Deformed special geometry and topological string theory

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Seminar at Kvali IPMU

with Bernard de Wit and Swapna Mahapatra,
arXiv:1406.5478, work in progress

with Thomas Mohaupt, arXiv:1511.06658

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In supersymmetric field theories and string theories:

class of terms in Wilsonian effective action that play a special role:

F-terms.

Example: Type II superstring theory on $M_4 \times CY_3$

N=2 supersymmetry in four dimensions.

For every integer $g \geq 0$: $\exists$ F-term at precisely g-loop order in string perturbation theory, schematically

$$\int d^4x\, d^4\theta \sum_{g\geq 0} F^{(g)}(Y^I)\,(W^2)^g$$

Y^I: conical affine special Kähler manifold $\mathcal{M}$, $\quad Y^I \to \lambda\, Y^I\,,\ \lambda \in \mathbb{C}^*$, $I = 0, \ldots, n = h^{1,1}$ (or $h^{2,1}$)

Special Kähler geometry. de Wit, van Proeyen

$F^{(0)}(Y)$ determines Kähler metric $N_{IJ} = 2\,\mathrm{Im}\left(\partial^2 F^{(0)}/\partial Y^I \partial Y^J\right)$.

Projective special Kähler manifold $\bar{\mathcal{M}} = \mathcal{M}/\mathbb{C}^* \quad (t^I = Y^I/Y^0)$.

Assemble coupling functions $F^{(g)}(Y)$ into $F(Y, \Upsilon)$:

$$F(Y,\Upsilon) = F^{(0)}(Y) + w(Y,\Upsilon) \;,\;\; w(Y,\Upsilon) = \sum_{g=1}^{\infty} F^{(g)}(Y)\,\Upsilon^g \quad , \quad \Upsilon \in \mathbb{C}$$

Wilsonian action encoded in symplectic vector ($Sp(2(n+1), \mathbb{R})$-transformations)

$$(Y^I, F_I) \qquad , \qquad F_I = \frac{\partial F(Y,\Upsilon)}{\partial Y^I} \quad , \quad I = 0, \ldots, n$$

Special significance of F-terms:

- determine entropy of half-BPS black holes `OSV, 2004`
- $F^{(g)}$ topological string amplitudes.

  Actually, the $F^{(g)}$ are not quite holomorphic ($g \geq 1$):

  holomorphic anomaly equation (recursive relation, $g \geq 2$)
  `Bershadsky, Cecotti, Ooguri, Vafa, 1993`

Q1: what is the precise relation between TST and the LEEA?

Q2: Consistent extension of special geometry for incorporating non-holomorphic corrections encoded in $F^{(g)}$, $g \geq 1$?

Deformed special geometry

- Q2: deformed special geometry, based on

$$F(Y,\bar{Y}) = F^{(0)}(Y) + 2i\,\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})\ , \qquad \Omega \quad \text{real}$$

Symplectic vector $\qquad (Sp(2(n+1),\mathbb{R})$-transformations)

$$(Y^I, F_I) \qquad , \qquad F_I = \frac{\partial F(Y,\bar{Y})}{\partial Y^I} \quad , \quad I = 0,\dots,n$$

LEEA. When Ω harmonic: Wilsonian limit

$$\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = w(Y,\Upsilon) + \bar{w}(\bar{Y},\bar{\Upsilon})$$

Natural deformation leads to perturbative topological string.

Deformed special geometry

- Q1: Generalized Hesse potential $H(\phi, \chi)$ Freed 1997

$$\phi^I = Y^I + \bar{Y}^I \qquad , \qquad \chi_I = F_I + \bar{F}_I$$

Obtained by Legendre transform with respect to $Y^I - \bar{Y}^I$:

$$H(\phi, \chi) = 4 \left[\mathrm{Im} F^{(0)}(Y) + \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \right] + i \, \chi_I \, (Y - \bar{Y})^I$$

Significance:

- H 'Hamiltonian' (Legendre transform of LEEA), symplectic function $\longrightarrow \Omega$ transforms in prescribed, non-trivially way.
- Want to understand the structure of the Hesse potential:

a unique subsector captures perturbative topological string.

Evaluating the Hesse potential

$$
\begin{aligned}
F(Y,\bar Y) &= F^{(0)}(Y) + 2i\,\Omega(Y,\bar Y,\Upsilon,\bar\Upsilon) \\
H(\phi,\chi) &= -i\left(\bar Y^I F_I - Y^I \bar F_{\bar I}\right) + 2\left(2\Omega - Y^I\Omega_I - \bar Y^{\bar I}\,\Omega_{\bar I}\right)
\end{aligned}
$$

Q1: New variables:

$$
\begin{pmatrix}\phi^I\\ \chi_I\end{pmatrix} = 2\,\mathrm{Re}\begin{pmatrix} Y^I \\ F_I(Y,\bar Y)\end{pmatrix} = 2\,\mathrm{Re}\begin{pmatrix}\mathcal{Y}^I\\ F_I^{(0)}(\mathcal{Y})\end{pmatrix}
$$

Relation between $\mathcal{Y}^I$ and (ϕ,χ) only involves $F^{(0)}$.

Y^I: sugra variables $\qquad$ $\mathcal{Y}^I$: new (stringy) variables

$$
\begin{aligned}
\mathcal{Y}^I &= Y^I + \Delta Y^I(\Omega) \quad , \quad \Omega \neq 0 \\
\mathcal{Y}^I &= Y^I \quad , \quad \Omega = 0\,.
\end{aligned}
$$

Evaluating the Hesse potential

Evaluate H in terms of $\mathcal{Y}^I \Rightarrow$ expansion of ΔY^I in powers of $\Omega(\mathcal{Y},\bar{\mathcal{Y}})$.

The Hesse potential transforms as a function under symplectic transformations: $\tilde{H}(\tilde{\phi},\tilde{\pi}) = H(\phi,\pi)$.

- H as series of symplectic functions, $H = \sum_{k=0}^{\infty} H^{(k)}(\mathcal{Y},\bar{\mathcal{Y}})$
- $H^{(0)} = -i\left[\bar{\mathcal{Y}}^I F_I^{(0)}(\mathcal{Y}) - \mathrm{c.c.}\right]$
- $H^{(1)}$ is the only one that contains $\Omega(\mathcal{Y},\bar{\mathcal{Y}})$ (the other $H^{(k)}$ contain derivatives thereof)

$$H^{(1)} = 4\Omega - 4N^{IJ}\left(\Omega_I\Omega_J + \Omega_{\bar{I}}\Omega_{\bar{J}}\right) + \mathcal{O}(\Omega^3)$$

$$N_{IJ} = -i\left(F_{IJ}^{(0)} - \bar{F}_{\bar{I}\bar{J}}^{(0)}\right)$$

- $N^{IJ} \to \ldots (N - \mathrm{i}\mathcal{Z})^{IJ}$

$$\tilde{\Omega}(\tilde{\mathcal{Y}},\bar{\tilde{\mathcal{Y}}}) = \Omega - \mathrm{i}\left(\mathcal{Z}^{IJ}\,\Omega_I\Omega_J - \bar{\mathcal{Z}}^{\bar{I}\bar{J}}\,\Omega_{\bar{I}}\Omega_{\bar{J}}\right) + \mathcal{O}(\Omega^3)$$

The holomorphic anomaly equation

Wilsonian set-up:

$$\Omega(\mathcal{Y}, \bar{\mathcal{Y}}) = w(\mathcal{Y}) + \bar{w}(\bar{\mathcal{Y}})$$

Preserved by symplectic transformations:

$$\tilde{\Omega}(\tilde{\mathcal{Y}}, \bar{\tilde{\mathcal{Y}}}) = \tilde{w}(\tilde{\mathcal{Y}}) + \bar{\tilde{w}}(\bar{\tilde{\mathcal{Y}}})$$

Now add non-holomorphic term whose variation is harmonic:

$$\Omega(\mathcal{Y}, \bar{\mathcal{Y}}) = w(\mathcal{Y}) + \bar{w}(\bar{\mathcal{Y}}) + \alpha \ln \det N_{IJ} + \mathcal{O}(\alpha^2) \quad , \quad \alpha \in \mathbb{R}$$

$$\ln \det \tilde{N} = \ln \det N - \ln \det \mathcal{S}(\mathcal{Y}) - \ln \det \bar{\mathcal{S}}(\bar{\mathcal{Y}})$$

Transformation law of Ω requires further modifications:

The holomorphic anomaly equation

$$\Omega(\mathcal{Y},\bar{\mathcal{Y}}) = w(\mathcal{Y}) + \bar{w}(\bar{\mathcal{Y}}) + \alpha \ln\det N_{IJ} + \Big[2\alpha\, N^{IJ}\, w_{IJ} - \alpha^2 \left(iF^{(0)}_{IJKL} - \tfrac{2}{3} F^{(0)}_{IKM} F^{(0)}_{JLN} N^{MN}\right) N^{IJ} N^{KL} + \mathrm{h.c.}\Big] + \dots$$

Double expansion: $w(\mathcal{Y}) = \sum_{g=1}^{\infty} w^{(g)}(\mathcal{Y})\,\beta^g \quad , \quad \beta = 1$

Expanding $H^{(1)}$ in powers of (α,β):

$$H^{(1)} = 4\Big[F^{(1)}(\mathcal{Y},\bar{\mathcal{Y}}) + \left(F^{(2)}(\mathcal{Y},\bar{\mathcal{Y}}) + \mathrm{h.c.}\right) + \left(\left(F^{(3)}(\mathcal{Y},\bar{\mathcal{Y}}) + \mathrm{h.c.}\right) + 4\,\alpha\, D_I F^{(1)}_J\, N^{IK}\, N^{JL}\, \bar{D}_{\bar{K}} \bar{F}^{(1)}_{\bar{L}}\right) + \mathcal{O}(\alpha^4)\Big]$$

Expansion in terms of symplectic functions $F^{(g)}$.

The holomorphic anomaly equation

$$
\begin{aligned}
F^{(1)}(\mathcal{Y},\bar{\mathcal{Y}}) &= w^{(1)}(\mathcal{Y}) + \bar{w}^{(1)}(\bar{\mathcal{Y}}) + \alpha \ln\det N_{IJ} \quad , \quad \text{real} \\
F^{(2)}(\mathcal{Y},\bar{\mathcal{Y}}) &= w^{(2)}(\mathcal{Y}) + 2\alpha\, N^{IJ}\, w^{(1)}_{IJ} \\
&\quad - i\alpha^2\, N^{IJ} N^{KL}\, F^{(0)}_{IJKL} + \tfrac{2}{3}\alpha^2 N^{IJ} N^{KP} N^{LQ} F^{(0)}_{IKL} F^{(0)}_{JPQ} \\
&\quad - N^{IJ} \left(w^{(1)}_I - i\alpha N^{KL} F^{(0)}_{IKL} \right) \left(w^{(1)}_J - i\alpha N^{PQ} F^{(0)}_{JPQ} \right) \\
F^{(3)}(\mathcal{Y},\bar{\mathcal{Y}}) &= 41 \quad \text{terms}
\end{aligned}
$$

$F^{(g)}(\mathcal{Y},\bar{\mathcal{Y}}),\ g \geq 2$: polynomials in N^{IJ}, degree $3g-3$.

Non-holomorphicity entirely through $N_{IJ} = -i\left(F^{(0)}_{IJ} - \bar{F}^{(0)}_{\bar{I}\bar{J}}\right)$.

$F^{(2)}$: Grimm+Klemm+Mariño+Weiss, hep-th/0702187

$F^{(3)}$: new.

The holomorphic anomaly equation

By explicit calculation, verify that the $F^{(g)}(\mathcal{Y}, \bar{\mathcal{Y}})$ $(g \geq 2)$ satisfy the holomorphic anomaly equation of topological string theory in big moduli space:

$$\partial_{\bar{I}} F^{(g)} = i\, \bar{F}^{(0)}_{\bar{I}\bar{P}\bar{Q}} N^{PJ} N^{QK} \left(2\alpha\, D_J \partial_K F^{(g-1)} + \sum_{r=1}^{g-1} \partial_J F^{(r)} \partial_K F^{(g-r)} \right)$$

where $\partial_{\bar{I}} F^{(g)} = \frac{\partial F^{(g)}(\mathcal{Y},\bar{\mathcal{Y}})}{\partial \bar{\mathcal{Y}}^I}$, etc.

Bershadsky, Cecotti, Ooguri, Vafa, hep-th/9309140

Consequence of symplectic covariance.

Hessian structure

Set $\alpha = 0$: geometric interpretation? In supergravity variables:

$$H(\phi, \chi) = 4\Big[\mathrm{Im} F^{(0)}(Y) + \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})\Big] + i\,\chi_I\,(Y - \bar{Y})^I \,,$$

$$\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = w(Y, \Upsilon) + \bar{w}(\bar{Y}, \bar{\Upsilon}) \;\;,\;\; w(Y, \Upsilon) = \sum_{g=1}^{\infty} F^{(g)}(Y)\,\Upsilon^g$$

- $\hat{\mathcal{M}} = \mathcal{M} \times \mathbb{C}$
- Coordinates $q^a = (\phi^I, \chi_I, \Upsilon, \bar{\Upsilon})$ on $\hat{\mathcal{M}}$.
- $\hat{\mathcal{M}}$: equipped with flat, torsion-free connection ∇.
- Hessian metric: $g^H \sim \partial^2 H / \partial q^a \partial q^b$
- Hessian structure (g^H, ∇): $S = \nabla g^H$ totally symm. rank 3-tensor $\longrightarrow$ $S_{\phi^I \Upsilon \Upsilon} = S_{\Upsilon \phi^I \Upsilon} \longrightarrow$ anomaly equation with $\alpha = 0$.
- Deformed affine special Kähler manifold: $\nabla\omega \neq 0, d_{\nabla} J \neq 0$.
- Extends to $\alpha \neq 0$.

Holomorphic anomaly $\Leftrightarrow$ existence of a Hessian structure.

Summary

- Described a consistent deformation of special geometry:
  - framework for incorporating non-holomorphic corrections into LEEA
  - $F(Y,\bar{Y}) = F^{(0)}(Y) + 2i\,\Omega(Y,\bar{Y})$
- Discussed structure of the associated Hesse potential H, by expanding it in new variables $\mathcal{Y}^I$.
- Identified a unique subsector of H that captures the topological string free energies $F^{(g)}(\mathcal{Y},\bar{\mathcal{Y}})$: $H^{(1)}$
- Taking as non-holomorphic deformation

$$\Omega(\mathcal{Y},\bar{\mathcal{Y}}) = w(\mathcal{Y}) + \bar{w}(\bar{\mathcal{Y}}) + \alpha \ln \det N_{IJ} + \ldots$$

  and expanding $H^{(1)}$ in powers of (w,α), obtained expansion functions $F^{(g)}(\mathcal{Y},\bar{\mathcal{Y}})$ that satisfy the holomorphic anomaly equation of TST. Diagrammatic expansion.

Going beyond a perturbative expansion

In projective coordinates $(\mathcal{Y}^0,\ t^I = \mathcal{Y}^I/\mathcal{Y}^0)$:

$F^{(g)}(\mathcal{Y}, \bar{\mathcal{Y}}) \sim (\mathcal{Y}^0)^{2(1-g)}\, F^{(g)}(t, \bar{t})$

$\Longrightarrow H^{(1)}$ is series expansion in powers of $(\mathcal{Y}^0)^{-2}$

Non-perturbative completion: $e^{-\mathcal{Y}^0}, \quad \ln \mathcal{Y}^0$ effects ?

Resort to models with exact duality symmetries:

STU-model $(\chi = 0)$ `Sen and Vafa, hep-th/9508064`

In supergravity variables Y^I:

$$F = F^{(0)}(Y) + 2\mathrm{i}\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = i(Y^0)^2\, STU + 2\mathrm{i}\Omega$$

S-, T-, U duality symmetry: $\Gamma(2) \subset SL(2, \mathbb{Z})$

Symplectic vector (Y^I, F_I).

Under S-duality: $T^a = T, U \quad , \quad \Delta_{\rm S} = {\rm i}cS + d$

$$S \to \frac{aS - {\rm i}b}{{\rm i}cS + d}\ , \quad T^a \to T^a + \frac{{\rm i}c}{\Delta_{\rm S}\,(Y^0)^2}\,\eta^{ab}\,\frac{\partial\Omega}{\partial T^b}\ , \quad Y^0 \to \Delta_{\rm S}\, Y^0\ .$$

Invariance under S-duality: GLC + de Wit + Mahapatra, 0808.2627

$$\begin{aligned}
\left(\frac{\partial\Omega}{\partial T^a}\right)'_{\rm S} &= \frac{\partial\Omega}{\partial T^a} \\
\left(\frac{\partial\Omega}{\partial S}\right)'_{\rm S} - {\Delta_{\rm S}}^2\,\frac{\partial\Omega}{\partial S} &= \frac{\partial({\Delta_{\rm S}}^2)}{\partial S}\left[-\tfrac{1}{2}\,Y^0\,\frac{\partial\Omega}{\partial Y^0} - \frac{{\rm i}c}{4\,\Delta_{\rm S}\,(Y^0)^2}\,\frac{\partial\Omega}{\partial T^a}\eta^{ab}\,\frac{\partial\Omega}{\partial T^b}\right] \\
\left(Y^0\,\frac{\partial\Omega}{\partial Y^0}\right)'_{\rm S} &= Y^0\,\frac{\partial\Omega}{\partial Y^0} + \frac{{\rm i}c}{\Delta_{\rm S}\,(Y^0)^2}\,\frac{\partial\Omega}{\partial T^a}\eta^{ab}\,\frac{\partial\Omega}{\partial T^b}
\end{aligned}$$

Non-linear in Ω. Similarly for T and U duality.

Ansatz for Ω: perturbative expansion in $(Y^0)^{-2}$

$$\Omega = \sum_{g=1}^{\infty} (Y^0)^{2-2g}\, \Omega^{(g)}(S, T, U, \bar{S}, \bar{T}, \bar{U})$$

with non-holomorphic input $\Omega^{(1)}(S, T, U, \bar{S}, \bar{T}, \bar{U})$.

$\Longrightarrow$ Obtain results for $\Omega^{(g)}$ consistent with perturbative TST.

$\Omega^{(g)}$ non-holomorphic.

Different ansatz for Ω: $\Omega = w(Y) + \bar{w}(\bar{Y})$ harmonic

$$w(Y) = \gamma \ln Y^0 + \sum_{n=1}^{\infty} (Y^0)^{2-2n}\, w^{(n)}(S, T, U) \, .$$

Presence of $\ln Y^0$-term makes this possible.

Not perturbative TST.

STU-model

Imposing S-, T-, U- duality invariance we obtain (to appear):

$$
\begin{aligned}
w^{(1)}(S,T,U) &= w^{(1)}(S) + w^{(1)}(T) + w^{(1)}(U)\ , \\
w^{(2)}(S,T,U) &= \frac{2}{\gamma}\,\frac{\partial w^{(1)}}{\partial S}\,\frac{\partial w^{(1)}}{\partial T}\,\frac{\partial w^{(1)}}{\partial U}\ , \\
w^{(3)}(S,T,U) &= -\frac{2}{\gamma}\left[\gamma\,\frac{\partial^2 w^{(1)}}{\partial S^2}\,\frac{\partial^2 w^{(1)}}{\partial T^2}\,\frac{\partial^2 w^{(1)}}{\partial U^2}\right. \\
&\qquad + \frac{\partial^2 w^{(1)}}{\partial T^2}\,\frac{\partial^2 w^{(1)}}{\partial U^2}\left(\frac{\partial w^{(1)}}{\partial S}\right)^2 \\
&\qquad + \frac{\partial^2 w^{(1)}}{\partial S^2}\,\frac{\partial^2 w^{(1)}}{\partial T^2}\left(\frac{\partial w^{(1)}}{\partial U}\right)^2 \\
&\qquad \left. + \frac{\partial^2 w^{(1)}}{\partial U^2}\,\frac{\partial^2 w^{(1)}}{\partial S^2}\left(\frac{\partial w^{(1)}}{\partial T}\right)^2\right] .
\end{aligned}
$$

Outlook

Duality invariance fixes $\gamma = 2$.

Term $\gamma \ln Y^0$ affects BPS BH entropy:

$\gamma = 2$ yields correct value for the logarithmic correction to BPS entropy

`Ashoke Sen, arXiv:1108.3842`

Non-perturbative effects e^{-Y^0} ?

Interpretation on TST side?

Thanks!