Entanglement, Conformal Field Theory, and Interfaces

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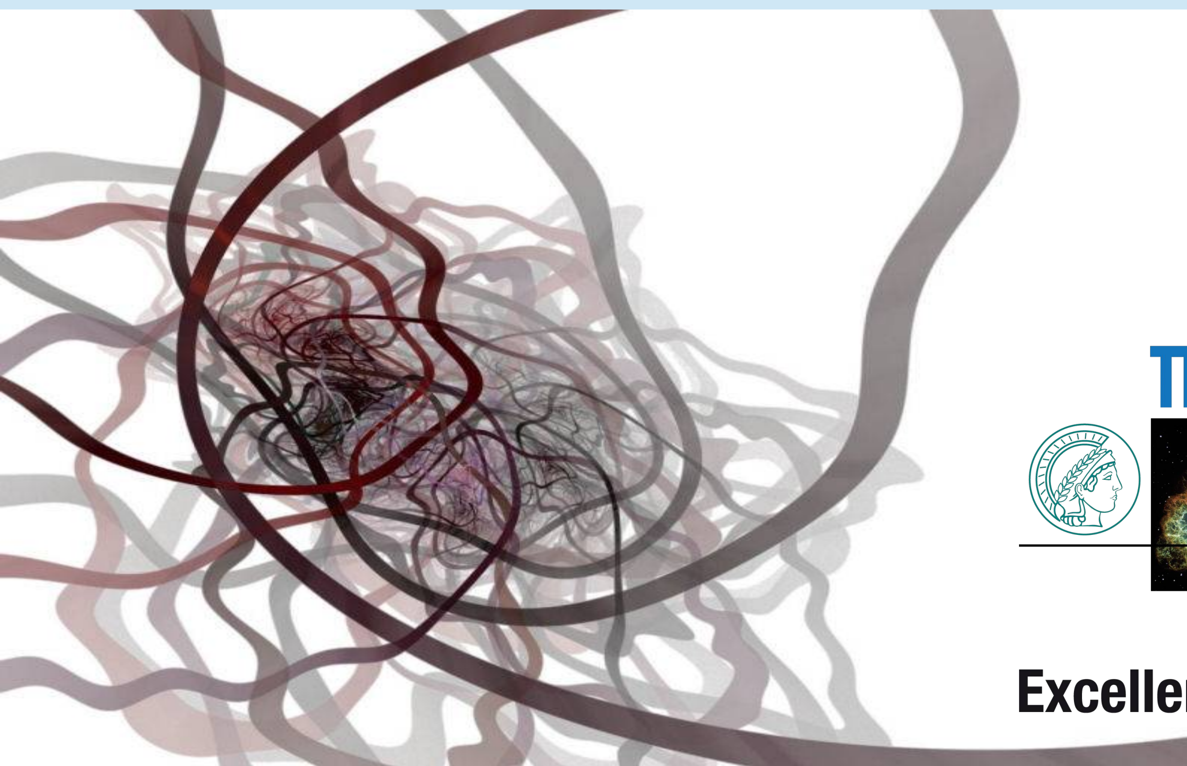
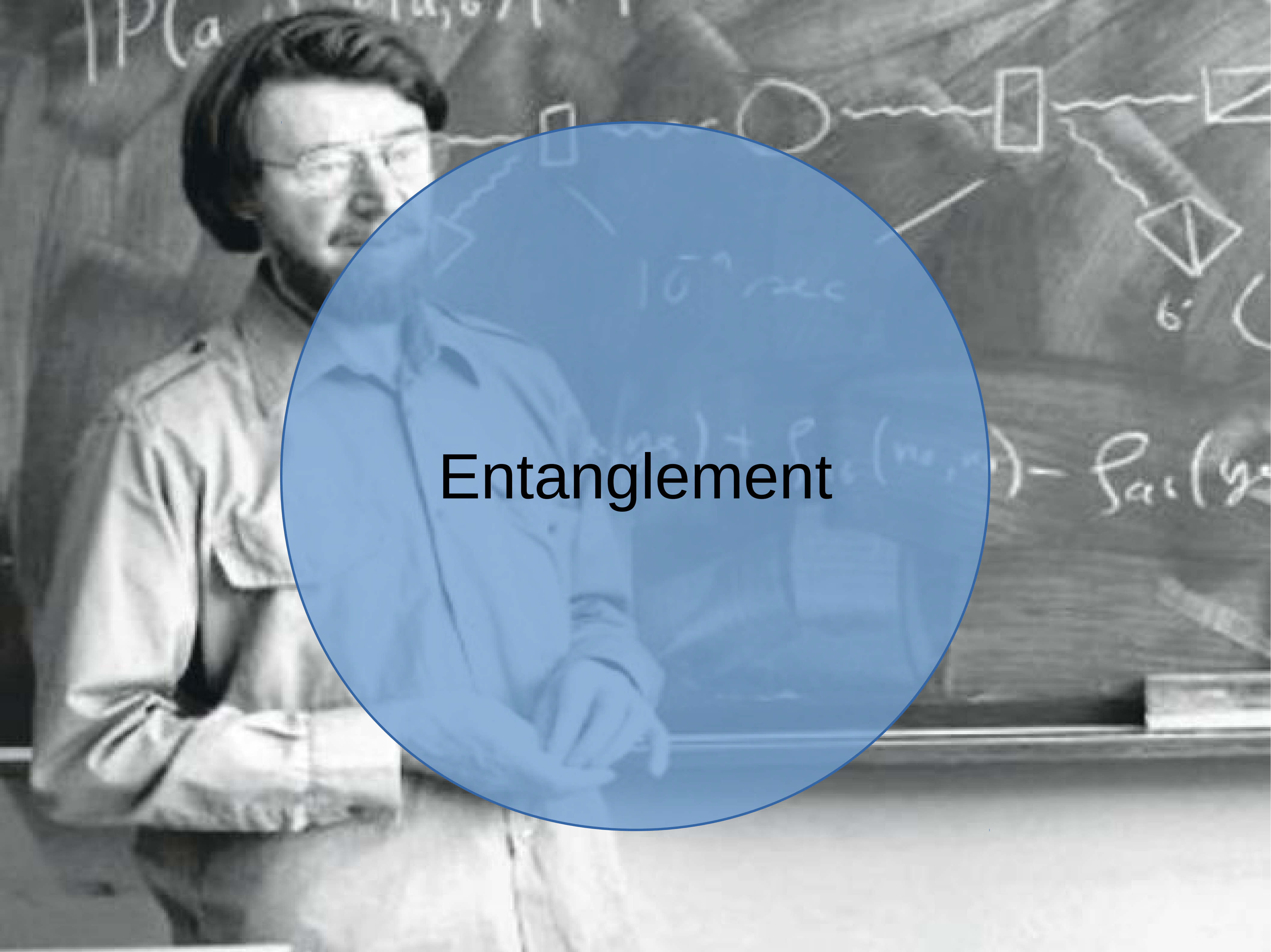
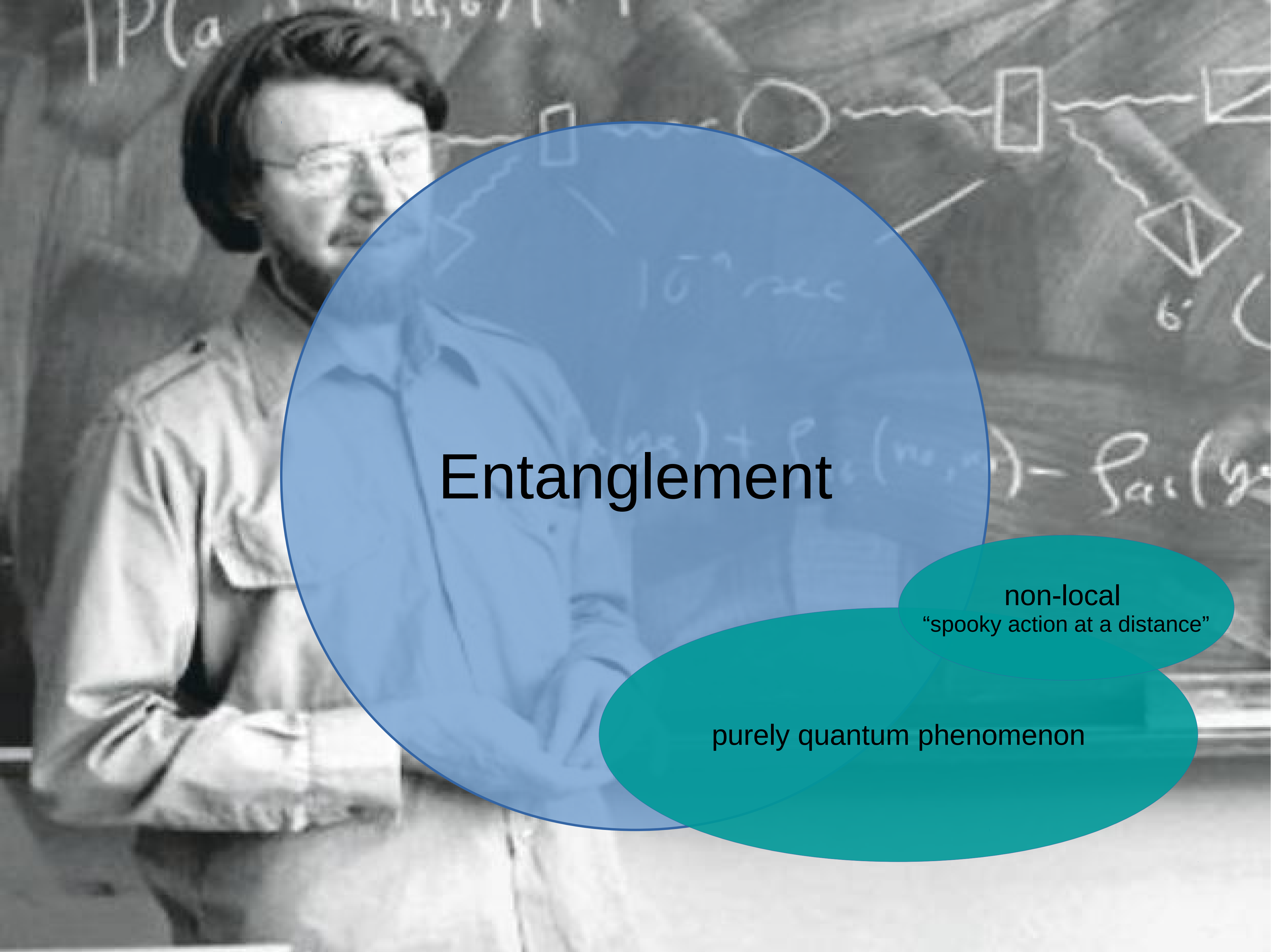


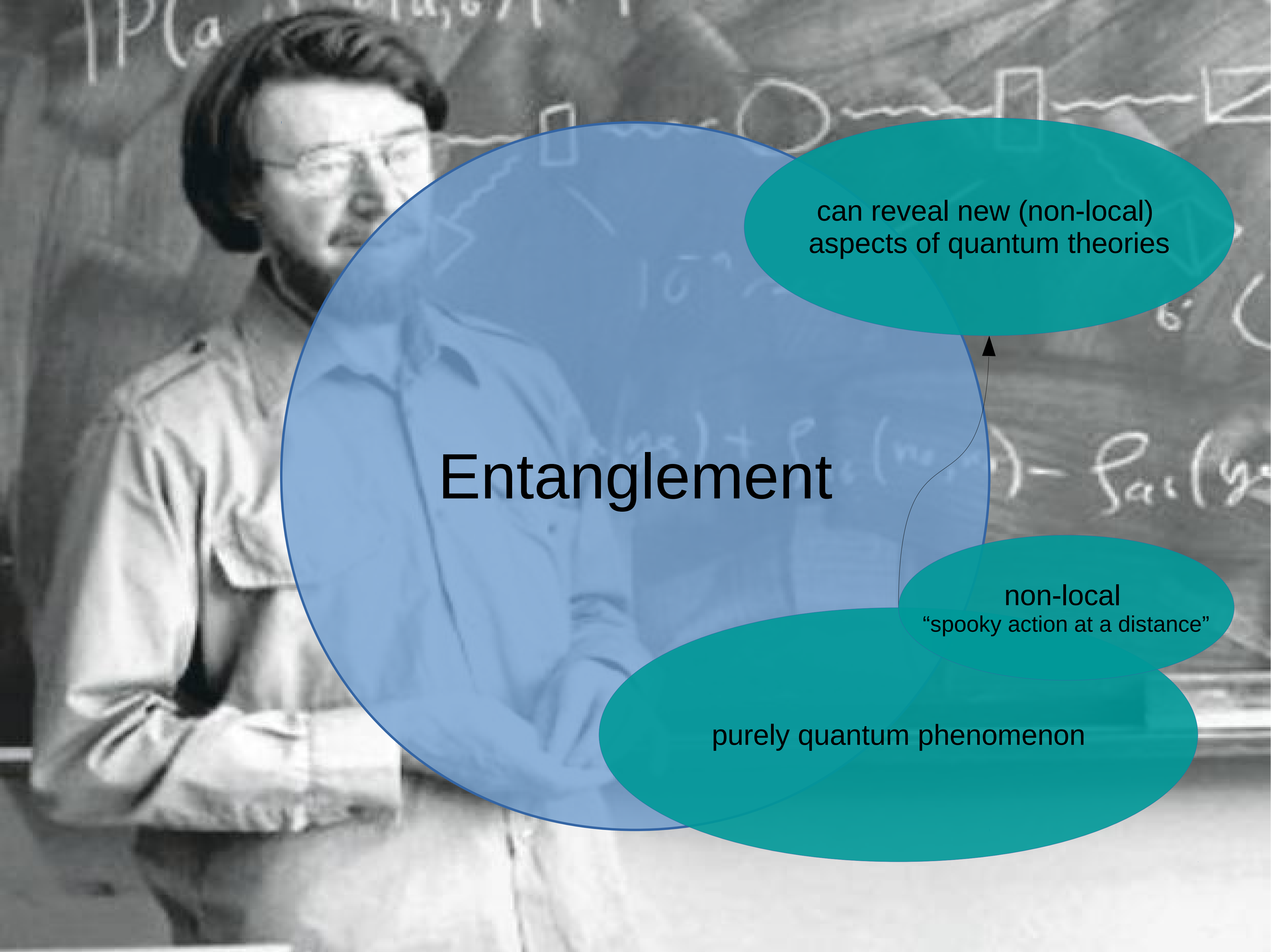
with Ilka Brunner,

Daniel Jaud,
Cornelius Schmidt Colinet

Excellence Cluster Universe


Entanglement


Entanglement
non-local
“spooky action at a distance”
purely quantum phenomenon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aspects of quantum theories
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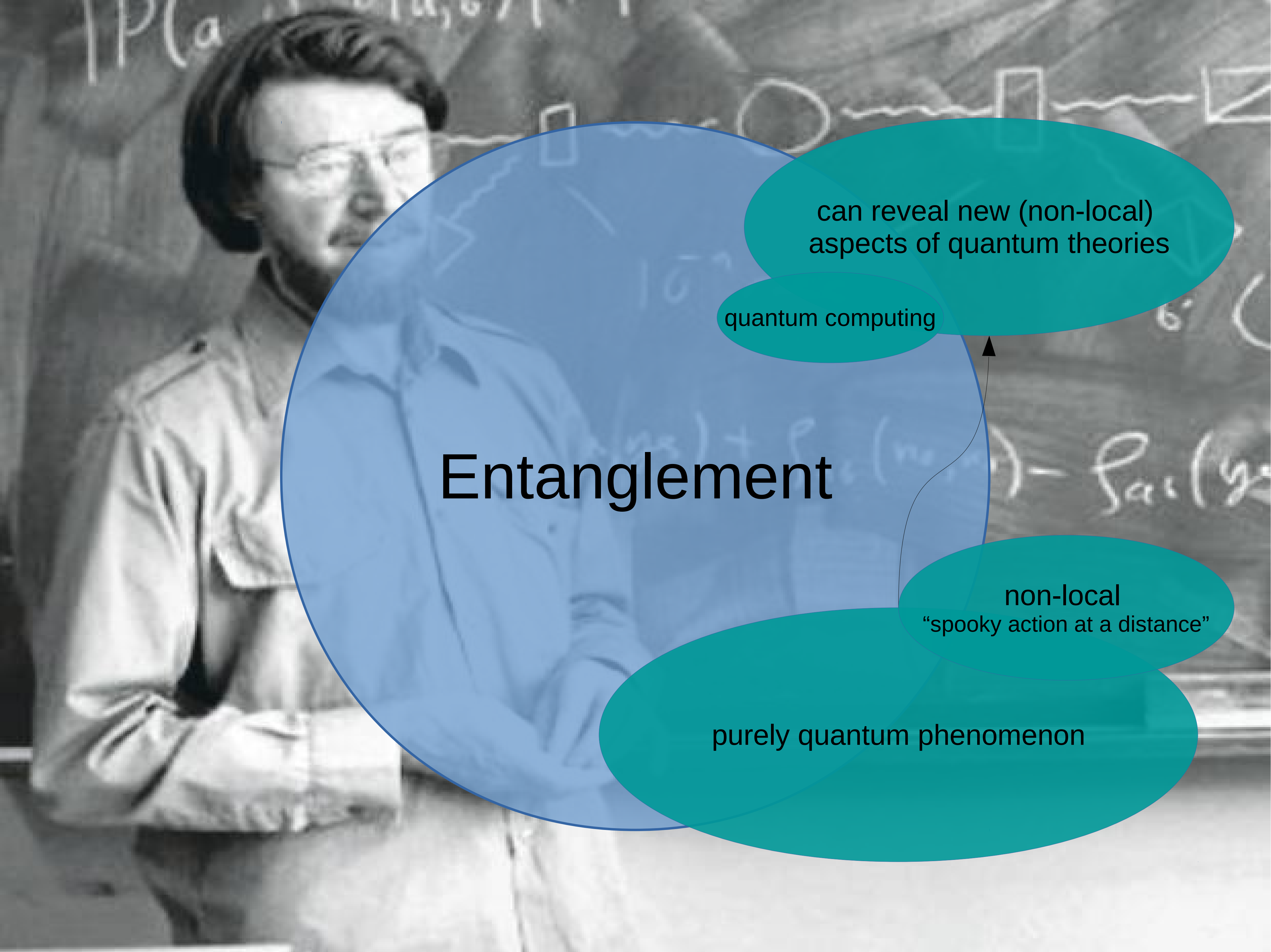
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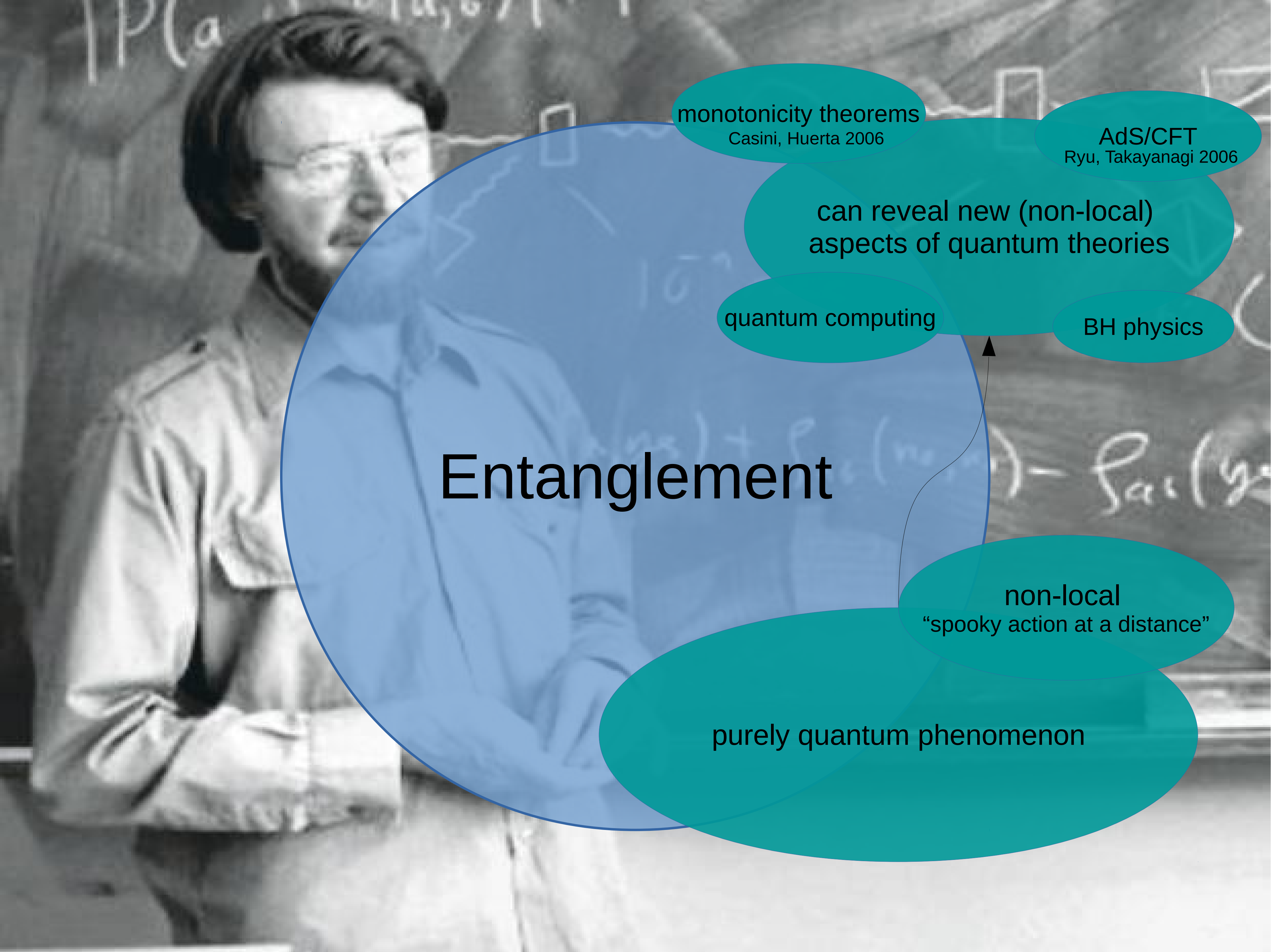
monotonicity theorems
Casini, Huerta 2006
AdS/CFT
Ryu, Takayanagi 2006
can reveal new (non-local) aspects of quantum theories
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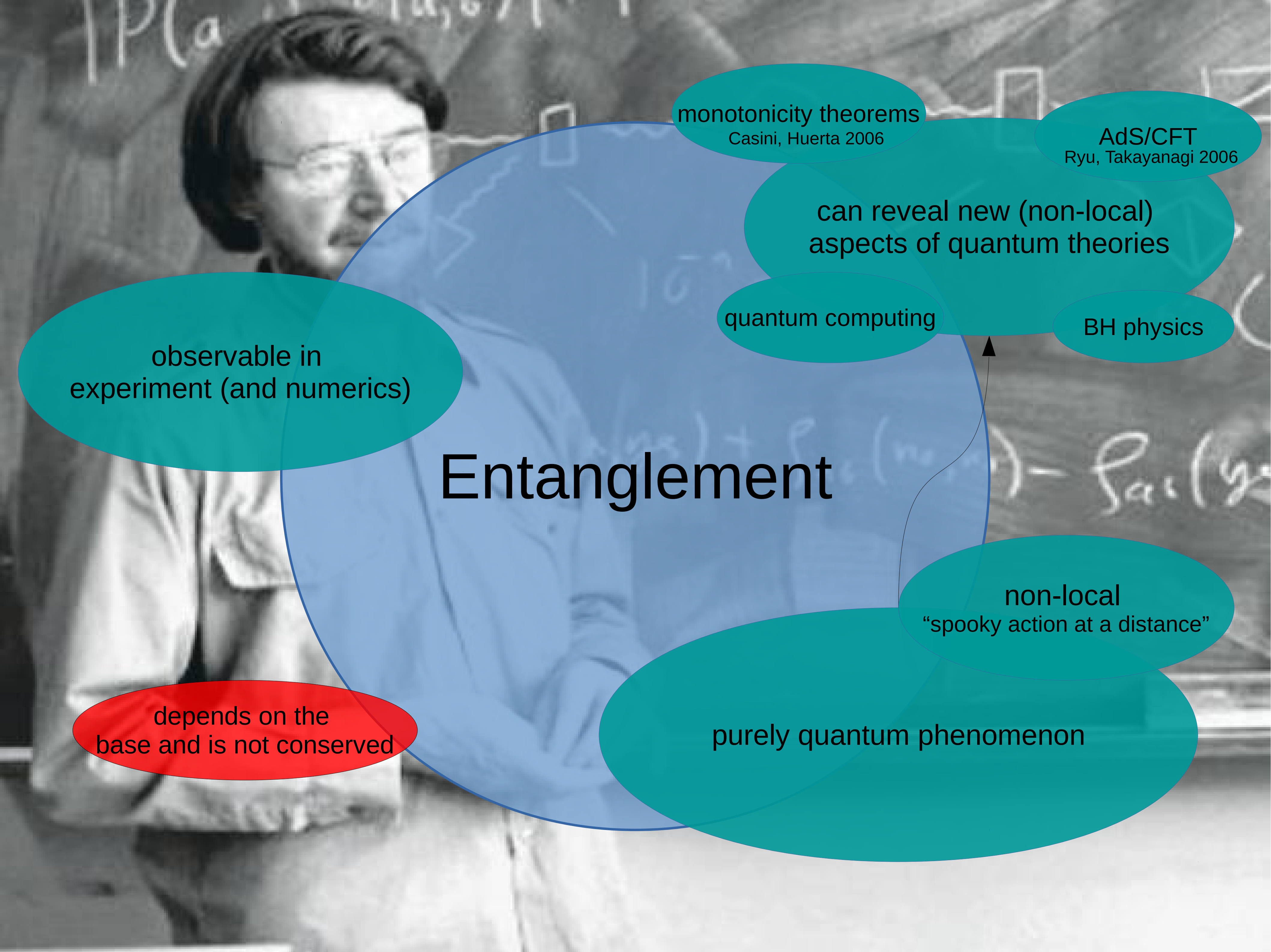
monotonicity theorems
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can reveal new (non-local) aspects of quantum theories
quantum computing
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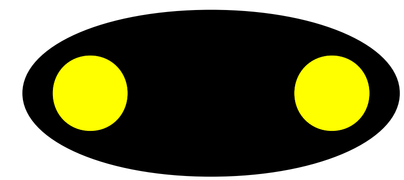
observable in experiment (and numerics)
Entanglement
non-local
“spooky action at a distance”
purely quantum phenomenon
depends on the base and is not conserved

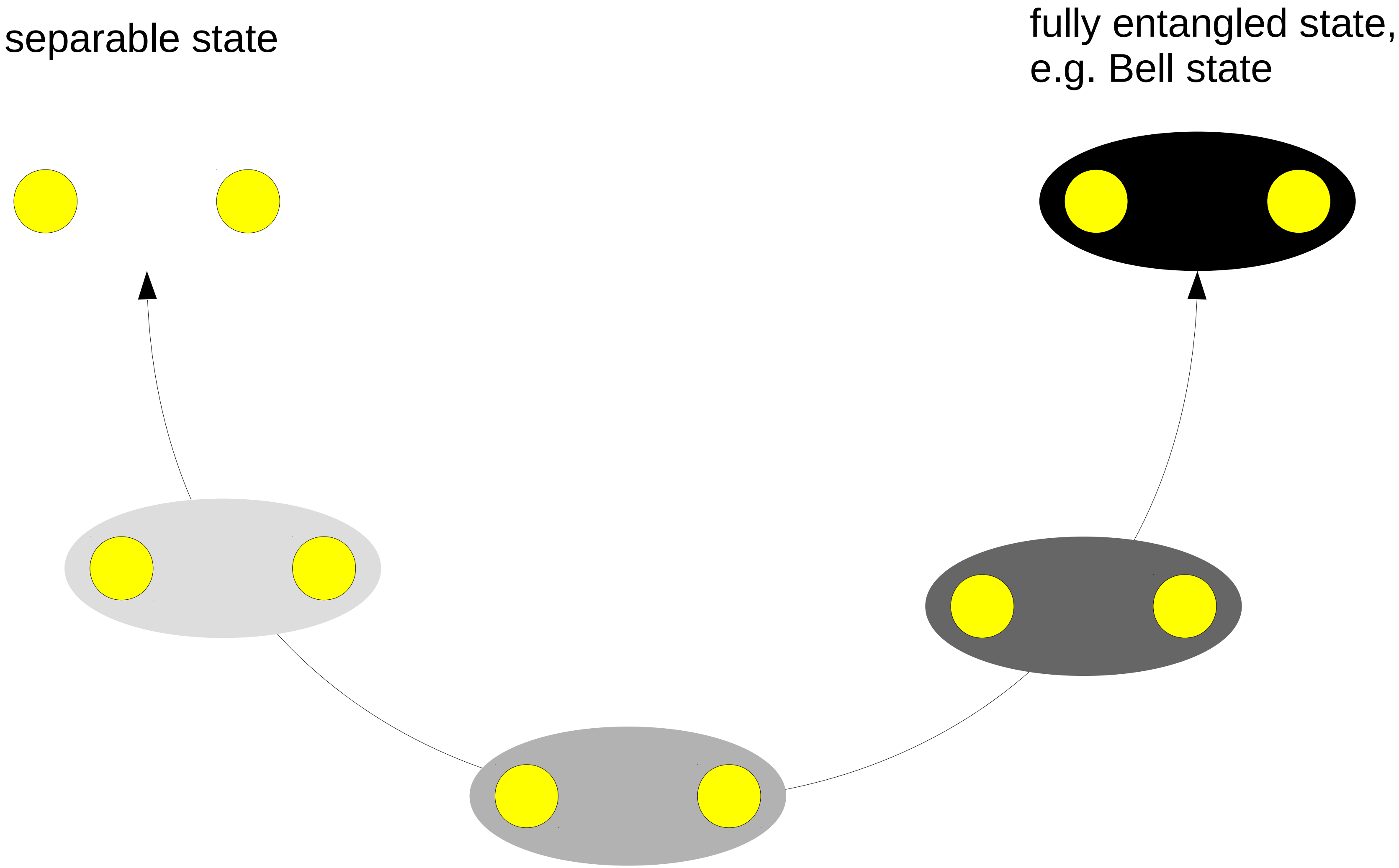
Measure of Entanglement

separable state

fully entangled state,
e.g. Bell state



Measure of Entanglement

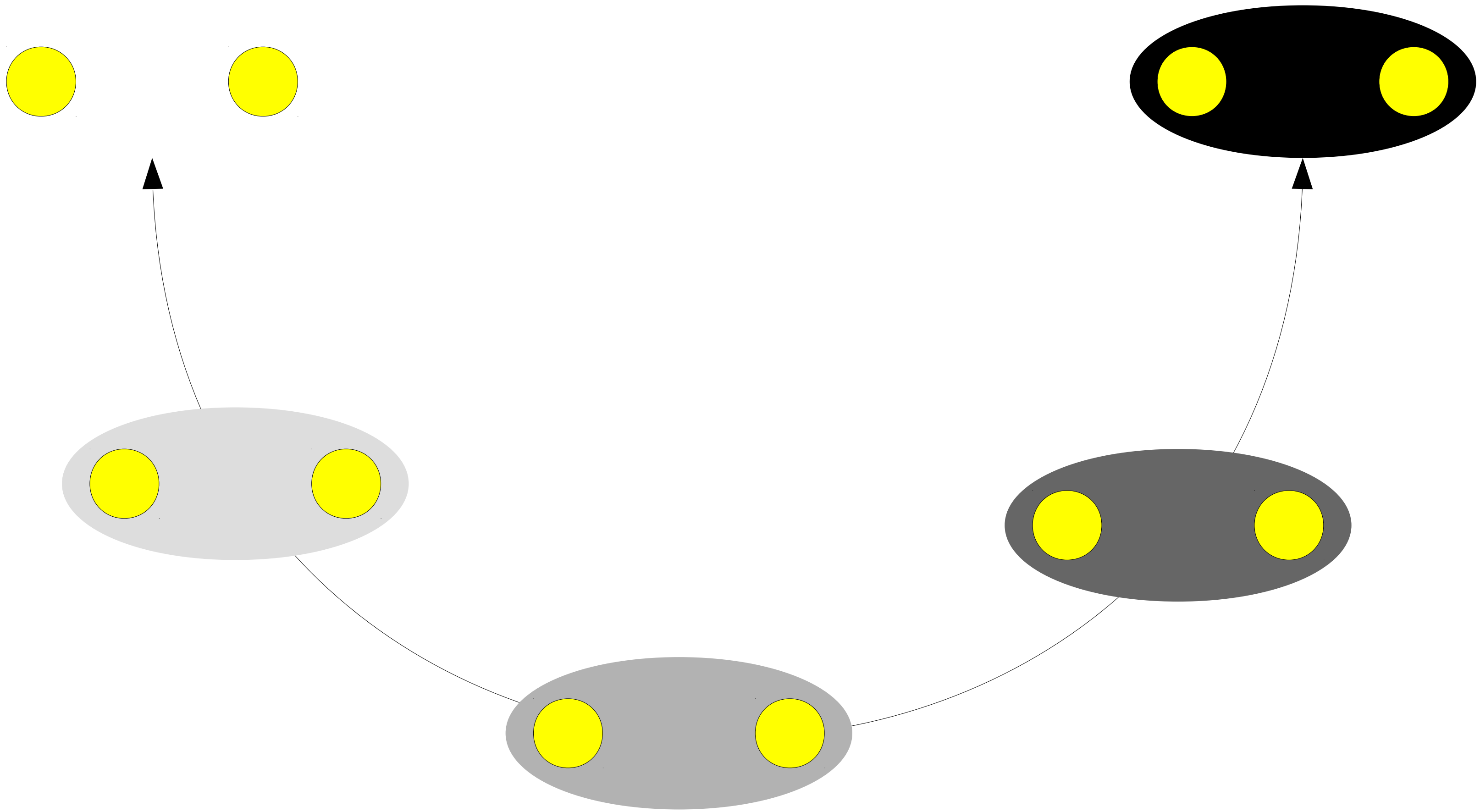


Measure of Entanglement

How to “order” these states?

separable state

fully entangled state,
e.g. Bell state



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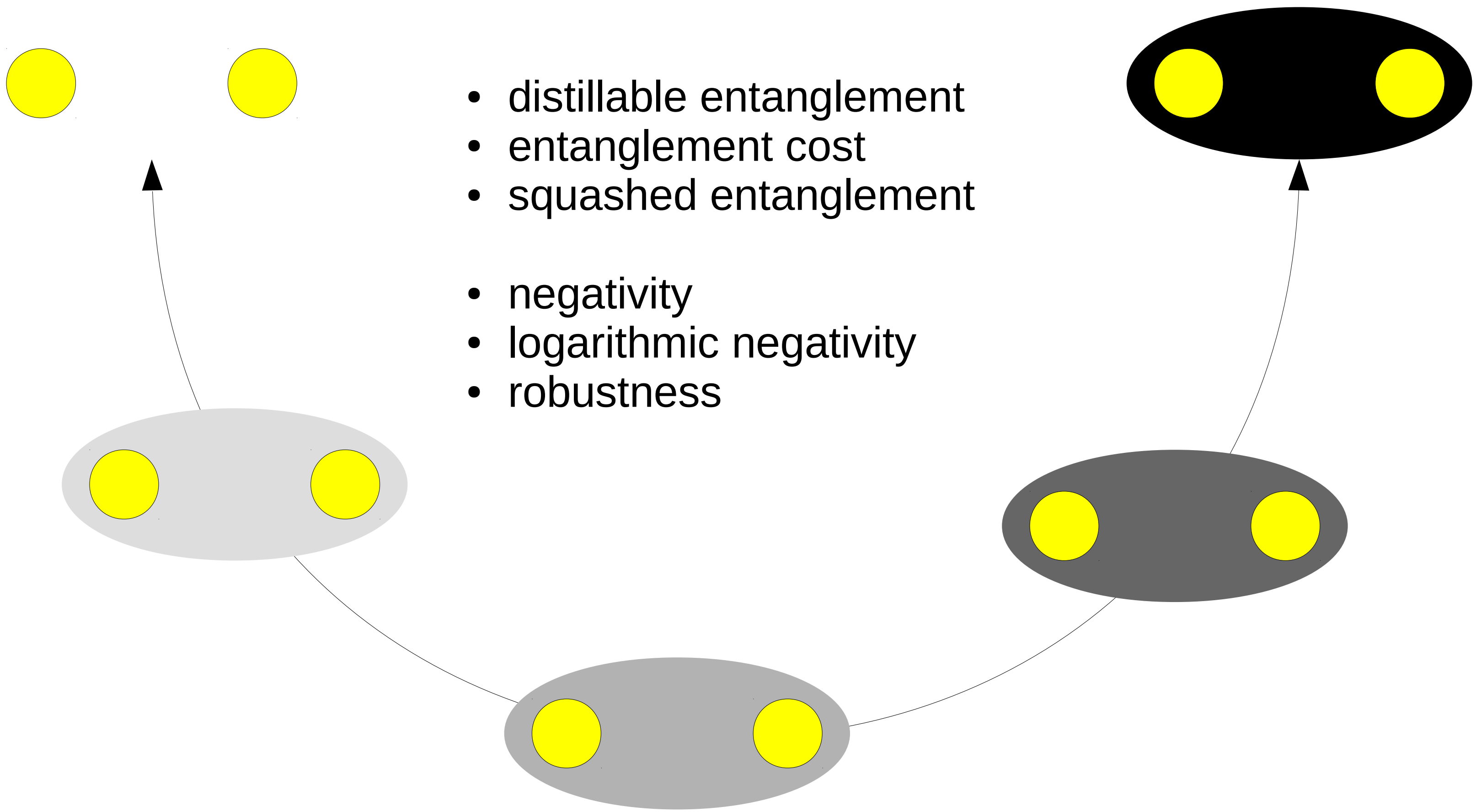


- distillable entanglement
- entanglement cost
- squashed entanglement

- negativity
- logarithmic negativity
- robustness

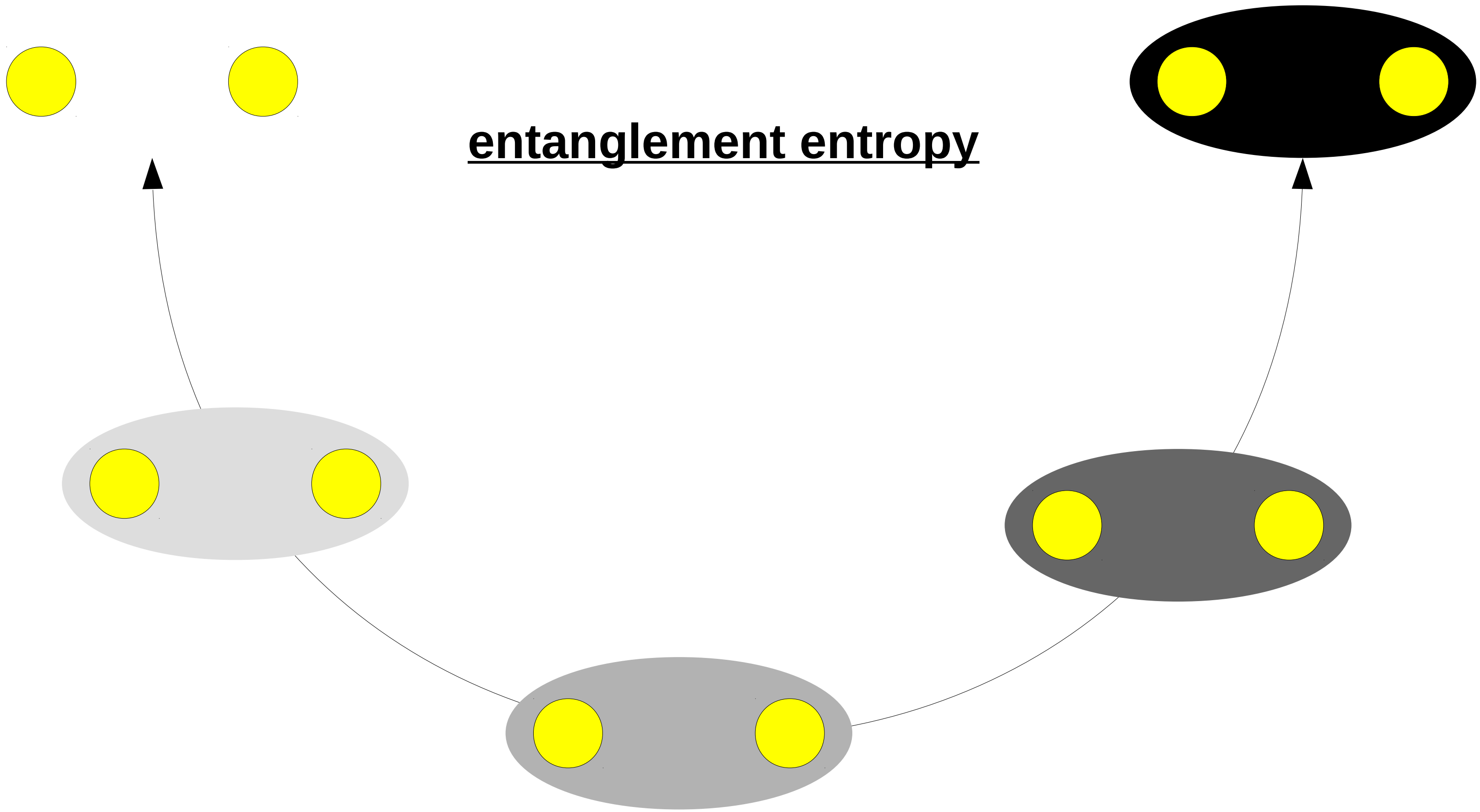
Measure of Entanglement

separable state

How to “order” these states?

fully entangled state,
e.g. Bell state

entanglement entropy

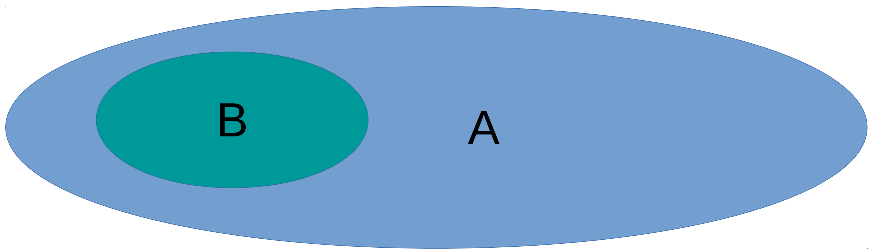


Entanglement Entropy

Definition: Let $\rho = |\psi\rangle\langle\psi|$ be the **density matrix** of a system in a pure quantum state $|\psi\rangle$. Let the Hilbert space be a direct product $\mathcal{H} = \mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}$. The reduced density matrix of A is $\rho_A = \mathrm{Tr}_\mathrm{B}\,\rho$. The **entanglement entropy** is the corresponding **von Neumann entropy**

$$S_A = -\mathrm{Tr}\,\rho_A \log \rho_A.$$

It measures the entanglement, i.e. quantum correlation, between the two sub-systems **A** and **B**.



Entanglement Entropy

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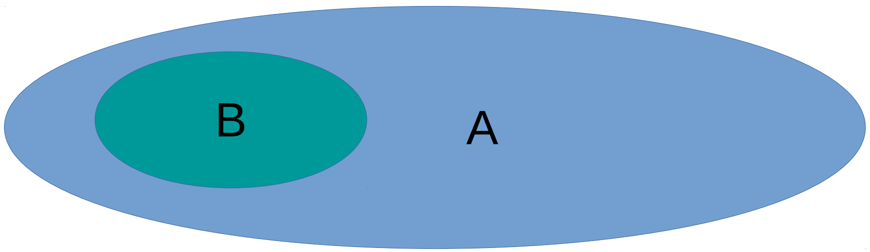
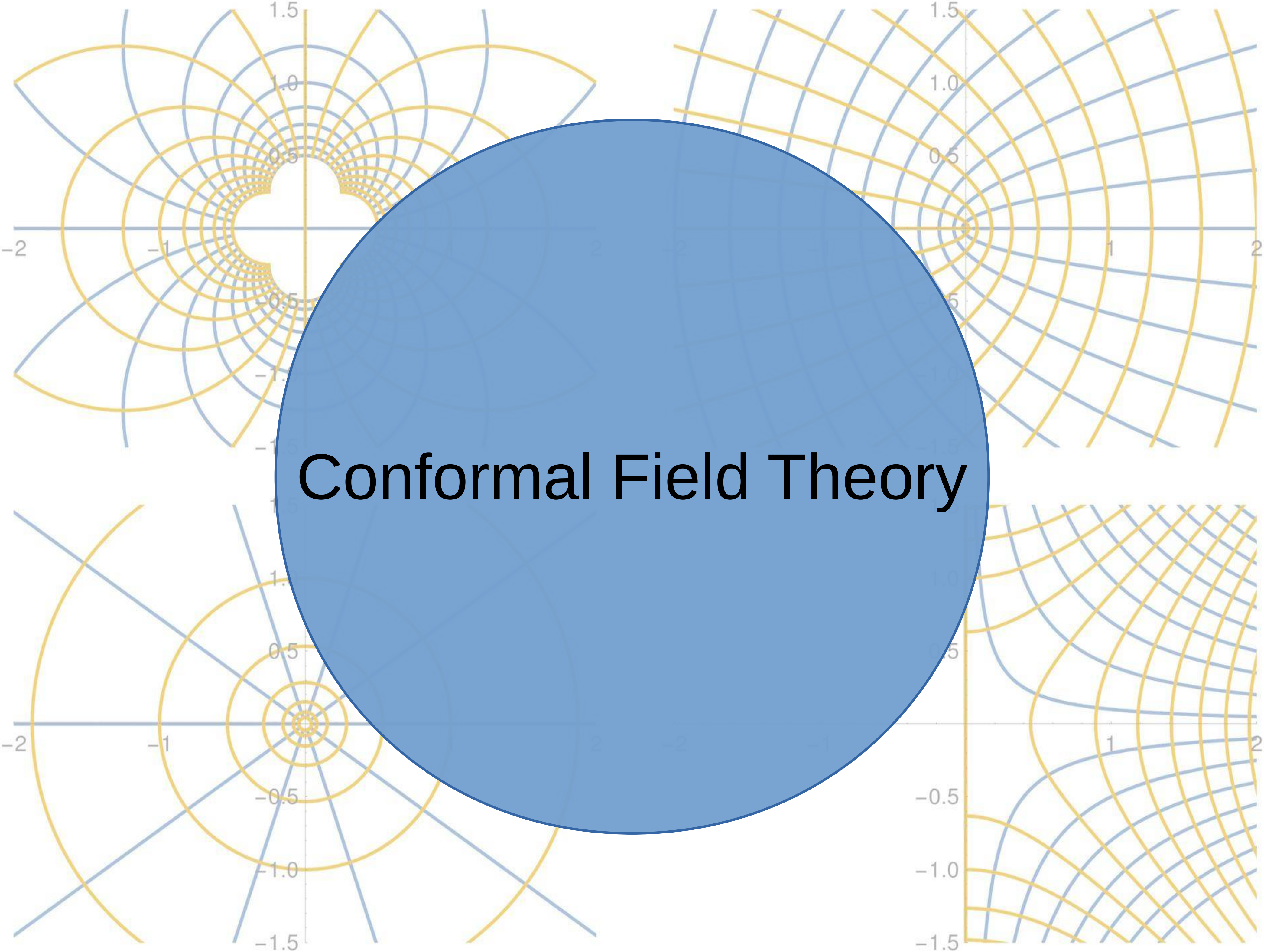
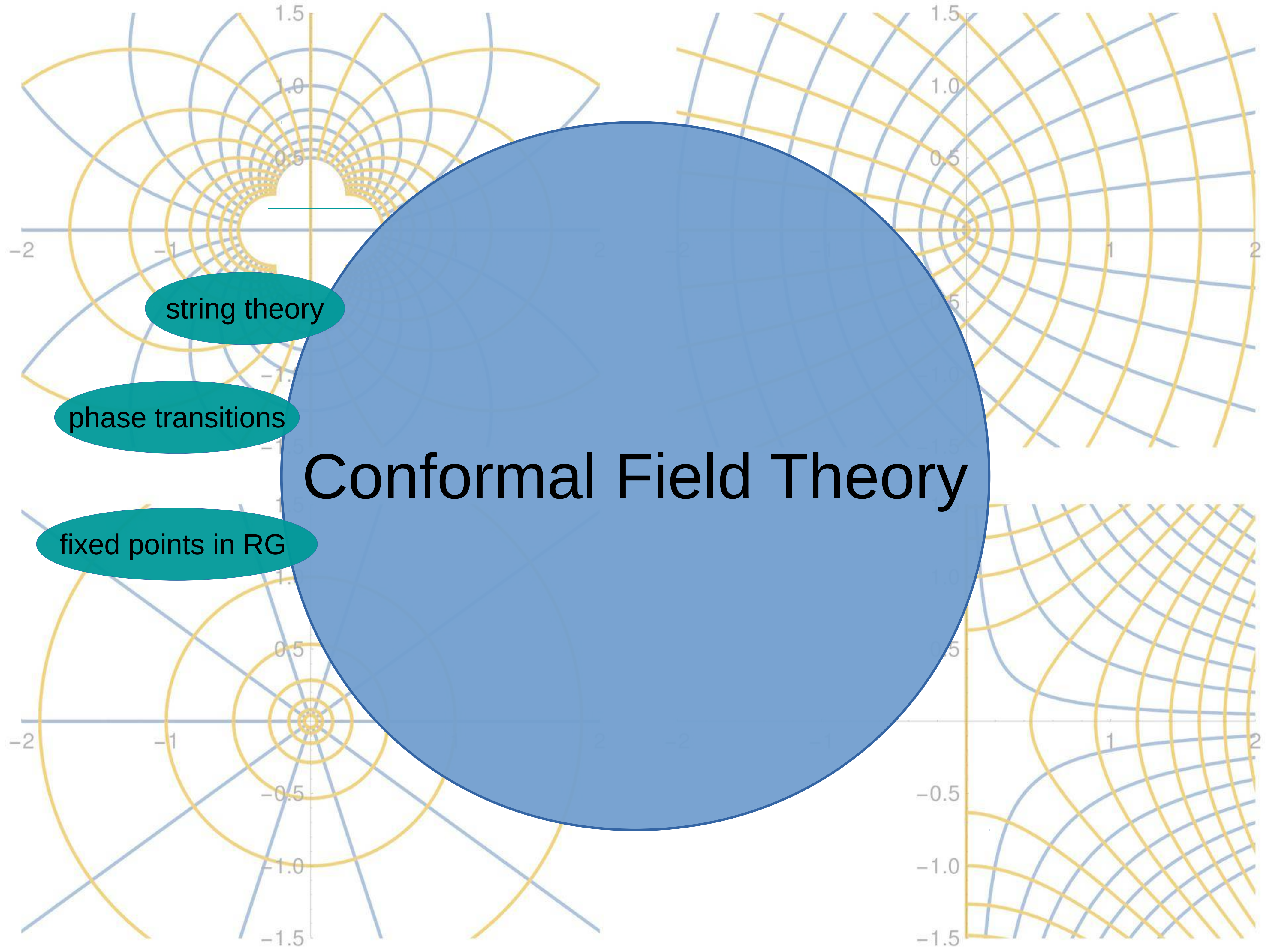


Replica trick...

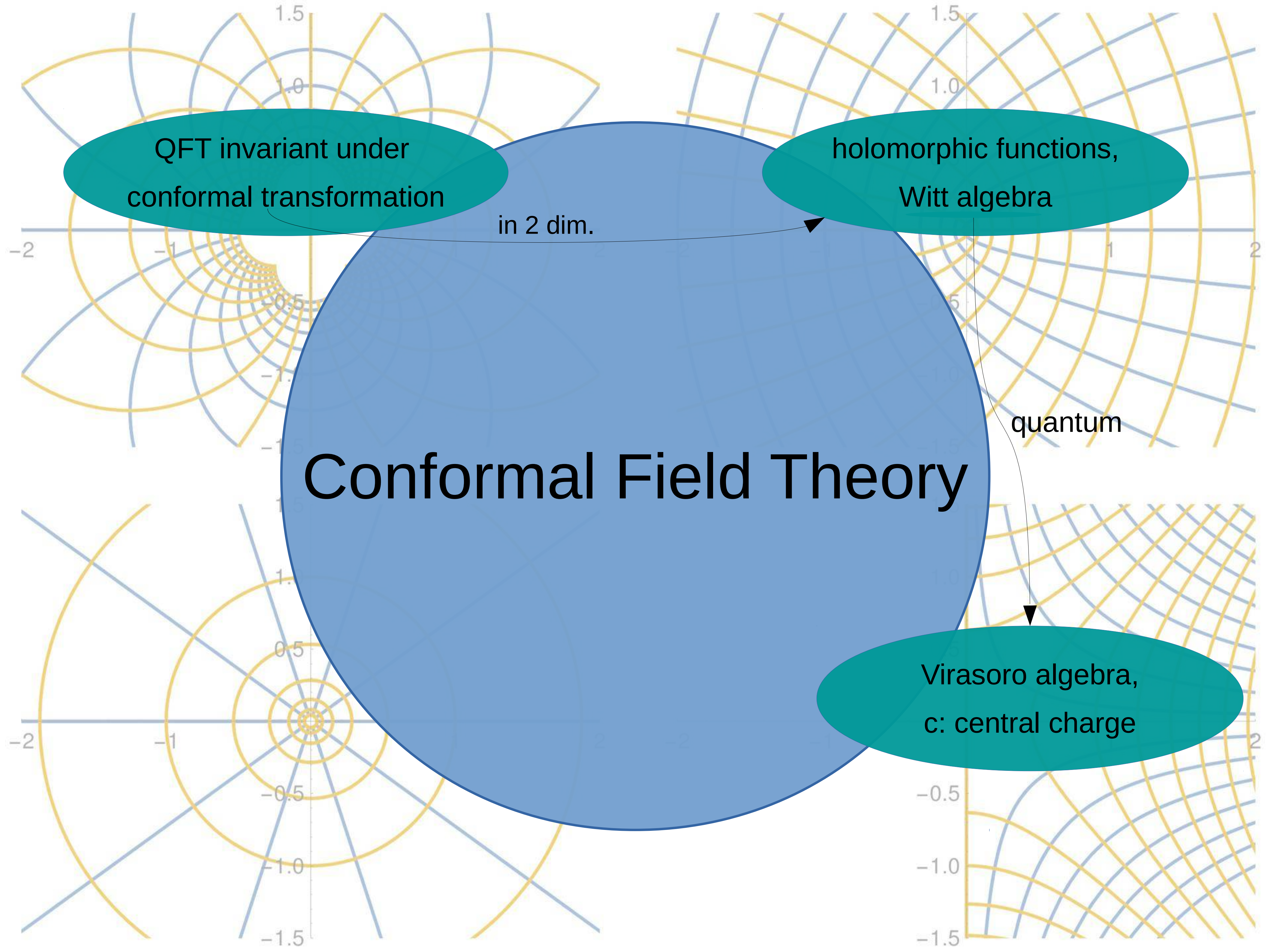
$$S_A = -\frac{\partial}{\partial n}\mathrm{Tr}\rho_A^n|_{n\to 1}$$

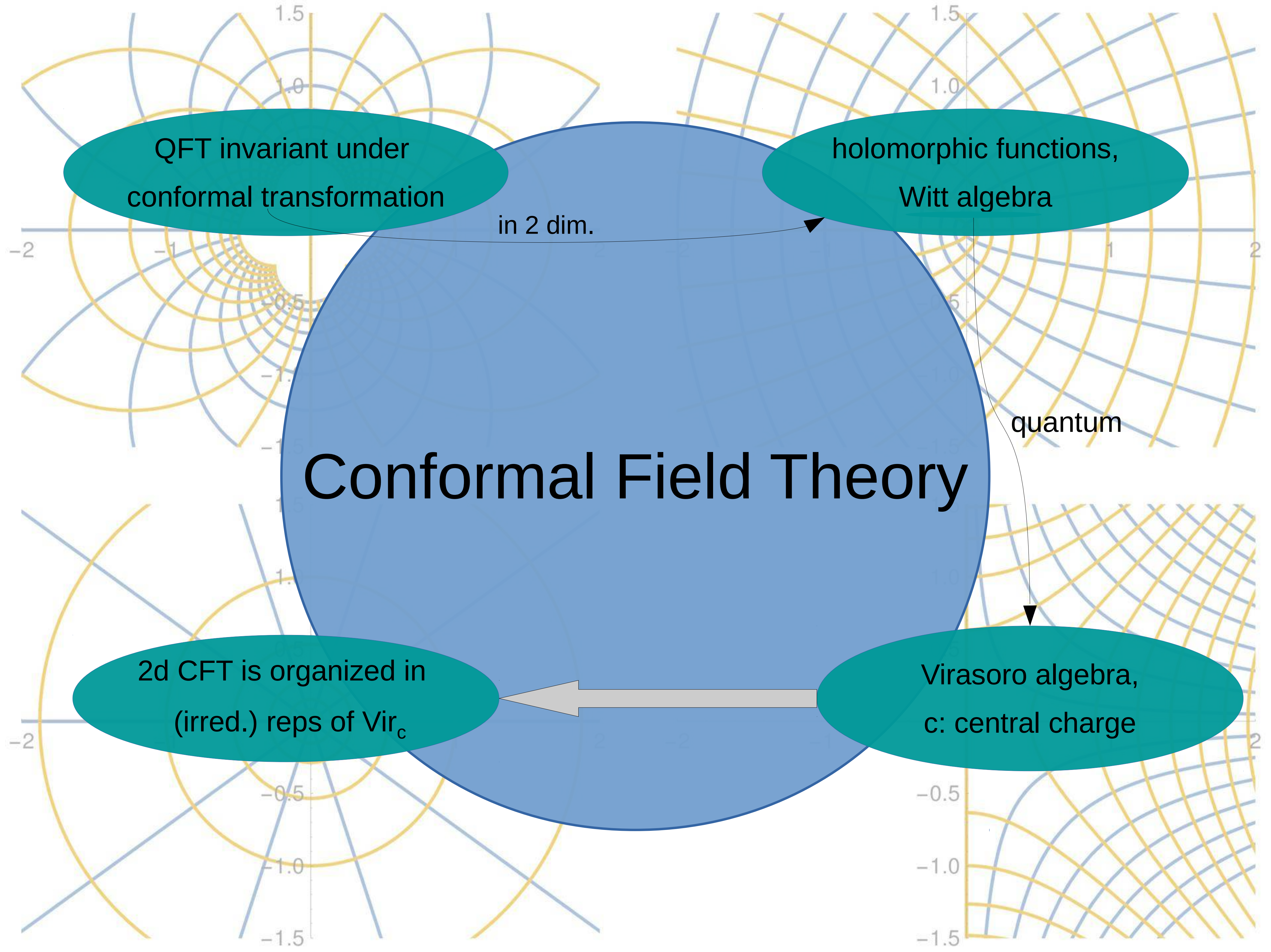
Conformal Field Theory



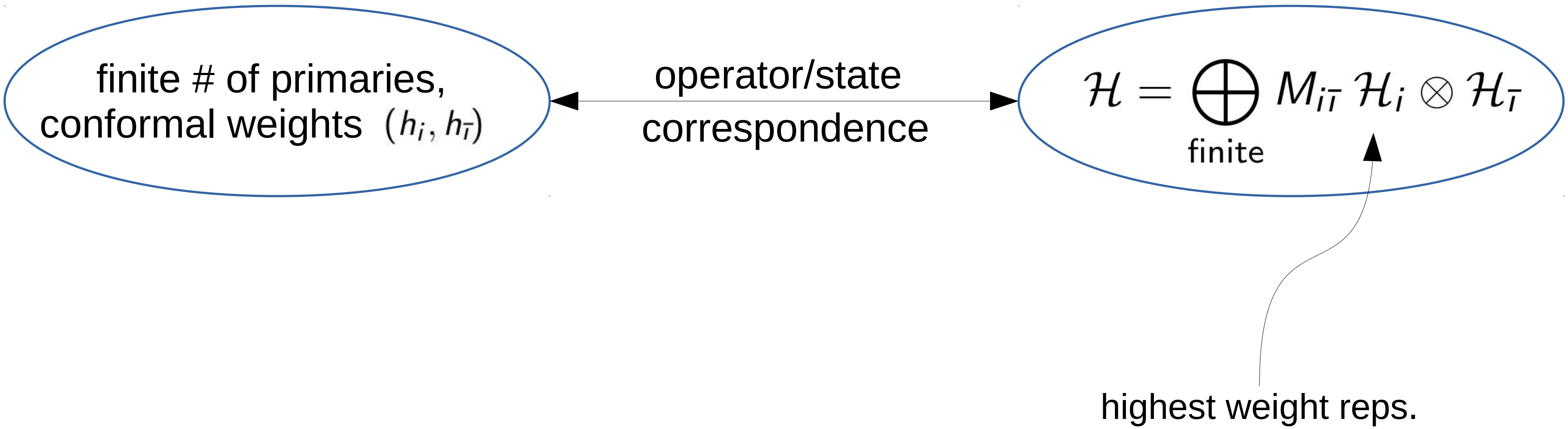


Conformal Field Theory
string theory
phase transitions
fixed points in RG


QFT invariant under
conformal transformation
in 2 dim.
holomorphic functions,
Witt algebra
quantum
Conformal Field Theory
Virasoro algebra,
c: central charge


Conformal Field Theory
QFT invariant under
conformal transformation
in 2 dim.
holomorphic functions,
Witt algebra
quantum
Virasoro algebra,
c: central charge
2d CFT is organized in
(irred.) reps of Vir_c

Rational Models



Rational Models

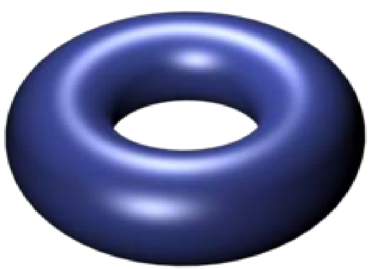
finite # of primaries,
conformal weights $(h_i, h_{\bar{\imath}})$

operator/state
correspondence

$$\mathcal{H} = \bigoplus_{\text{finite}} M_{i\bar{\imath}}\, \mathcal{H}_i \otimes \mathcal{H}_{\bar{\imath}}$$

highest weight reps.

their characters



torus partition function

$$Z(\tau,\bar{\tau}) = \sum_{(i\bar{\imath})} M_{(i\bar{\imath})}\ \chi_i(q)\chi_{\bar{\imath}}(\bar{q}) = \sum_{(i\bar{\imath})} M_{(i\bar{\imath})}\, S_{ij} S_{\bar{\imath}\bar{\jmath}}\ \chi_j(\tilde{q})\chi_{\bar{\jmath}}(\bar{\tilde{q}})$$

Example: Critical Ising Model

-1	1	1	1	-1
1	-1	1	-1	-1
1	-1	1	-1	-1

at some critical value:

second order phase transition

→ **scale invariance**

Example: Critical Ising Model

-1	1	1	1	-1
1	-1	1	-1	-1
1	-1	1	-1	-1

at some critical value:

second order phase transition

→ **scale invariance**

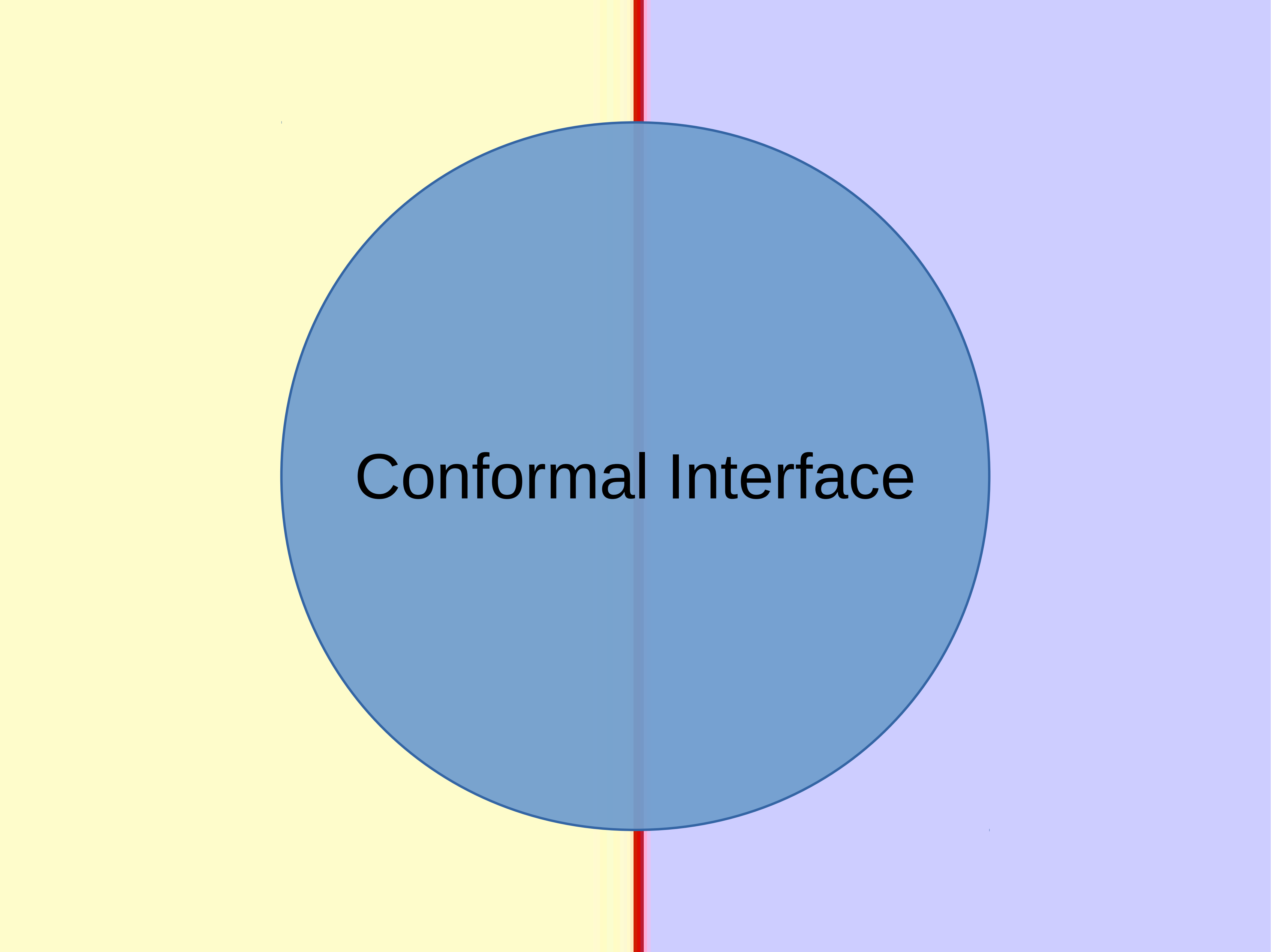
in continuum limit:

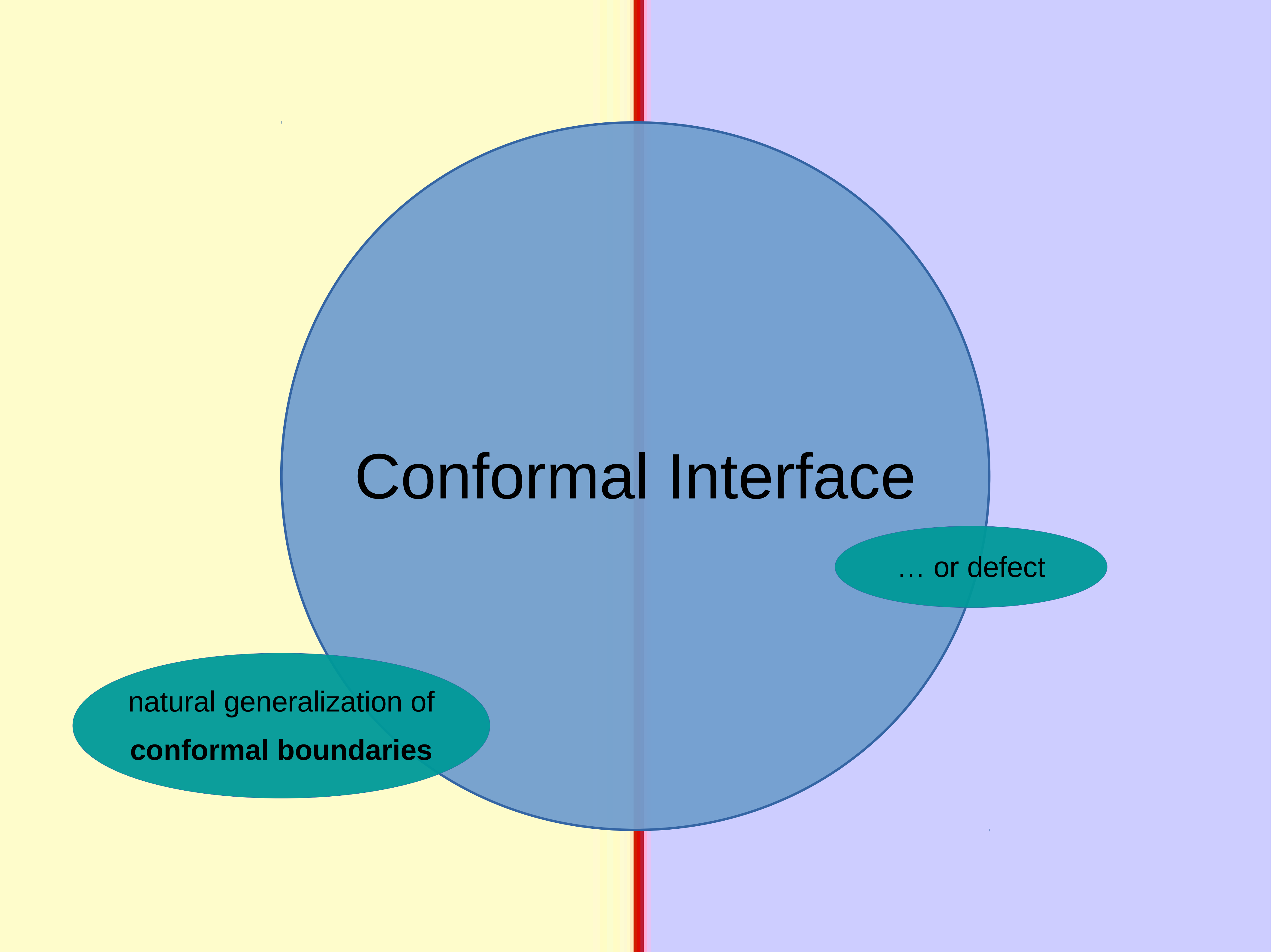
free Majorana fermions projected on even fermion numbers

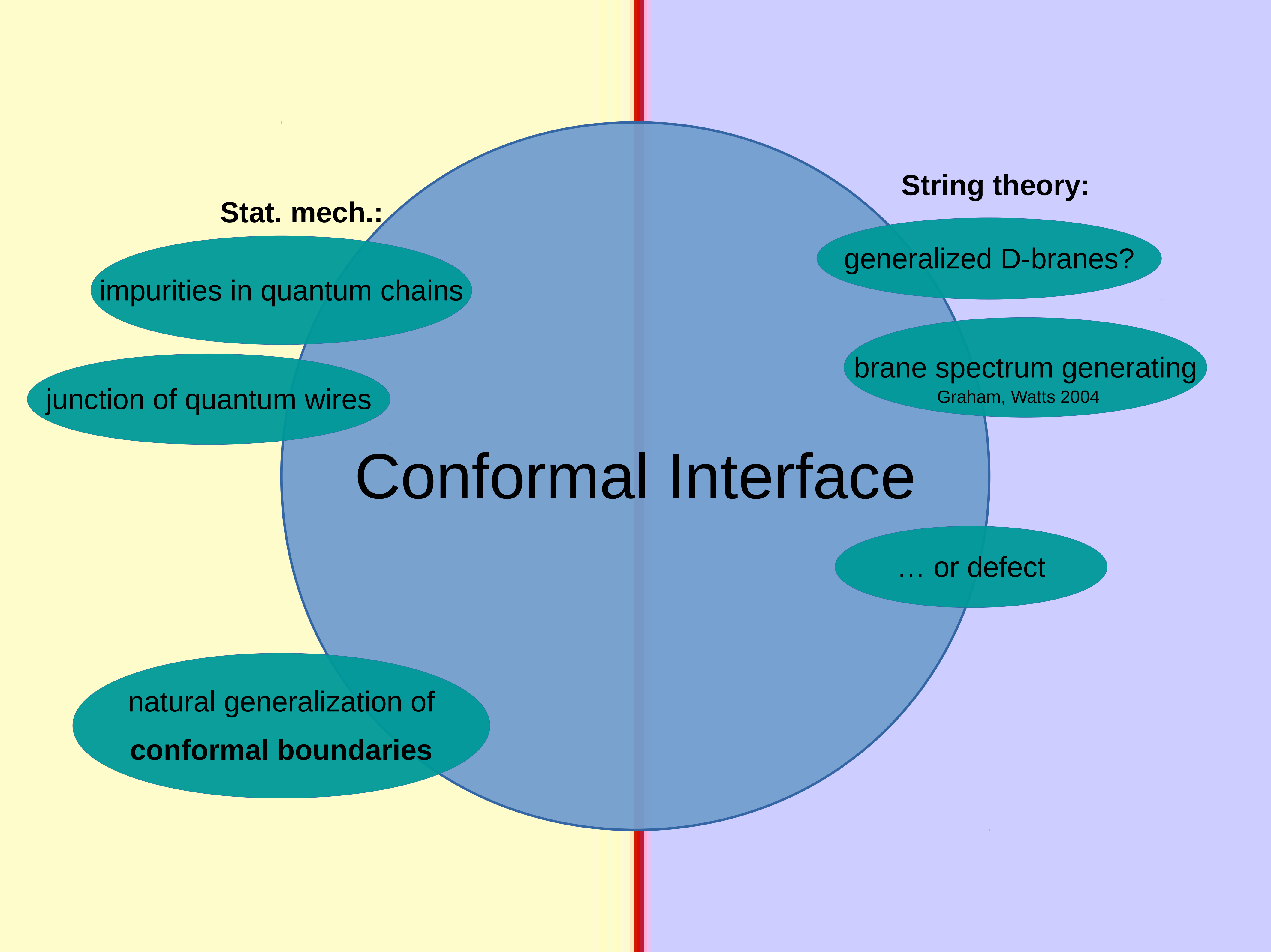
↕

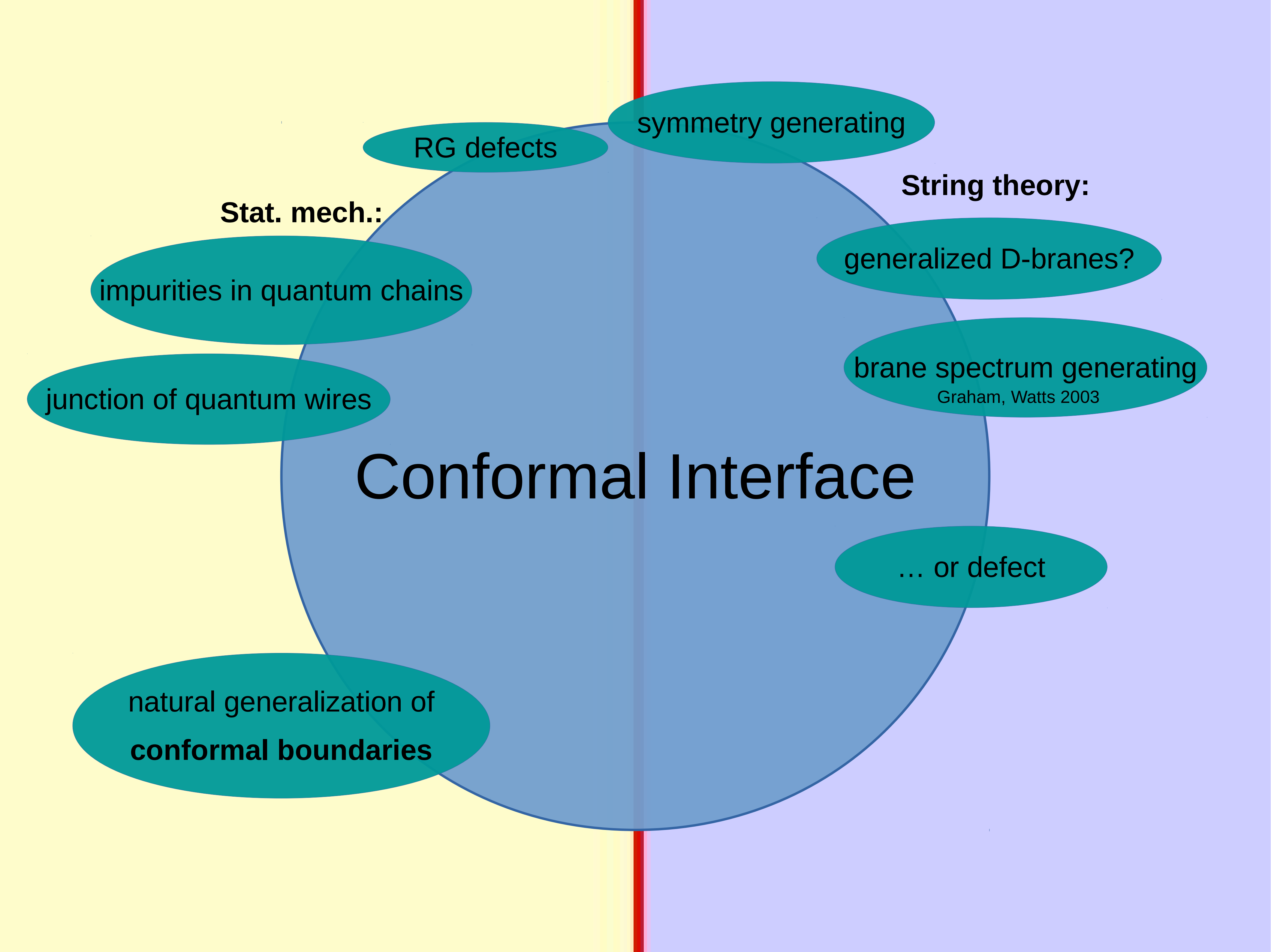
rational model consisting of **3 primaries:**

primary	conformal weight
id	(0,0)
ε	(1/2, 1/2)
σ	(1/16,1/16)


Conformal Interface


Conformal Interface
… or defect
natural generalization of
conformal boundaries


Stat. mech.:
impurities in quantum chains
junction of quantum wires
natural generalization of
conformal boundaries
Conformal Interface
String theory:
generalized D-branes?
brane spectrum generating
Graham, Watts 2004
… or defect



Conformal Interface
Stat. mech.:
RG defects
impurities in quantum chains
junction of quantum wires
natural generalization of
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symmetry generating
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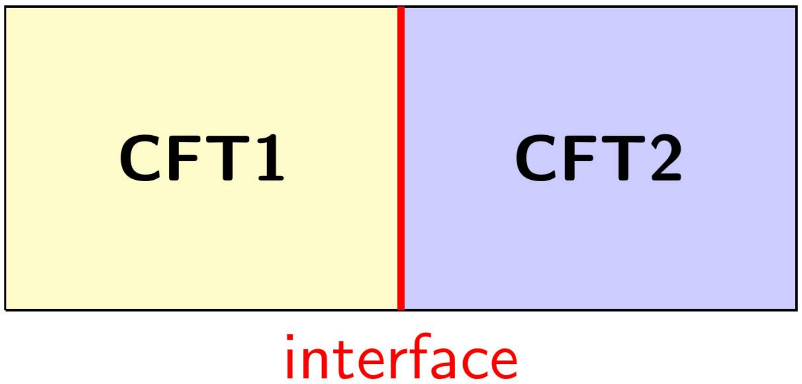
… or defect

Conformal Interfaces

Bachas et al 2002

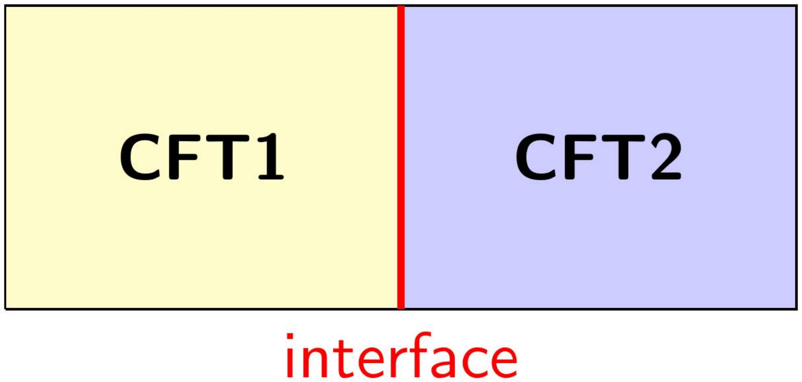


operator mapping states from on CFT to the other

$I_{1,2}$

Conformal Interfaces

Bachas et al 2002



gluing condition:

$$T^1(z) - \bar{T}^1(\bar{z}) = T^2(z) - \bar{T}^2(\bar{z})$$

operator mapping states from on CFT to the other

$I_{1,2}$

Conformal Interfaces

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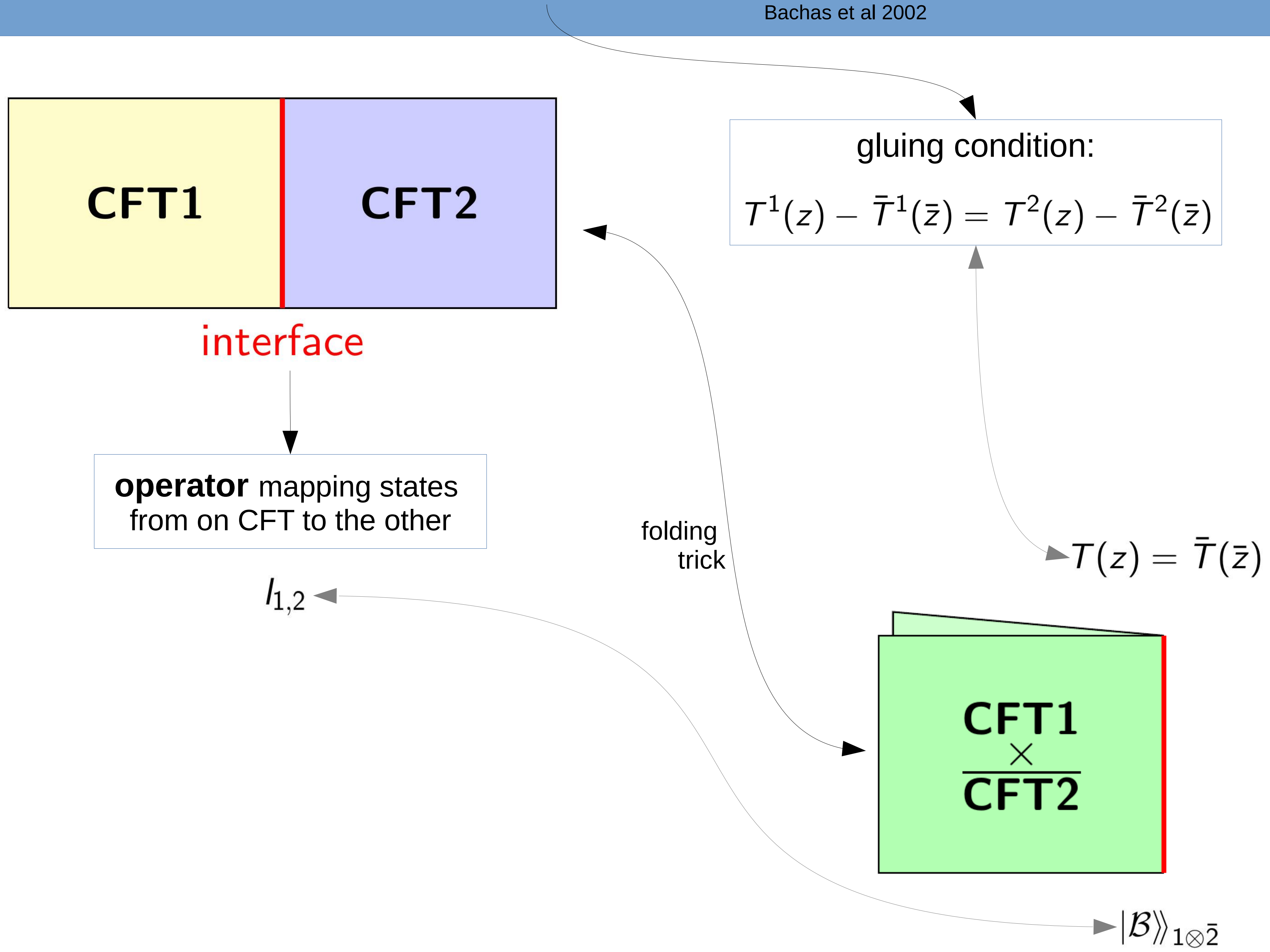


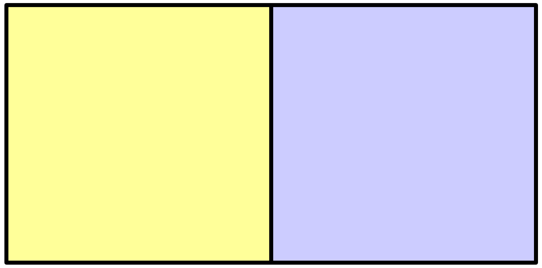
Special Gluing Conditions

$$T^1(z) - \bar{T}^1(\bar{z}) = T^2(z) - \bar{T}^2(\bar{z})$$

- Both sides vanish independently:

$$T^i(z) = \bar{T}^i(\bar{z})$$

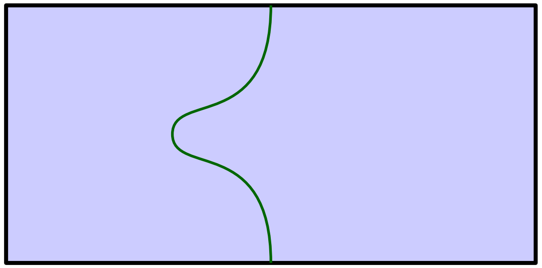
- **<u>separate boundary conditions</u>**
- In particular happens when one of the CFTs is **trivial**



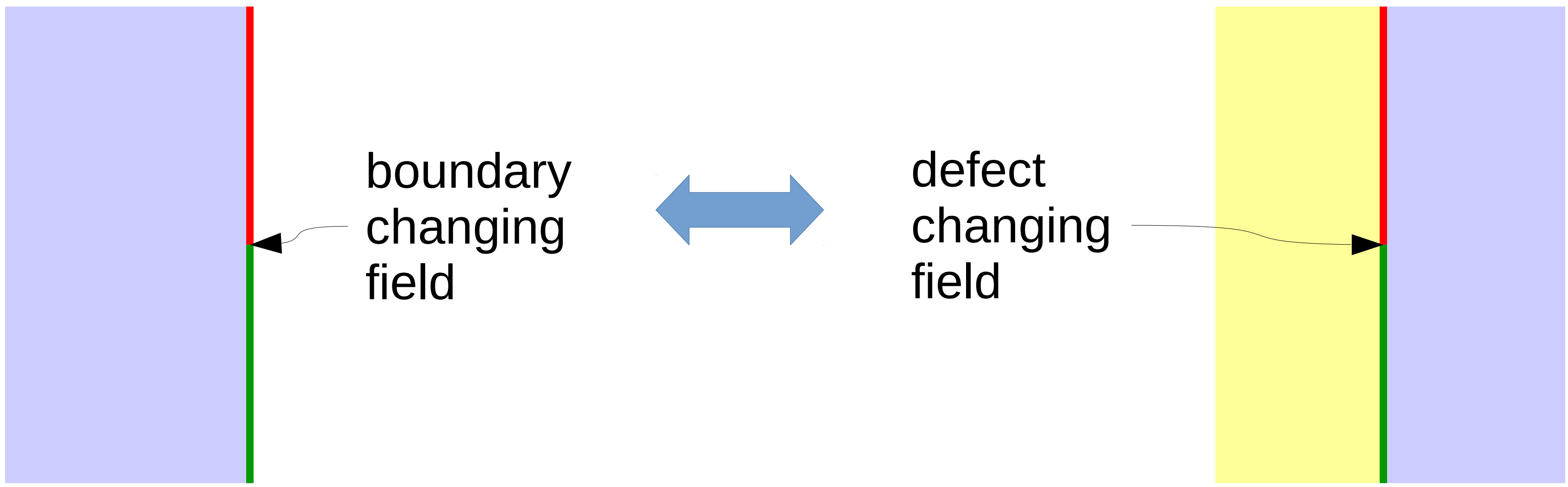
- The two components equal independently:

$$T^1(z) = T^2(z)\,, \quad \bar{T}^1(\bar{z}) = \bar{T}^2(\bar{z})$$

- $I_{1,2}$ also commutes with the Hamiltonian
- The interface can be moved around without cost of energy or momentum
- This is called a **<u>topological interface</u>**



What makes the difference?



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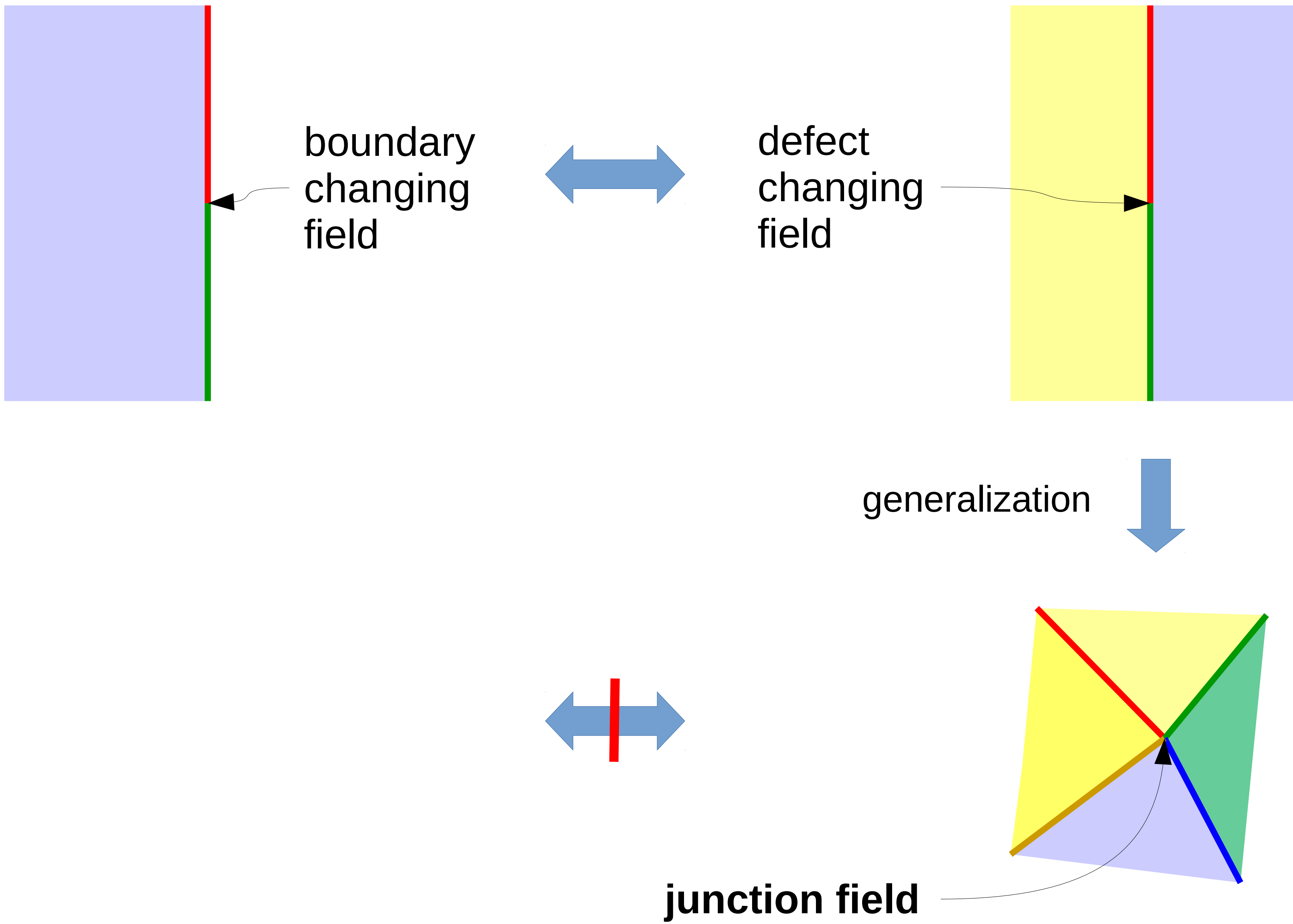
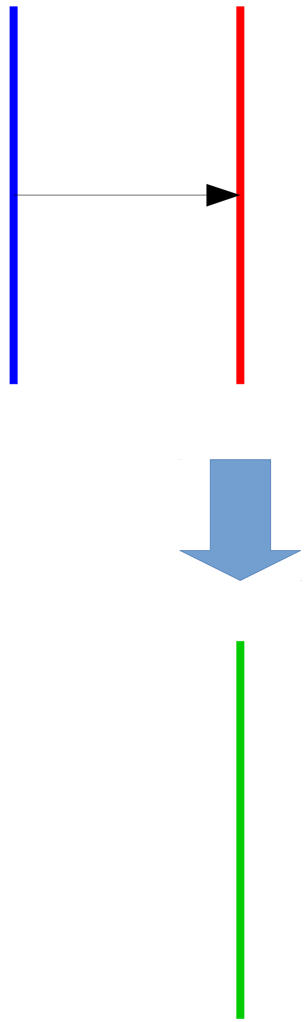


What else makes the difference?

Fusion of topological defects with other defects or boundaries:



The set of topological defects form a (Frobenius) **algebra**

Fröhlich et al 2007

Topological Interfaces in a CFT

acts as a **constant map** between isomorphic **Virasoro representations**

Petkova, Zuber 2000

⇕

$$I_A = \sum_{\mathbf{i}=(i\bar{\imath})} d_{A\mathbf{i}} \, ||\mathbf{i}||$$

Topological Interfaces in a CFT

acts as a **constant map** between isomorphic **Virasoro representations**

Petkova, Zuber 2000

$$I_A = \sum_{\mathbf{i}=(i\bar{\imath})} d_{A\mathbf{i}} \, \|\mathbf{i}\|$$

invariance under S-trafo

$$\sum_{\mathbf{i}} S_{ij} S_{\bar{\imath}\bar{\jmath}} \, \mathrm{Tr}\, d_{A^*\mathbf{i}} d_{A\mathbf{i}} = \mathcal{N}_{j\bar{\jmath}} \in \mathbb{N}$$

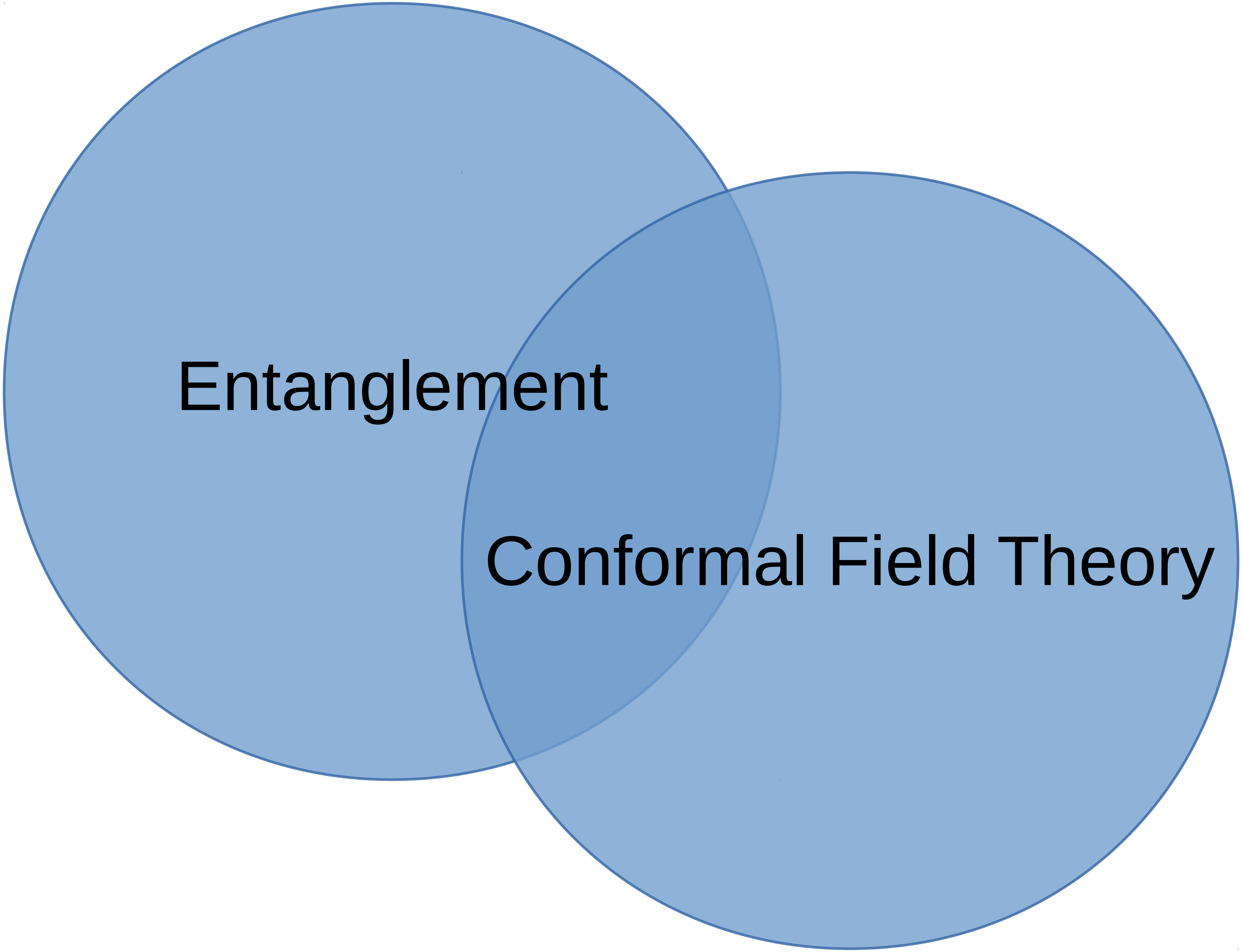
example: diagonal rational theories

$$I_a = \sum_i \frac{S_{ai}}{S_{0i}} \, \|i\|$$

Example: Topological Interfaces of the Ising model

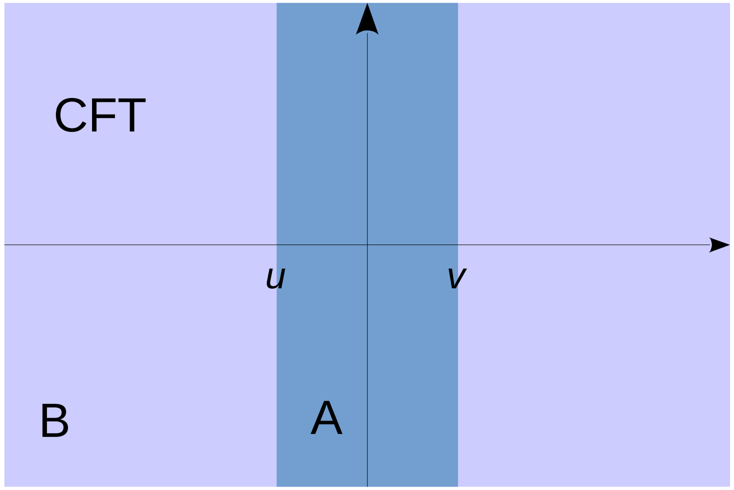
$$S_{ij} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

$$\begin{aligned} I_{id} &= \|id\| + \|\epsilon\| + \|\sigma\| \\ I_{\epsilon} &= \|id\| + \|\epsilon\| - \|\sigma\| \\ I_{\sigma} &= \sqrt{2}\|id\| - \sqrt{2}\|\epsilon\| \end{aligned}$$


Entanglement
Conformal Field Theory

Entanglement Entropy of a Finite Interval

Cardy, Calabrese 2009



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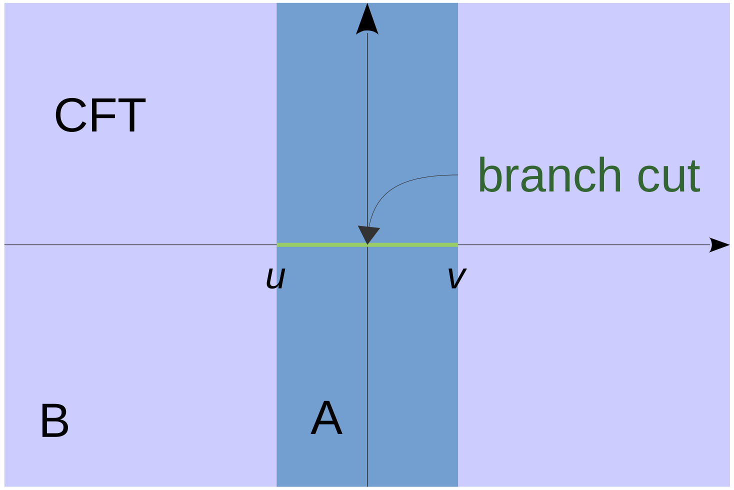
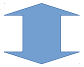


remember replica trick:

$\mathrm{Tr}\rho_A^n$



partition function *Z(n)* on a complicated Riemann surface

Entanglement Entropy of a Finite Interval

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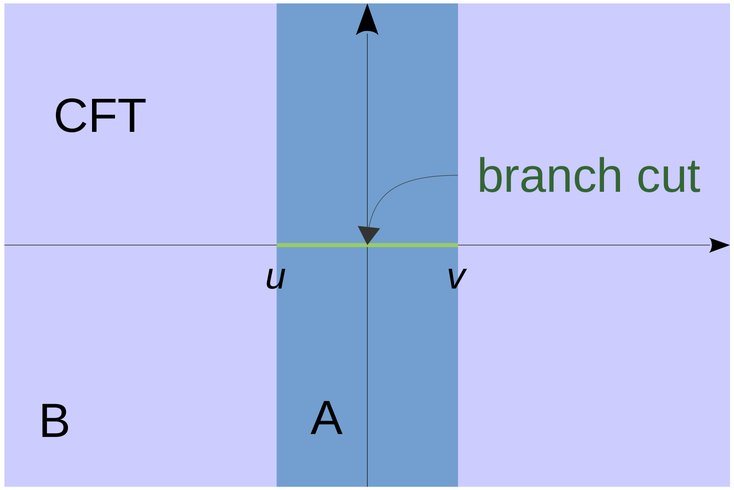

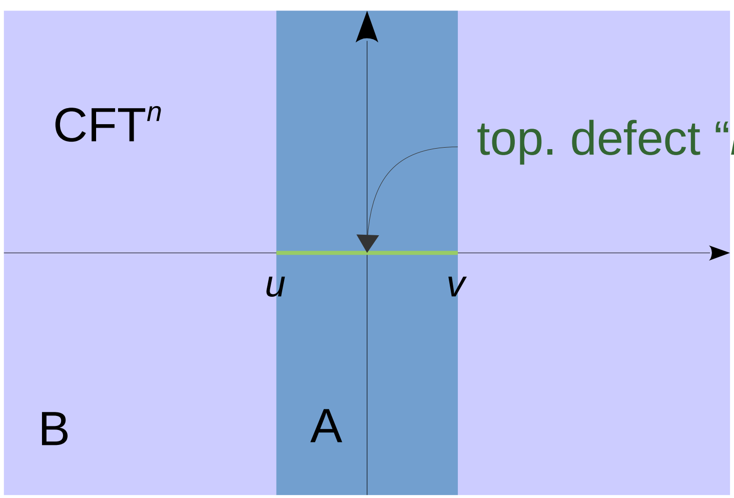

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$\Updownarrow$

partition function *Z(n)* on a complicated Riemann surface

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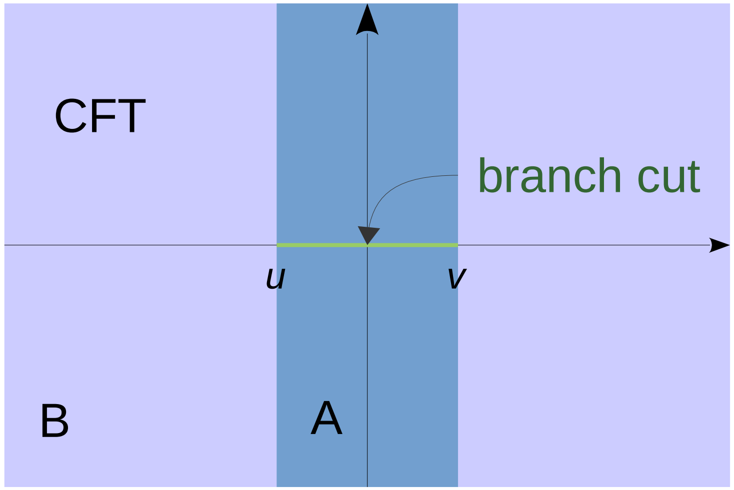

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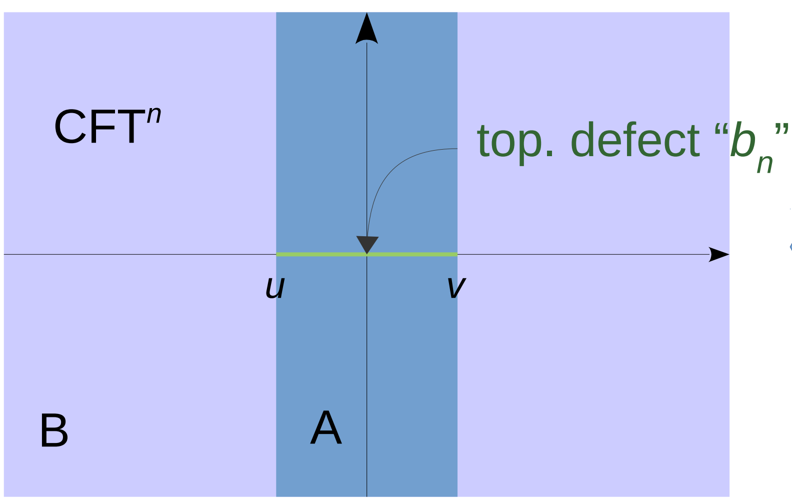
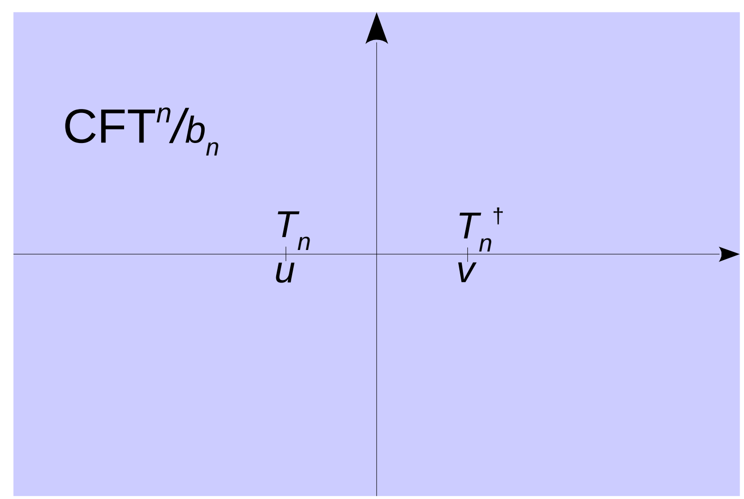
partition function *Z(n)* on a complicated Riemann surface

2-point function of twist fields

$\langle T_n(u) T_n^\dagger(v) \rangle$





EE of a Finite Interval

2-point function of **twist fields**

$$\langle T_n(u) T_n^{\dagger}(v) \rangle$$

“junction field” of lowest conformal weight

EE of a Finite Interval

2-point function of **twist fields**

$$\langle T_n(u) T_n^\dagger(v) \rangle$$

"junction field" of lowest conformal weight

state-operator correspondence

leading order for large L

$$q^{h_n - \frac{n\,c}{12}} = \langle T_n | q^{H^n_{b_n}} | T_n \rangle = Z_{\mathcal{H}^n_{b_n}}(\tau \gg 1)$$

Cardy condition

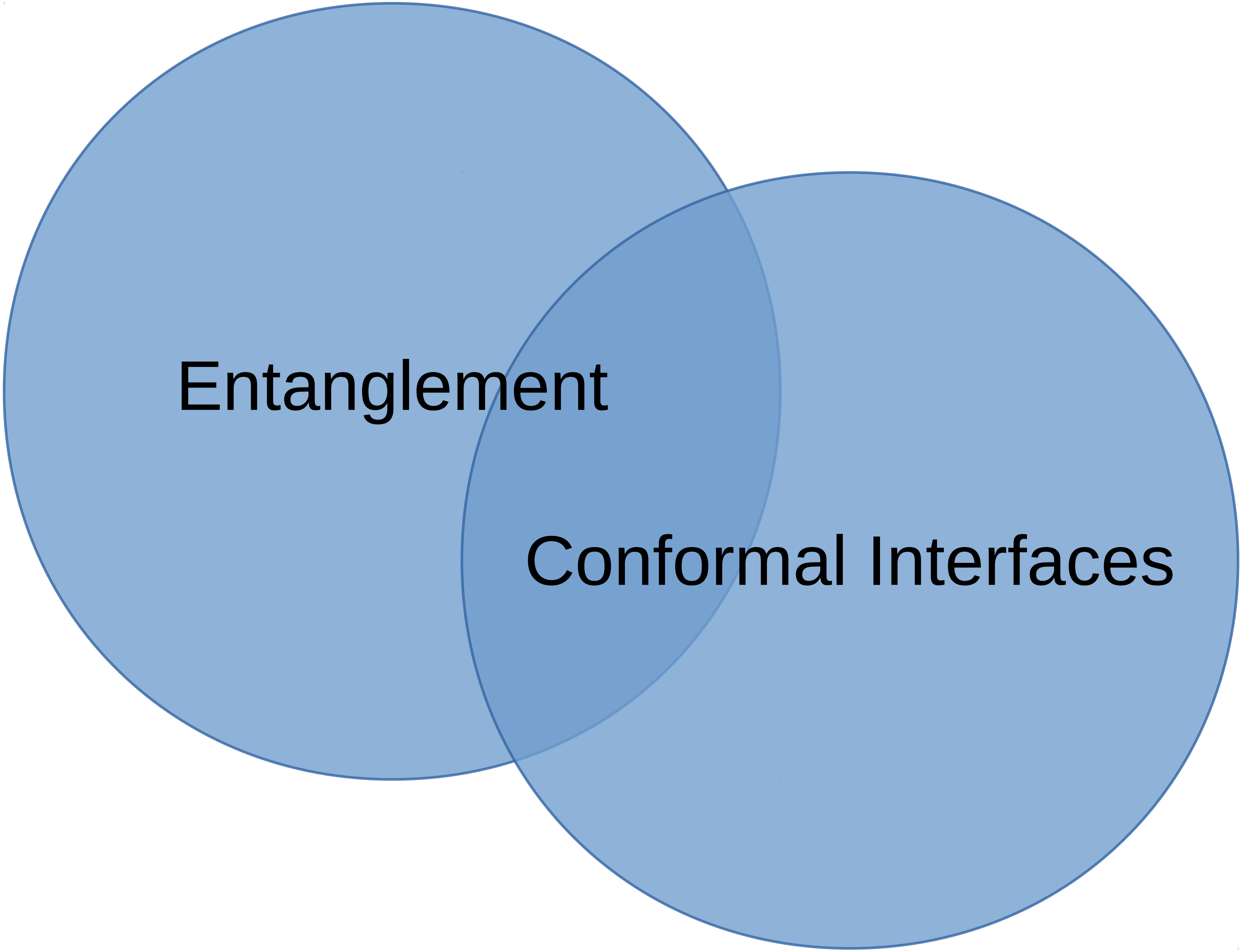
$$= \mathrm{Tr}(b_n \tilde{q}^{H^n}) = \mathrm{Tr}(\tilde{q}^{nH}) = \sum_{(i\bar{\imath})} \chi_i(\tilde{q}^n)\chi_{\bar{\imath}}(\tilde{q}^n)$$

S-trafo & leading order

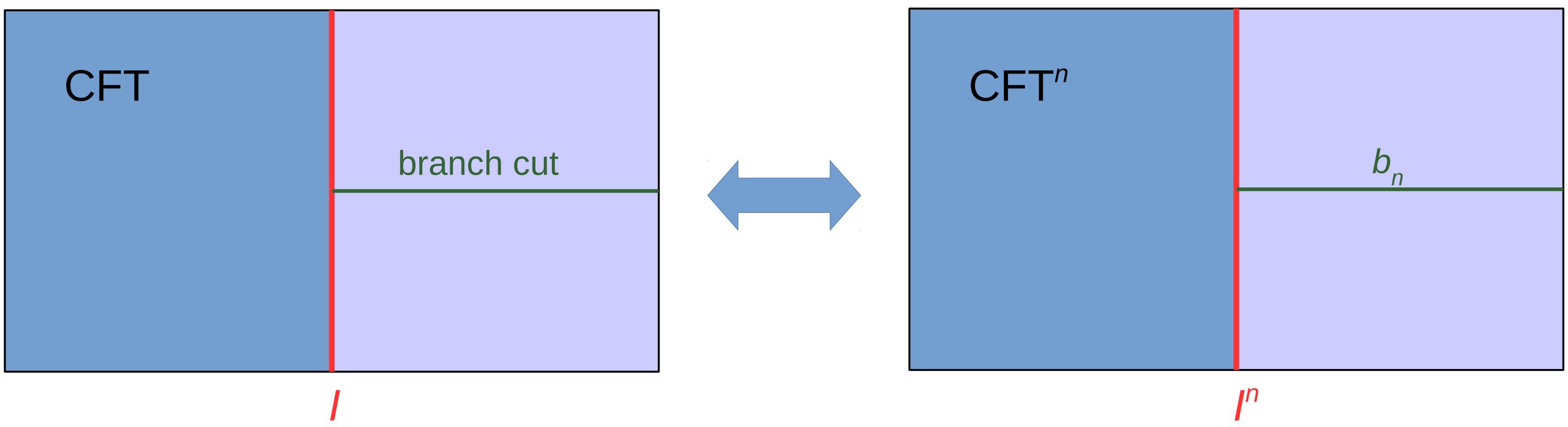
$$= q^{-\frac{c}{12n}}$$

$$h_n = \frac{c}{12}\left(n - \frac{1}{n}\right), \quad L = |v - u| \gg 1$$

$$S_A = \frac{c}{3} \log L + c_0$$


Entanglement
Conformal Interfaces

Entanglement through Conformal Interfaces



$$Z(n) = \mathrm{Tr}(b_n\, q^{H^n/4}\, I^n\, q^{H^n/2}\, (I^n)^\dagger\, q^{H^n/4})$$

$$= \mathrm{Tr}(I\, q^{H/2}\, I^\dagger\, q^{H/2})^n$$

Entanglement through Topological Interfaces

Remember: $I_A = \sum_{\mathbf{i}=(i\bar{\imath})} d_{A\mathbf{i}} \, ||\mathbf{i}||$ and $[I_A, H] = 0$

$$Z(n) = \mathrm{Tr}\left(\left(I_A I_A^\dagger\right)^n q^{nH}\right) = \sum_{(i\bar{\imath})} \mathrm{Tr}(d_{A\mathbf{i}} d_{A^*\mathbf{i}})^n \chi_i(q^n)\,\chi_{\bar{\imath}}(\bar{q}^n)$$

$$= \underbrace{\sum_{(i,\bar{\imath})} \mathrm{Tr}\,(d_{A^*\mathbf{i}}\, d_{A\mathbf{i}})^n \, S_{i0} S_{\bar{\imath}0}}_{\equiv A(n)} \;\; \tilde{q}^{-\frac{c}{12n}}$$

S-trafo & leading order

Entanglement through Topological Interfaces

Remember: $I_A = \sum_{\mathbf{i}=(i\bar{\imath})} d_{A\mathbf{i}} \, ||\mathbf{i}||$ and $[I_A, H] = 0$

$$Z(n) = \mathrm{Tr}\left(\left(I_A I_A^\dagger\right)^n q^{nH}\right) = \sum_{(i\bar{\imath})} \mathrm{Tr}(d_{A\mathbf{i}} d_{A^*\mathbf{i}})^n \chi_i(q^n)\, \chi_{\bar{\imath}}(\bar{q}^n)$$

S-trafo & leading order

$$= \underbrace{\sum_{(i,\bar{\imath})} \mathrm{Tr}\,(d_{A^*\mathbf{i}}\, d_{A\mathbf{i}})^n\, S_{i0} S_{\bar{\imath}0}}_{\equiv A(n)} \;\; \tilde{q}^{-\frac{c}{12n}}$$

no change in the log term of the EE

$$\frac{c}{3} \log L$$

contributes to sub-leading term in the EE:

$$s(I_A) = -\sum_{(i,\bar{\imath})} \mathrm{Tr}\, p_{\mathbf{i}}^A \, \log \frac{p_{\mathbf{i}}^A}{p_{\mathbf{i}}^{id}}$$

with

$$p_{\mathbf{i}}^A = \frac{d_{A^*\mathbf{i}}\, d_{A\mathbf{i}}\, S_{i0} S_{\bar{\imath}0}}{\mathcal{N}_{\mathbf{0}A}^{\;A}}$$

Entanglement through Topological Interfaces

relative entropy / Kullback–Leibler divergence:

$$s(I_A) = -\sum_{(i,\bar{\imath})} \mathrm{Tr}\, p_{\mathbf{i}}^{A} \log \frac{p_{\mathbf{i}}^{A}}{p_{\mathbf{i}}^{id}}$$

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diagonal RCFTs

$$p_i^a = |S_{ia}|^2$$

$$s(I_a) = -\sum_i |S_{ia}|^2 \log \left| \frac{S_{ia}}{S_{i0}} \right|^2$$

Ising

$$s(I_a) = \begin{cases} -\log 2\,, & a = \sigma \\ 0\,, & a = id, \epsilon \end{cases}$$

$su(2)_{k\gg1}$

$$s(I_a) = -\frac{a}{a+1} \qquad (a \ll k)$$

Entanglement through Non-Topological Interfaces

they affect the leading order contribution ⇔ change the conformal weight of the twist field

Example: General interfaces of the Ising model

→ interfaces of the free fermion theory:

$$I_{1,2}(O) = \prod_{n>0} I^{n}_{1,2}(\mathcal{O})\, I^{0}_{1,2}(\mathcal{O})$$

Entanglement through Non-Topological Interfaces

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Example: General interfaces of the Ising model

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$$I_{1,2}(O) = \prod_{n>0} I^n_{1,2}(\mathcal{O})\, I^0_{1,2}(\mathcal{O})$$

$$I^n_{1,2}(\mathcal{O}) = \exp\left(-i\psi^1_{-n}\mathcal{O}_{11}\bar{\psi}^1_{-n} + \psi^1_{-n}\mathcal{O}_{12}\psi^2_n + \bar{\psi}^1_{-n}\mathcal{O}_{21}\bar{\psi}^2_n + i\psi^2_n\mathcal{O}_{22}\bar{\psi}^2_n\right)$$

$$I^0_{1,2}(\mathcal{O}) = \begin{cases} |0\rangle\langle 0| & \text{NS} \\ \sqrt{2}\,(\cos(\phi)|+\rangle\langle+| + \sin(\phi)|-\rangle\langle-|) & \text{R} \end{cases}$$

$$\mathcal{O} = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \pm\sin 2\phi & \mp\cos 2\phi \end{pmatrix}$$

$\phi = 0$ sep. boundaries

$\phi = \pi/4$ topological

Entanglement through Non-Topological Interfaces

they affect the leading order contribution ⟺ change the conformal weight of the twist field

Example: General interfaces of the Ising model

→projection on even fermion number:

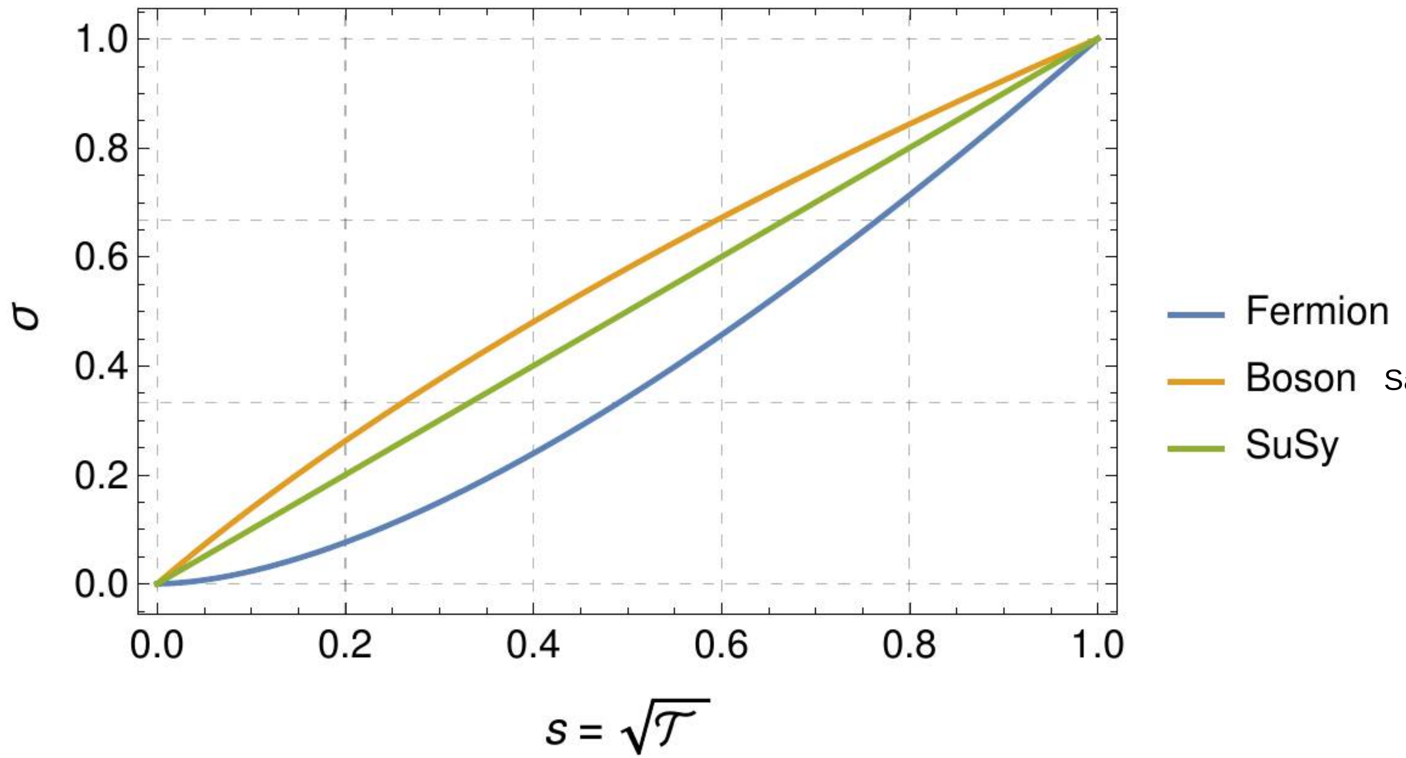
$$I^{\pm}(\Lambda) = \frac{1}{2}\left(I^{\mathrm{NS}}(\mathcal{O}) \pm I^{\mathrm{R}}(\mathcal{O})\right) + (\phi \to -\phi)$$

$$I^{\mathrm{n.}}(\Lambda) = \frac{1}{\sqrt{2}} I^{\mathrm{NS}}(\mathcal{O}) + (\phi \to -\phi)$$

Entanglement through Non-Topological Interfaces

$$S = \sigma(\mathcal{T})\,\frac{c}{3}\,\log L + s$$

$\mathcal{T} = \sin^2 2\phi$ transmission coefficient



Sakai, Satoh 2008

Entanglement through Non-Topological Interfaces

they affect the leading order contribution ⟺ change the conformal weight of the twist field

Some interesting questions:

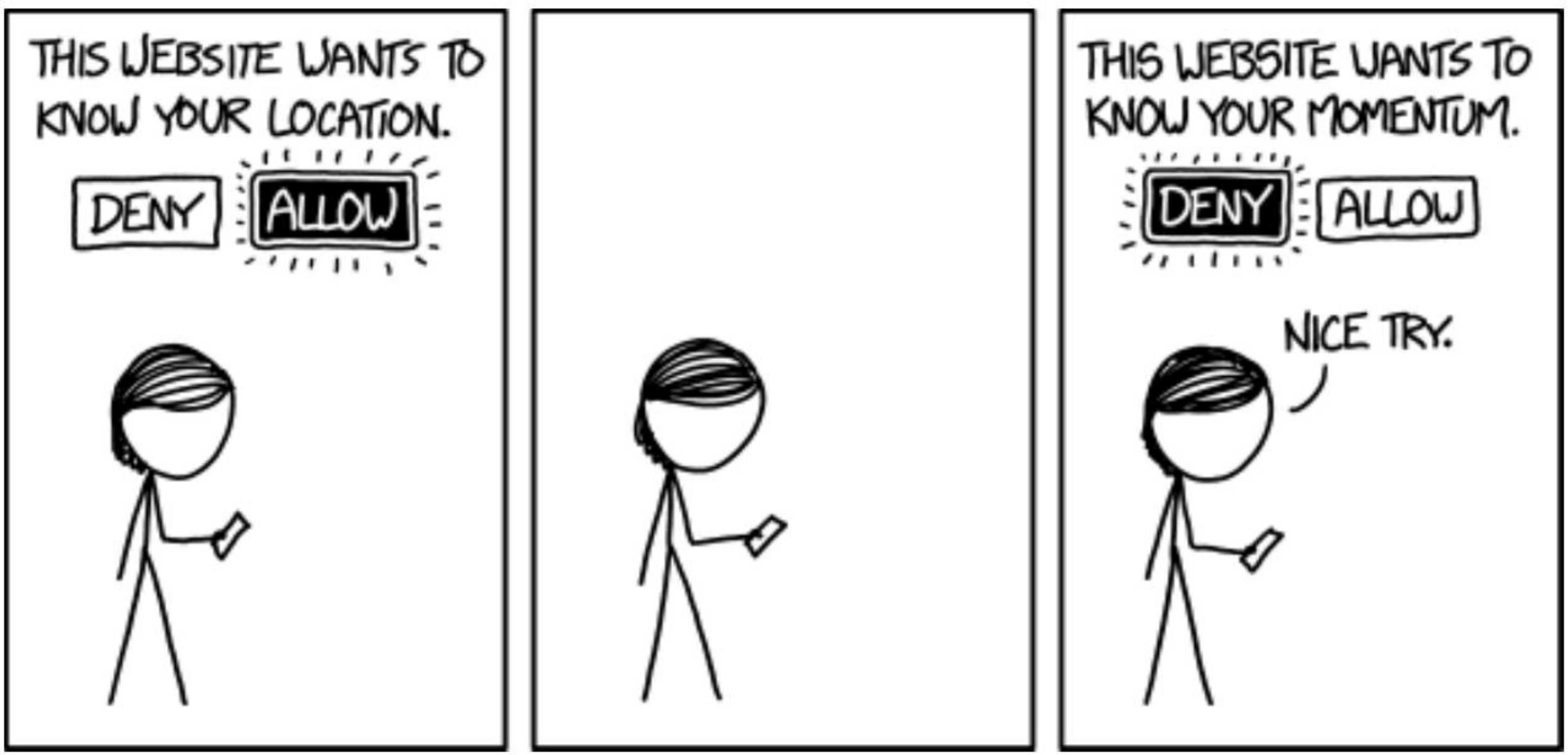
- How does the EE behave for **general non-topological defects**?
- On which **features** of a general conformal defect does it depend? Keywords: **transmission coefficient**; **Casimir energy**; topological data**.**
- Is the sub-leading term always constant under **non-topological deformations** of a topological defect?

Final Words and Thoughts

- By **unfolding a boundary** one may interpret it as a **top. defect in a chiral theory**
  - one can use the same techniques to derive the **left-right entanglement** at a boundary

- The entanglement through the defect is a **feature** of the defect itself.
- It might be possible to define more **structure** to the space of 2d CFTs
  - **define distances** between CFTs, by the help of conformal defects and the EE through them? (in the spirit of ideas of Bachas et al 2014)
  - the infinitesimal limit of the Kullback–Leibler divergence yields the **Fisher information metric**

Thank You!



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More about relative entropy

Using the constrains for $d_{A\mathbf{i}}$:

$$\sum_{(i,\bar{\imath})} \mathrm{Tr}\, p_{\mathbf{i}}^{A} = 1$$

so they form a probability distribution.

$$s \leq \log\left(\sum_{(i,\bar{\imath})} T_{i\bar{\imath}} S_{i0} S_{\bar{\imath}0}\right)$$

$T_{i\bar{\imath}}$: $\min(M^{1}_{i\bar{\imath}}, M^{2}_{i\bar{\imath}})$

If the two CFTs are not the same: Their exists a defect s.t. the Kullback-Leibler divergence vanishes iff the **spectra are identical**.

Results for higher torus models

$$\mathcal{I}_{12}(\Lambda) = \sum_{\gamma \in \Gamma_{12}^{\Lambda}} d_{\Lambda\gamma} ||\gamma||$$

Bachas et al 2012

$$\Gamma_{12}^{\Lambda} = \{\gamma \in \Gamma_1 \,|\, \Lambda\gamma \in \Gamma_2\} = \Gamma_1 \cap \Lambda^{-1}\Gamma_2 \subset \Gamma_1$$

$$S = (1 - \partial_K) \log(Z(K))\big|_{K=1} = \frac{c}{3} \log(L) - \log |\Gamma_1 / \Gamma_{12}^{\Lambda}|$$

is also the g-factor of the interface