Resurgence and Non-Perturbative Physics: Decoding the Path Integral

Gerald Dunne

University of Connecticut

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GD, M. Ünsal, G. Başar, 1210.2423, 1210.3646, 1306.4405, 1401.5202, 1501.05671
1505.07803 1509.05046 1511.05977

GD, lectures at CERN 2014 Winter School

GD, lectures at Schladming 2015 Winter School

Some Physical Motivation

- improved asymptotics in QFT
- infrared renormalon puzzle in asymptotically free QFT
- non-perturbative physics without instantons: physical meaning of non-BPS saddle configurations

Bigger Picture

- non-perturbative definition of nontrivial QFT in continuum
- analytic continuation of path integrals (Lefschetz thimbles)
- dynamical and non-equilibrium physics from path integrals

Some Mathematical Motivation

Resurgence: ‘new’ idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory $\longrightarrow$ divergent (asymptotic) series
- formal series expansion $\longrightarrow$ *trans-series* expansion
- trans-series ‘well-defined under analytic continuation’
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, fluids, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift:

  view asymptotics/semiclassics as potentially exact

Views of Divergent Series

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever ... That most of these things [summation of divergent series] are correct, in spite of that, is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.

N. Abel, 1802 – 1829

The series is divergent; therefore we may be able to do something with it.

O. Heaviside, 1850 – 1925

Series don't diverge for no reason; it is not a capricious thing. The divergence of a series must reflect its cause.

M. Berry, Stokes & the Rainbow (2003)

Resurgent Trans-Series

- trans-series expansion in QM and QFT applications:

$$f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k−instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

> *No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms*
>
> *G. H. Hardy, Divergent Series, 1949*

- J. Écalle (1980): set of functions closed under:

(Borel transform) + (analytic continuation) + (Laplace transform)

- "any reasonable function" has a trans-series expansion

Resurgent Trans-Series

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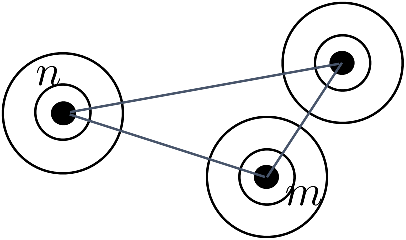
- *trans-monomial elements*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, are familiar
- "multi-instanton calculus" in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: exponential asymptotics (Olver, Kruskal, Segur, Costin, ...)

Resurgence

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



local analysis encodes more global information than one might naïvely think

recap: rough basics of Borel summation

(i) divergent, alternating:

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, g^{2n} = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1+g^2\, t}$$

(ii) divergent, non-alternating:

$$\sum_{n=0}^{\infty} n!\, g^{2n} = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1-g^2\, t}$$

$\Rightarrow$ ambiguous imaginary non-pert. term: $\pm\frac{i\,\pi}{g^2}\, e^{-1/g^2}$

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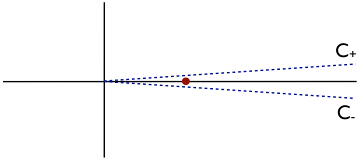
avoid singularities on $\mathbb{R}^+$: *directional* Borel sums:



$\theta = 0^\pm \longrightarrow$ non-perturbative ambiguity: $\pm \mathrm{Im}[\mathcal{B}f(g^2)]$

challenge: use physical input to resolve ambiguity

Resurgent relations within Trans-series

- trans-series (neglecting logs for now for simplicity)

$$
\begin{aligned}
F(g^2) \quad \sim \quad & \left(f_0^{(0)} + f_1^{(0)} g^2 + f_2^{(0)} g^4 + \dots\right) \\
& + \sigma\, e^{-S/g^2} \left(f_0^{(1)} + f_1^{(1)} g^2 + f_2^{(1)} g^4 + \dots\right) \\
& + \sigma^2\, e^{-2S/g^2} \left(f_0^{(2)} + f_1^{(2)} g^2 + f_2^{(2)} g^4 + \dots\right) \\
& + \dots
\end{aligned}
$$

- basic idea: ambiguous imaginary non-perturbative contributions from Borel summation of non-alternating divergent series in one sector must cancel against terms in some other non-perturbative sector

- implies very strong relations between trans-series expansion coefficients in different non-perturbative sectors

Trans-series Example: Stirling expansion $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots + \frac{174611}{6600z^{20}} - \ldots$$

- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$ ✓
- reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot(\pi z)$

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formal series $\Rightarrow$ $\operatorname{Im} \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2}$

reflection $\Rightarrow$ $\operatorname{Im} \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y}$

"raw" asymptotics inconsistent with analytic continuation

- resurgence fixes this: series $\rightarrow$ trans-series
- infinite number of exponential terms in trans-series

QFT Application: Euler-Heisenberg 1935

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

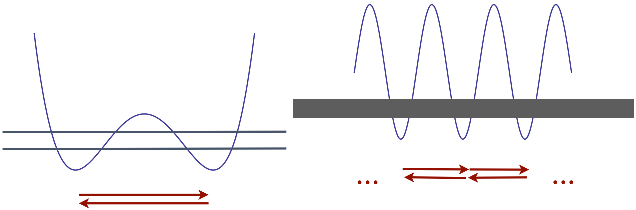
$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{h c}\int_0^\infty e^{-\eta}\frac{\mathrm{d}\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}.$$

($\mathfrak{E}, \mathfrak{B}$ Kraft auf das Elektron.
$|\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2} =$ „Kritische Feldstärke“.)

- EH effective action $\sim$ Barnes function $\sim \int \ln\Gamma(x)$
- resurgent trans-series: analytic continuation $B \longleftrightarrow E$

Hint of Resurgence in QM Spectral Problems

- QM analog of IR renormalon problem in QFT



- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

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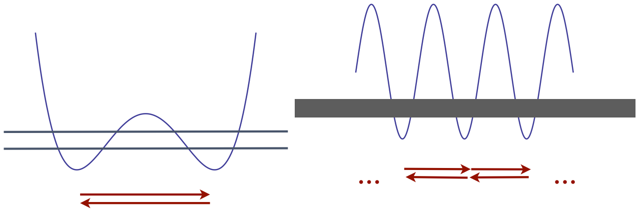


- degenerate vacua: double-well, Sine-Gordon, ...

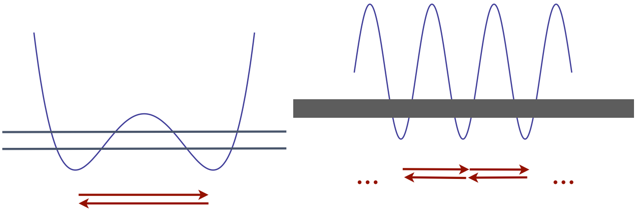
splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel-summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

"Bogomolny/Zinn-Justin mechanism"

Bogomolny 1980; Zinn-Justin 1980



- degenerate vacua: double-well, Sine-Gordon, ...
  1. perturbation theory non-Borel-summable: ill-defined/incomplete
  2. instanton gas picture ill-defined/incomplete: $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract
- regularize *both* by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

"resurgence" $\Rightarrow$ cancellation to all orders

Resurgence and Matrix Models, Topological Strings

Mariño, Schiappa, Weiss: *Nonperturbative Effects and the Large-Order Behavior of Matrix Models and Topological Strings* 0711.1954; Mariño, *Nonperturbative effects and nonperturbative definitions in matrix models and topological strings* 0805.3033

• resurgent Borel-Écalle analysis of partition functions etc in matrix models

$$Z(g_s, N) = \int dU \, \exp\left[\frac{2}{g_s} \mathrm{tr}\, V(U)\right]$$

• two variables: g_s and N ('t Hooft coupling: $\lambda = g_s N$)

• e.g. Gross-Witten-Wadia: $V = U + U^{-1}$

• 3rd order phase transition at $\lambda = 1$, associated with condensation of instantons (Neuberger)

• double-scaling limit: Painlevé II

Towards Resurgence in QFT: Renormalons

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

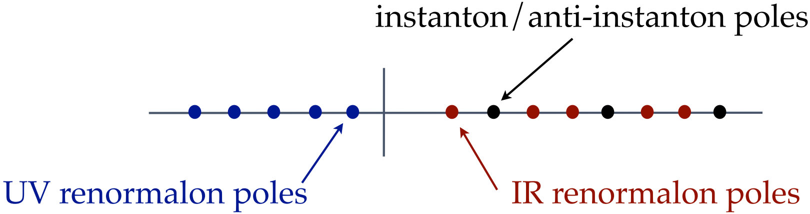
QFT: new physical effects occur, due to running of couplings with momentum

- asymptotically free QFT
- faster source of divergence: “renormalons”
- both positive and negative Borel poles

IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: $\longrightarrow \quad \pm i\, e^{-\frac{2S}{\beta_0\, g^2}}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow \quad \pm i\, e^{-\frac{2S}{g^2}}$



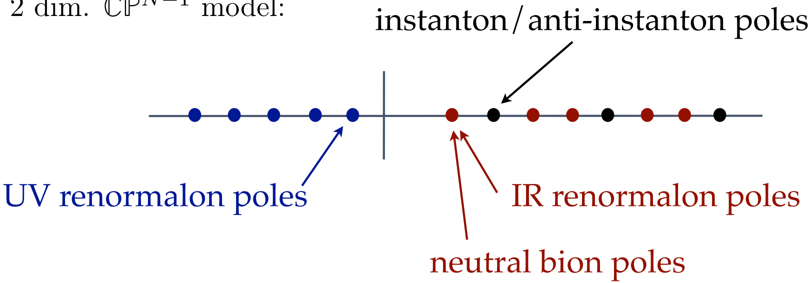
appears that BZJ cancellation cannot occur

asymptotically free theories remain inconsistent

't Hooft, 1980; David, 1981

resolution: there is another problem with the non-perturbative instanton gas analysis: scale modulus of instantons

- spatial compactification and principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model:



cancellation occurs !

$\rightarrow$ semiclassical realization of IR renormalons

BZJ cancellation in Spatially Compactified $\mathbb{CP}^{N-1}$

- spatial compactification, continuity $\rightarrow$ $\mathbb{Z}_N$ twisted b.c.s

$$\Rightarrow \quad \text{instantons fractionalize} : S_{\rm inst} \rightarrow \frac{S_{\rm inst}}{N} = \frac{S_{\rm inst}}{\beta_0}$$

- non-perturbative sector: bion amplitudes

$$\left[\mathcal{I}_i\bar{\mathcal{I}}_i\right]_\pm = \left(\ln\left(\frac{g^2\,N}{8\pi}\right) - \gamma\right)\frac{16}{g^2\,N}\,e^{-\frac{8\pi}{g^2\,N}} \pm i\pi\,\frac{16}{g^2\,N}\,e^{-\frac{8\pi}{g^2\,N}}$$

- perturbative sector: lateral Borel summation

$$B_\pm\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\, B\mathcal{E}(t)\,e^{-t/g^2} = \mathrm{Re}\,B\mathcal{E}(g^2) \mp i\pi\,\frac{16}{g^2\,N}\,e^{-\frac{8\pi}{g^2\,N}}$$

exact cancellation ! (GD, Ünsal, 1210.2423, 1210.3646)

explicit application of resurgence to nontrivial QFT

Non-perturbative Physics Without Instantons Dabrowski/GD 1306.0921

- $O(N)$ & principal chiral model have no instantons !

- but have non-BPS finite action solutions

negative fluctuation modes

twisted b.c.s $\rightarrow$ fractionalize (Cherman et al, 1308.0127, 1403.1277, Nitta et al, ...) saddles play role of bions in resurgence structure

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

- Yang-Mills, $\mathbb{CP}^{N-1}$, $U(N)$ PCM, $O(N)$, $Gr(N,M)$, ... have 'unstable' finite action non-BPS saddles

- what do these mean physically ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

open problem: non-BPS saddle classification/fluctuations

Resurgence and Localization

Mariño, 1104.0783; Aniceto, Russo, Schiappa, 1410.5834 Wang, Wang, Huang, 1409.4967, Grassi, Hatsuda, Mariño, 1410.7658, ...)

- certain protected quantities in especially symmetric QFTs can be reduced to matrix models $\Rightarrow$ resurgent asymptotics
- 3d Chern-Simons on $\mathbb{S}^3 \to$ matrix model

$$Z_{CS}(N,g) = \frac{1}{\text{vol}(U(N))} \int dM \exp\left[-\frac{1}{g}\,\text{tr}\left(\frac{1}{2}(\ln M)^2\right)\right]$$

- ABJM: $\mathcal{N}=6$ SUSY CS, $G = U(N)_k \times U(N)_{-k}$

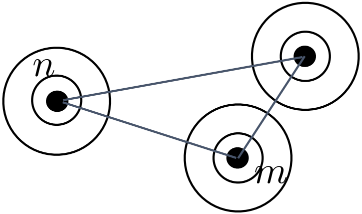
$$Z_{ABJM}(N,k) = \sum_{\sigma \in S_N} \frac{(-1)^{\epsilon(\sigma)}}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi k} \frac{1}{\prod_{i=1}^{N} 2\text{ch}\left(\frac{x_i}{2}\right)\, \text{ch}\left(\frac{x_i - x_{\sigma(i)}}{2k}\right)}$$

- $\mathcal{N}=4$ SUSY Yang-Mills on $\mathbb{S}^4$

$$Z_{SYM}(N,g^2) = \frac{1}{\text{vol}(U(N))} \int dM \exp\left[-\frac{1}{g^2}\,\text{tr} M^2\right]$$

The Bigger Picture: Decoding the Path Integral

what is the origin of resurgent behavior in QM and QFT ?



to what extent are (all?) multi-instanton effects encoded in perturbation theory? And if so, why?

- QM & QFT: basic property of all-orders steepest descents integrals
- Lefschetz thimbles: analytic continuation of path integrals

All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals:

<u>*hyperasymptotics*</u> (Berry/Howls 1991, Howls 1992)

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(g^2) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T^{(m)}\left(\frac{F_{nm}}{v}\right)$$

- exact resurgent relation between fluctuations about n^{th} saddle and about neighboring saddles m

$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots \right.$$

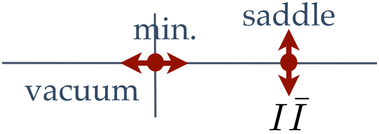
- universal factorial divergence of fluctuations (Darboux)
- fluctuations about *neighboring* saddles explicitly related!

All-Orders Steepest Descents: Darboux Theorem

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^\pi dz\, e^{-\frac{1}{g^2}\sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$T_r^{(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)}$$

$$\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \dots\right.$$

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- low order coefficients about saddle z_1:

$$T^{(1)}(g^2) \sim i\sqrt{\pi}\left(1 - \frac{1}{4}g^2 + \frac{9}{32}g^4 - \frac{75}{128}g^6 + \dots\right)$$

- fluctuations about the two saddles are explicitly related
- could something like this work for path integrals ?
- multi-dimensional case is already non-trivial and interesting

Picard/Lefschetz; Pham (1965); Delabaere/Howls (2002); ...

Resurgence in (Infinite Dim.) Path Integrals (GD, Ünsal, 1401.5202)

- periodic potential: $V(x) = \frac{1}{g^2} \sin^2(g\, x)$
- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\operatorname{Im} E \sim \pi\, e^{-2\frac{1}{2g^2}} \left(1 - \frac{5}{2} \cdot g^2 - \frac{13}{8} \cdot g^4 - \dots \right)$$

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- double-well potential: $V(x) = x^2(1 - gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6} \cdot \frac{1}{3} \cdot \frac{1}{n} - \frac{1277}{72} \cdot \frac{1}{3^2} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\text{Im}\, E \sim \pi\, e^{-2\frac{1}{6g^2}} \left(1 - \frac{53}{6} \cdot g^2 - \frac{1277}{72} \cdot g^4 - \dots \right)$$

Graded Resurgence Triangle

saddle points labelled by: $[n, m]$

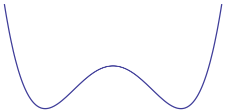
$$n = n_{\text{instanton}} + n_{\text{anti-instanton}} \quad , \quad m = n_{\text{instanton}} - n_{\text{anti-instanton}}$$

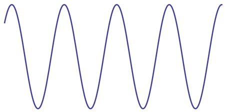
$$\begin{array}{ccccccccc}
 & & & & [0,0] & & & & \\
 & & & [1,1] & & [1,-1] & & & \\
 & & [2,2] & & [2,0] & & [2,-2] & & \\
 & [3,3] & & [3,1] & & [3,-1] & & [3,-3] & \\
[4,4] & & [4,2] & & [4,0] & & [4,-2] & & [4,-4] \\
\ddots & & & & \vdots & & & & \ddots
\end{array}$$

there is even more resurgent structure ...

Uniform WKB & Resurgent Trans-series (GD/MÜ:1306.4405, 1401.5202)

$$-\frac{\hbar^2}{2}\frac{d^2}{dx^2}\psi + V(x)\psi = E\,\psi$$





- weak coupling: degenerate harmonic classical vacua

$\Rightarrow$ uniform WKB: $\quad \psi(x) = \dfrac{D_\nu\left(\frac{1}{\sqrt{\hbar}}\varphi(x)\right)}{\sqrt{\varphi'(x)}}$

- non-perturbative effects: $\quad g^2 \leftrightarrow \hbar \quad \Rightarrow \quad \exp\left(-\frac{S}{\hbar}\right)$

- trans-series structure follows from exact quantization condition $\to E(N, \hbar) =$ trans-series

- Zinn-Justin, Voros, Pham, Delabaere, Aoki, Takei, Kawai, Koike, ...

Connecting Perturbative and Non-Perturbative Sector

Zinn-Justin/Jentschura conjecture:

generate *entire trans-series* from just two functions:

(i) perturbative expansion $E = E_{\rm pert}(\hbar, N)$
(ii) single-instanton fluctuation function $\mathcal{P}(\hbar, N)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

$$E(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} e^{-S/\hbar} \, \mathcal{P}(\hbar, N) + \ldots$$

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generate *entire trans-series* from just two functions:

(i) perturbative expansion $E = E_{\rm pert}(\hbar, N)$
(ii) single-instanton fluctuation function $\mathcal{P}(\hbar, N)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

$$E(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} e^{-S/\hbar} \mathcal{P}(\hbar, N) + \dots$$

in fact ... (GD, Ünsal, 1306.4405, 1401.5202) fluctuation factor:

$$\mathcal{P}(\hbar, N) = \frac{\partial E_{\rm pert}}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3} \left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S} \right) \right]$$

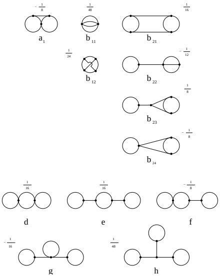
$\Rightarrow$ perturbation theory $E_{\rm pert}(\hbar, N)$ encodes everything !

Resurgence at work

- fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle are determined by those about the vacuum saddle, to all fluctuation orders

- "QFT computation": 3-loop fluctuation about $\mathcal{I}$ for double-well and Sine-Gordon:

Escobar-Ruiz/Shuryak/Turbiner 1501.03993, 1505.05115



$$\text{DW}: \quad e^{-\frac{S_0}{\hbar}}\left[1-\frac{71}{72}\,\hbar-0.607535\,\hbar^2-\ldots\right]$$

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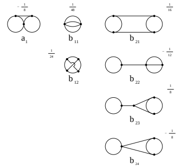

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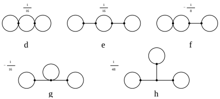


$$\text{resurgence}: e^{-\frac{S_0}{\hbar}}\left[1 + \frac{1}{72}\,\hbar\left(-102N^2 - 174N - 71\right)\right.$$

$$\left. + \frac{1}{10368}\hbar^2\left(10404N^4 + 17496N^3 - 2112N^2 - 14172N - 6299\right) + \dots\right]$$

- known for all N and to essentially any loop order, directly from perturbation theory !

- diagramatically mysterious ...

Graded Resurgence Triangle

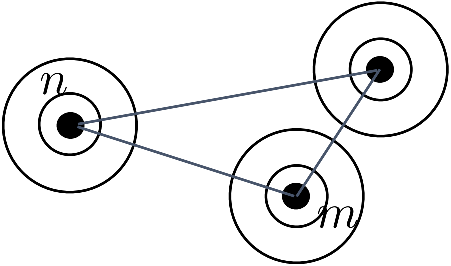
saddle points labelled by: $[n, m]$

$$n = n_{\text{instanton}} + n_{\text{anti-instanton}} \quad , \quad m = n_{\text{instanton}} - n_{\text{anti-instanton}}$$

$$
\begin{array}{ccccccccc}
 & & & & [0,0] & & & & \\
 & & & [1,1] & & [1,-1] & & & \\
 & & [2,2] & & [2,0] & & [2,-2] & & \\
 & [3,3] & & [3,1] & & [3,-1] & & [3,-3] & \\
[4,4] & & [4,2] & & [4,0] & & [4,-2] & & [4,-4] \\
\ddots & & & & \vdots & & & & \ddots
\end{array}
$$

Connecting Perturbative and Non-Perturbative Sector

all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}\, S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2} S_{\text{real}}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ... gradient flow

Stokes phenomenon: intersection numbers $\mathcal{N}_k$ can change with phase of parameters

Thimbles from Gradient Flow

gradient flow to generate steepest descent thimble:

$$\frac{\partial}{\partial\tau}A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

- keeps $Im[S]$ constant, and $Re[S]$ is monotonic

$$\frac{\partial}{\partial\tau}\left(\frac{S-\bar{S}}{2i}\right) = -\frac{1}{2i}\int\left(\frac{\delta S}{\delta A}\frac{\partial A}{\partial\tau} - \overline{\frac{\delta S}{\delta A}}\,\overline{\frac{\partial A}{\partial\tau}}\right) = 0$$

$$\frac{\partial}{\partial\tau}\left(\frac{S+\bar{S}}{2}\right) = -\int\left|\frac{\delta S}{\delta A}\right|^2$$

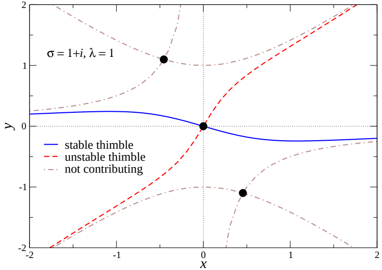
- Chern-Simons theory (Witten 2001)
- complex Langevin (Aarts 2013; ... ; Hayata, Hidaka, Tanizaki, 2015)
- lattice (Cristoforetti et al 2013, 2014; Fujii, Honda et al, 2013; Nishimura et al, 2014; Mukherjee et al 2014; Fukushima et al, 2015; Kanazawa et al, 2015; ...)

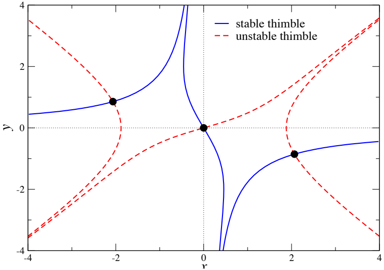
resurgence: asymptotics about different saddles related

Thimbles, Gradient Flow and Resurgence

$$Z = \int_{-\infty}^{\infty} dx \, \exp\left[-\left(\frac{\sigma}{2}\, x^2 + \frac{x^4}{4}\right)\right]$$

(Aarts, 2013; GD, Unsal, ...)



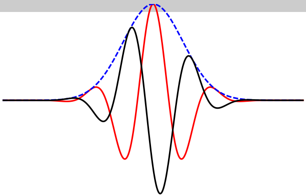


- contributing thimbles change with phase of σ
- need all three thimbles when $Re[\sigma] < 0$
- integrals along thimbles are related (resurgence)
- resurgence: preferred unique "field" choice

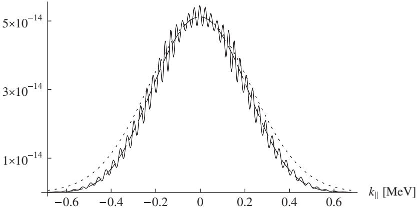
Carrier Phase Effect in Schwinger Effect

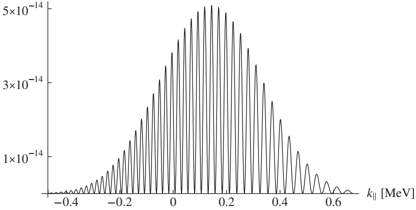
Hebenstreit, Alkofer, GD, Gies, PRL 2009

$$E(t) = \mathcal{E}\,\exp\left(-\frac{t^2}{\tau^2}\right)\,\cos\left(\omega t + \varphi\right)$$



- sensitivity to carrier phase φ ?

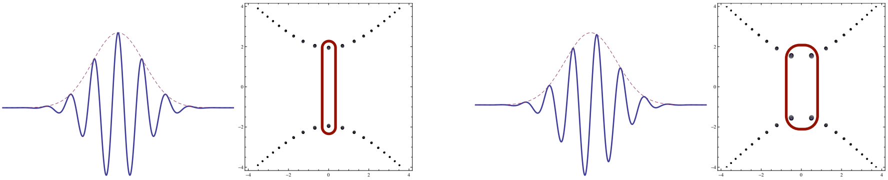




$\varphi = 0$

$\varphi = \frac{\pi}{2}$

Carrier Phase Effect and Stokes Phenomenon Dumlu, GD, 1004.2509



- interference produces momentum spectrum structure
- worldline effective action has complex saddles Dumlu, GD, 1110.1657

$$\Gamma = -\int d^4x \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint_x \mathcal{D}x \, \exp\left[-\int_0^T d\tau \left(\dot{x}_\mu^2 + A_\mu \, \dot{x}_\mu\right)\right]$$

$P \approx 4 \sin^2(\theta) \, e^{-2\,\mathrm{Im} W}$ $\qquad\qquad$ θ: interference phase

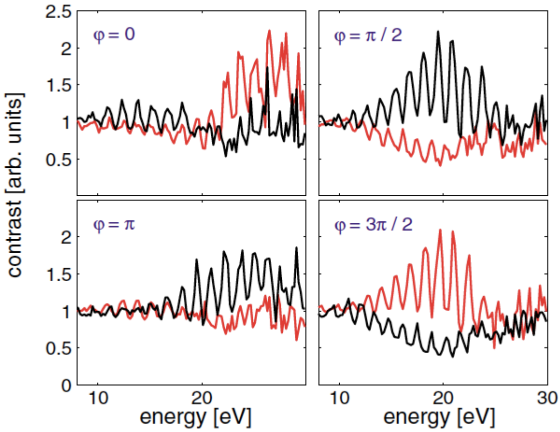
- double-slit interference, in time domain, from vacuum
- Ramsey effect: N alternating sign pulses $\Rightarrow$ N-slit system $\Rightarrow$ coherent N^2 enhancement (Akkermans, GD, PRL 2012)

Experimental effects of complex instantons

PRL 95, 040401 (2005) PHYSICAL REVIEW LETTERS week ending 22 JULY 2005

Attosecond Double-Slit Experiment

F. Lindner,[1] M. G. Schätzel,[1] H. Walther,[1,2] A. Baltuška,[1] E. Goulielmakis,[1] F. Krausz,[1,2,3] D. B. Milošević,[4] D. Bauer,[5] W. Becker,[6] and G. G. Paulus[1,2,7]

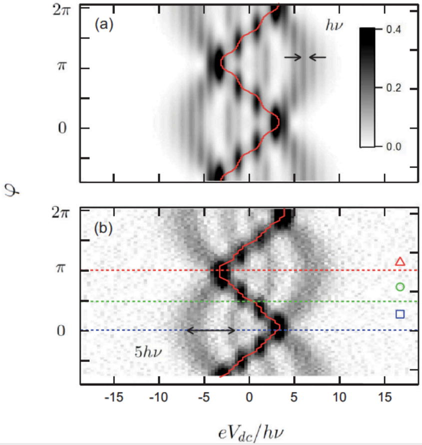


Experimental effects of complex instantons

PHYSICAL REVIEW B **87**, 075403 (2013)

Shaping a time-dependent excitation to minimize the shot noise in a tunnel junction

Julien Gabelli[1] and Bertrand Reulet[1,2]
[1] *Laboratoire de Physique des Solides, Univ. Paris-Sud, CNRS, UMR 8502, F-91405 Orsay Cedex, France*
[2] *Université de Sherbrooke, Sherbrooke, Québec J1K 2R1, Canada*

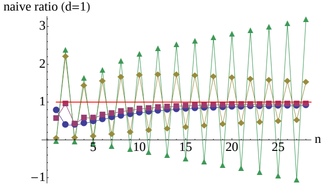


Ghost Instantons: Analytic Continuation of Path Integrals

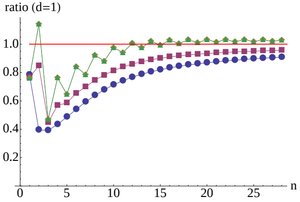
$$\mathcal{Z}(g^2|m) = \int \mathcal{D}x\, e^{-S[x]} = \int \mathcal{D}x\, e^{-\int d\tau \left(\frac{1}{4}\dot{x}^2 + \frac{1}{g^2}\,\mathrm{sd}^2(g\,x|m)\right)}$$

- doubly periodic potential: *real* & *complex* instantons

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$



without ghost instantons



with ghost instantons

- complex instantons directly affect perturbation theory, even though they are not in the original path integral measure

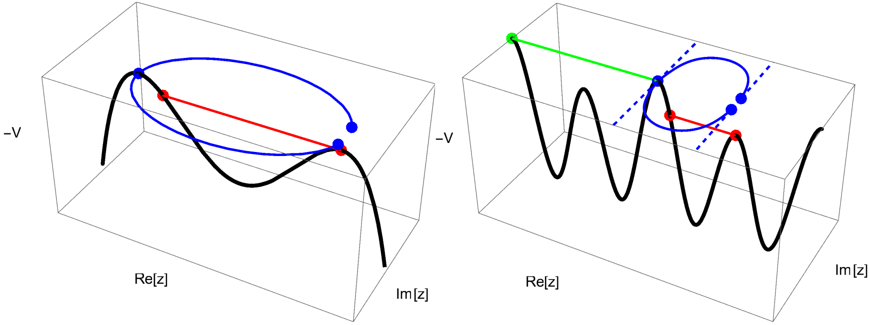
(Başar, GD, Ünsal, arXiv:1308.1108)

Necessity of Complex Saddles

(Behtash, GD, Schäfer, Sulejmanpasic, Ünsal (1510.00978), (1510.03435)

SUSY QM: $g\,\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\left(W'\right)^2 \pm \frac{g}{2}W''$

- $W = \frac{1}{3}x^3 - x \to$ tilted double-well
- $W = \cos\frac{x}{2} \to$ double Sine-Gordon
- new (exact) complex saddles



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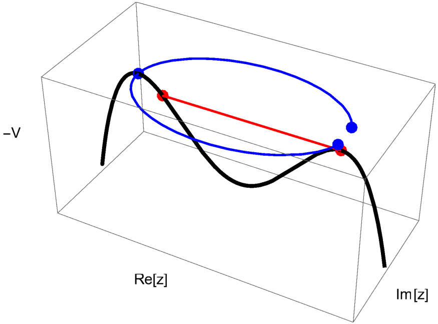
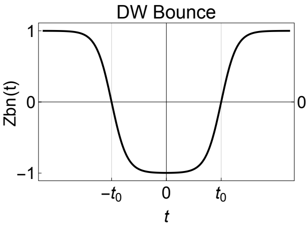
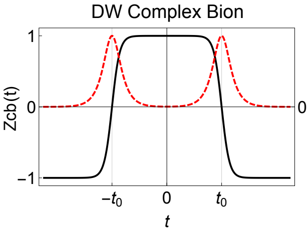
tilted double well







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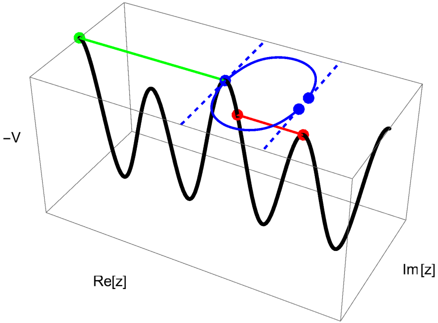
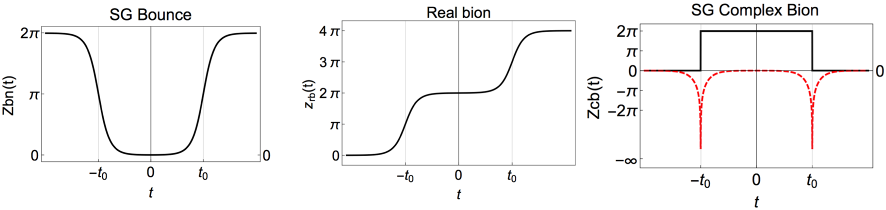
(Behtash, GD, Schäfer, Sulejmanpasic, Ünsal (1510.00978), (1510.03435)

double Sine Gordon





Necessity of Complex Saddles

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SUSY QM: $g\,\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\left(W'\right)^2 \pm \frac{g}{2} W''$

- complex saddles have complex action:

$$S_{\text{complex bion}} \sim 2S_I + i\pi$$

- $W = \cos\frac{x}{2} \rightarrow$ double Sine-Gordon

$$E_{\text{ground state}} \sim 0 - 2\,e^{-2S_I} - 2\,e^{-i\pi} e^{-2S_I} = 0 \qquad \checkmark$$

- $W = \frac{1}{3}x^3 - x \rightarrow$ tilted double-well

$$E_{\text{ground state}} \sim 0 - 2\,e^{-i\pi} e^{-2S_I} > 0 \qquad \checkmark$$

semiclassics: complex saddles required for SUSY algebra

- similar effects in QFT, also non-SUSY

Resurgence and Hydrodynamics (Heller/Spalinski 2015; Başar/GD, 2015)

- resurgence: generic feature of differential equations
- boost invariant conformal hydrodynamics
- second-order hydrodynamics: $T^{\mu\nu} = \mathcal{E}\, u^\mu\, u^\nu + T^{\mu\nu}_\perp$

$$\begin{aligned}
\tau \frac{d\mathcal{E}}{d\tau} &= -\frac{4}{3}\mathcal{E} + \Phi \\
\tau_{II}\, \frac{d\Phi}{d\tau} &= \frac{4}{3}\frac{\eta}{\tau} - \Phi - \frac{4}{3}\frac{\tau_{II}}{\tau}\,\Phi - \frac{1}{2}\frac{\lambda_1}{\eta^2}\Phi^2
\end{aligned}$$

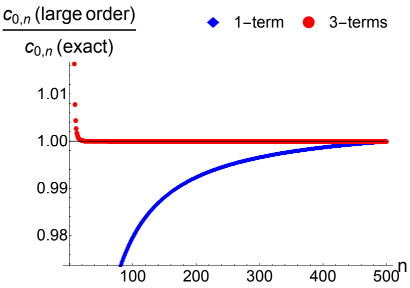
- asymptotic hydro expansion: $\mathcal{E} \sim \frac{1}{\tau^{4/3}}\left(1 - \frac{2\eta_0}{\tau^{2/3}} + \ldots\right)$
- formal series $\rightarrow$ trans-series

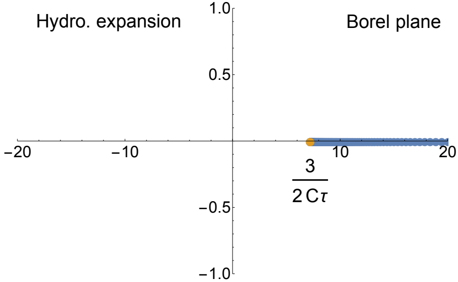
$$\mathcal{E} \sim \mathcal{E}_{\text{pert}} + e^{-S\tau^{2/3}} \times (\text{fluc}) + e^{-2S\tau^{2/3}} \times (\text{fluc}) + \ldots$$

- non-hydro modes clearly visible in the asymptotic hydro series

- study large-order behavior

$$c_{0,k} \sim S_1 \frac{\Gamma(k+\beta)}{2\pi i\, S^{k+\beta}} \left(c_{1,0} + \frac{S\, c_{1,1}}{k+\beta-1} + \frac{S^2\, c_{1,2}}{(k+\beta-1)(k+\beta-2)} + \ldots \right)$$





- resurgent large-order behavior and Borel structure verified to 4-instanton level

- ⇒ trans-series for metric coefficients in AdS

Connecting weak and strong coupling

important physics question:

does weak coupling analysis contain enough information to extrapolate to strong coupling ?

... even if the degrees of freedom re-organize themselves in a very non-trivial way?

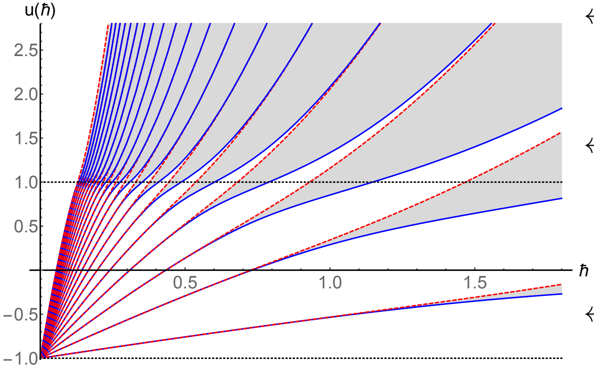
what about a QFT in which the vacuum re-arranges itself in a non-trivial manner?

classical (Poincaré) asymptotics is clearly not enough:

is resurgent asymptotics enough?

Resurgence in $\mathcal{N}=2$ and $\mathcal{N}=2^*$ Theories (Başar, GD, 1501.05671)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



⟵ electric sector (convergent)

⟵ magnetic sector

⟵ dyonic sector (divergent)

- energy: $u=u(N,\hbar)$; 't Hooft coupling: $\lambda\equiv N\,\hbar$
- very different physics for $\lambda\gg 1$, $\lambda\sim 1$, $\lambda\ll 1$

- Mathieu & Lamé encode Nekrasov twisted superpotential

Resurgence of $\mathcal{N}=2$ SUSY SU(2)

- moduli parameter: $u = \langle \mathrm{tr}\, \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{W}}{\partial a}$
- Nekrasov twisted superpotential $\mathcal{W}(a, \hbar, \lambda)$:
- Mathieu equation:

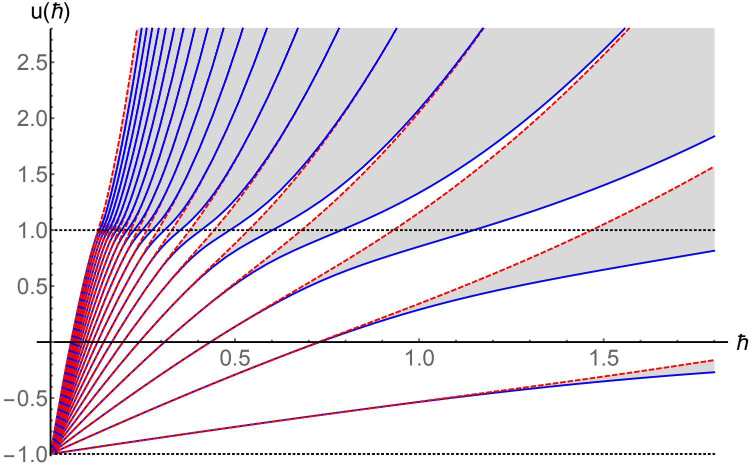
$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \Lambda^2 \cos(x)\, \psi = u\, \psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

- Matone relation:

$$u(a, \hbar) = \frac{i\pi}{2} \Lambda \frac{\partial \mathcal{W}(a, \hbar, \Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

Mathieu Equation Spectrum: ($\hbar$ plays role of g^2)

$$-\frac{\hbar^2}{2}\,\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



Mathieu Equation Spectrum

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$

- small N: divergent, non-Borel-summable $\rightarrow$ trans-series

$$u(N,\hbar) \sim -1+\hbar\left[N+\frac{1}{2}\right]-\frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right] -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right]-\ldots$$

- large N: convergent expansion: $\longrightarrow$?? trans-series ??

$$u(N,\hbar)\sim\frac{\hbar^2}{8}\left(N^2+\frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4+\frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 +\frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12}+\ldots\right)$$

Resurgence of $\mathcal{N}=2$ SUSY SU(2) (Başar, GD, 1501.05671)

- $N\hbar \ll 1$, deep inside wells: resurgent trans-series

$$u^{(\pm)}(N,\hbar) \sim \sum_{n=0}^{\infty} c_n(N)\hbar^n \pm \frac{32}{\sqrt{\pi}\, N!}\left(\frac{32}{\hbar}\right)^{N-1/2} e^{-\frac{8}{\hbar}} \sum_{n=0}^{\infty} d_n(N)\hbar^n + \dots$$

- Borel poles at two-instanton location
- $N\hbar \gg 1$, far above barrier: convergent series

$$u^{(\pm)}(N,\hbar) = \frac{\hbar^2 N^2}{8}\sum_{n=0}^{N-1}\frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8}\frac{\left(\frac{2}{\hbar}\right)^{2N}}{(2^{N-1}(N-1)!)^2}\sum_{n=0}^{N-1}\frac{\beta_n(N)}{\hbar^{4n}} + \dots$$

(Basar, GD, Ünsal, 2015)

- coefficients have poles at O(two-(complex)-instanton)
- $N\hbar \sim \frac{8}{\pi}$, near barrier top: "instanton condensation"

$$u^{(\pm)}(N,\hbar) \sim 1 \pm \frac{\pi}{16}\hbar + O(\hbar^2)$$

Uniform Expansions: Small $\hbar$ and Large N

- mathematical analogy: Bessel functions

$$I_N\left(\frac{1}{\hbar}\right) = I_N\left(N\frac{1}{N\hbar}\right) \sim \begin{cases} \sqrt{\frac{\hbar}{2\pi}}\, e^{1/\hbar} & , \quad \hbar \to 0,\ N \text{ fixed} \\ \frac{1}{\sqrt{2\pi N}}\left(\frac{e}{2N\hbar}\right)^N & , \quad N \to \infty,\ \hbar \text{ fixed} \end{cases}$$

- uniform asymptotics:

$$I_N\left(N\frac{1}{N\hbar}\right) \sim \frac{\exp\left[\sqrt{N^2+\frac{1}{\hbar^2}}\right]}{\sqrt{2\pi}\left(N^2+\frac{1}{\hbar^2}\right)^{\frac{1}{4}}}\left(\frac{\frac{1}{N\,\hbar}}{1+\sqrt{1+\frac{1}{(N\,\hbar)^2}}}\right)^N$$

- physical analogy: Schwinger pair production

$E(t) = \mathcal{E}\cos(\omega t)$: adiabaticity parameter: $\gamma \equiv \frac{m\,c\,\omega}{e\,\mathcal{E}}$

$$P_{\rm QED} \sim \begin{cases} \exp\left[-\pi\,\frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \left(\frac{e\,\mathcal{E}}{\omega\,m\,c}\right)^{4mc^2/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis
- trans-series 'encode' all information, and expansions about different saddles are intimately related
- local analysis encodes more than one might think
- matrix models, large N, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ SUSY gauge theory
- hydrodynamical equations
- fundamental property of steepest descents expansion
- analytic continuation for path integrals