ deformations, we have an integrand with singularities on more general divisors of $\mathbb{C}^r$:

$$D_\gamma = \{\sigma_a \,|\, P_\gamma(\sigma) = 0\} \ ,$$

which intersect at the origin only.

The Jeffrey-Kirwan-Grothendieck residue

To define the relevant Jeffrey-Kirwan-Grothendieck (JKG) residue, we introduce the data $\mathbf{P} = \{P_\gamma\}$ and $\mathbf{Q} = \{Q_\gamma\}$ of divisors D_γ and associated gauge charges Q_γ. The residue is defined by its action on the holomorphic forms:

$$\omega_S = d\sigma_1 \wedge \cdots \wedge d\sigma_r \, P_0 \prod_{b \in S} \frac{1}{P_b} \ ,$$

with $S = \{\gamma_1, \cdots, \gamma_r\}$, which is

$$\text{JKG-Res}[\eta] \, \omega_S = \begin{cases} \text{sign}\,(\det(Q_S)) \; \text{Res}_{(0)} \, \omega_S & \text{if} \quad \eta \in \text{Cone}(Q_S) \ , \\ 0 & \text{if} \quad \eta \notin \text{Cone}(Q_S) \ , \end{cases}$$

with $\text{Res}_{(0)}$ the (local) Grothendieck residue at the origin.

The Jeffrey-Kirwan-Grothendieck residue

The Grothendieck residue itself is defined as:

$$\text{Res}_{(0)}\, \omega_S = \frac{1}{(2\pi i)^r} \oint_{\Gamma_\varepsilon} d\sigma_1 \wedge \cdots \wedge d\sigma_r \, \frac{P_0}{P_{\gamma_1} \cdots P_{\gamma_r}} \ ,$$

with the real r-dimensional contour:

$$\Gamma_\varepsilon = \left\{ \sigma \in \mathbb{C}^r \, \Big| \, |P_{\gamma_1}| = \varepsilon_1 \ , \ \cdots \ , |P_{\gamma_r}| = \varepsilon_r \right\} \ ,$$

and it is eminently computable.

Finally, we should take $\eta = \xi_{\text{eff}}^{\text{UV}}$ to cancel the “boundary contributions” from infinity on the Coulomb branch.

Example: $\mathbb{C}P^1 \times \mathbb{C}P^1$ with deformed tangent bundle

Consider a theory with gauge group $U(1)^2$, two neutral chiral multiplets Σ_1, Σ_2 and four pairs of chiral and Fermi multiplets:

$$\Phi_i, \Lambda_i\ ,\ i = 1,2 \quad Q_i = (1,0)\ , \qquad \Phi_j, \Lambda_j\ ,\ j = 1,2 \quad Q_j = (0,1)\ ,$$

with holomorphic potentials $J_i = J_j = 0$ and

$$\mathcal{E}_i = \sigma_1 (A\phi)_i + \sigma_2 (B\phi)_i\ , \qquad \mathcal{E}_j = \sigma_1 (C\phi)_j + \sigma_2 (D\phi)_j\ .$$

with A, B, C, D arbitrary 2×2 constant matrices. This realizes a deformation of the tangent bundle to the holomorphic bundle $\mathbf{E}$ described by the cokernel:

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{\begin{pmatrix} A & B \\ C & D \end{pmatrix}} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathbf{E} \longrightarrow 0$$

$\mathbb{C}P^1 \times \mathbb{C}P^1$, continued.

We have two sets $\gamma = 1, 2$:

$$\det M_1 = \det(A\sigma_1 + B\sigma_2)\ , \qquad \det M_2 = \det(C\sigma_1 + D\sigma_2)\ .$$

The Coulomb branch residue formula gives

$$\langle \sigma_1^{p_1}\sigma_2^{p_2}\rangle = \sum_{k_1,k_2\in\mathbb{Z}} q_1^{k_1} q_2^{k_2} \oint_{\text{JKG}} d\sigma_1 d\sigma_2\, \frac{\sigma_1^{p_1}\sigma_2^{p_2}}{(\det M_1)^{1+k_1}(\det M_2)^{1+k_2}}$$

This can be checked against independent mathematical computations of sheaf cohomology groups.

This result also implies the "quantum sheaf cohomology relations":

$$\det M_1 = q_1\ , \qquad \det M_2 = q_2\ ,$$

in the $A/2$-ring. This can also be derived from a standard argument on the Coulomb branch. [McOrist, Melnikov, 2008]

Conclusions

- We studied $\mathcal{N} = (2,2)$ supersymmetric GLSMs on the Ω-deformed sphere, S^2_Ω.
- We derived a simple Coulomb branch formula for the S^2_Ω observables.
- When $\epsilon_\Omega = 0$, this gives a simple, general formula for A-twisted GLSM correlation functions.
  - Some correlators could not be computed with other methods, such as the ones involving $\mathrm{Tr}(\sigma^n)$ in a non-Abelian theory.
  - Even when other methods are possible (*e.g.* mirror symmetry), the Coulomb branch formula is much simpler.
- The formula is valid in any phase in FI parameter space (away from boundaries), geometric or not.
- Surprisingly, it generalizes off the $(2,2)$ locus, leading to very interesting new results for some $(0,2)$ models and the corresponding heterotic geometries.