Planar Zeros in Gauge Theories and Gravity

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Based on: D. Medrano Jiménez, A. Sabio Vera & M.Á.V.-M., JHEP 1609 (2016) 006 and work in progress.

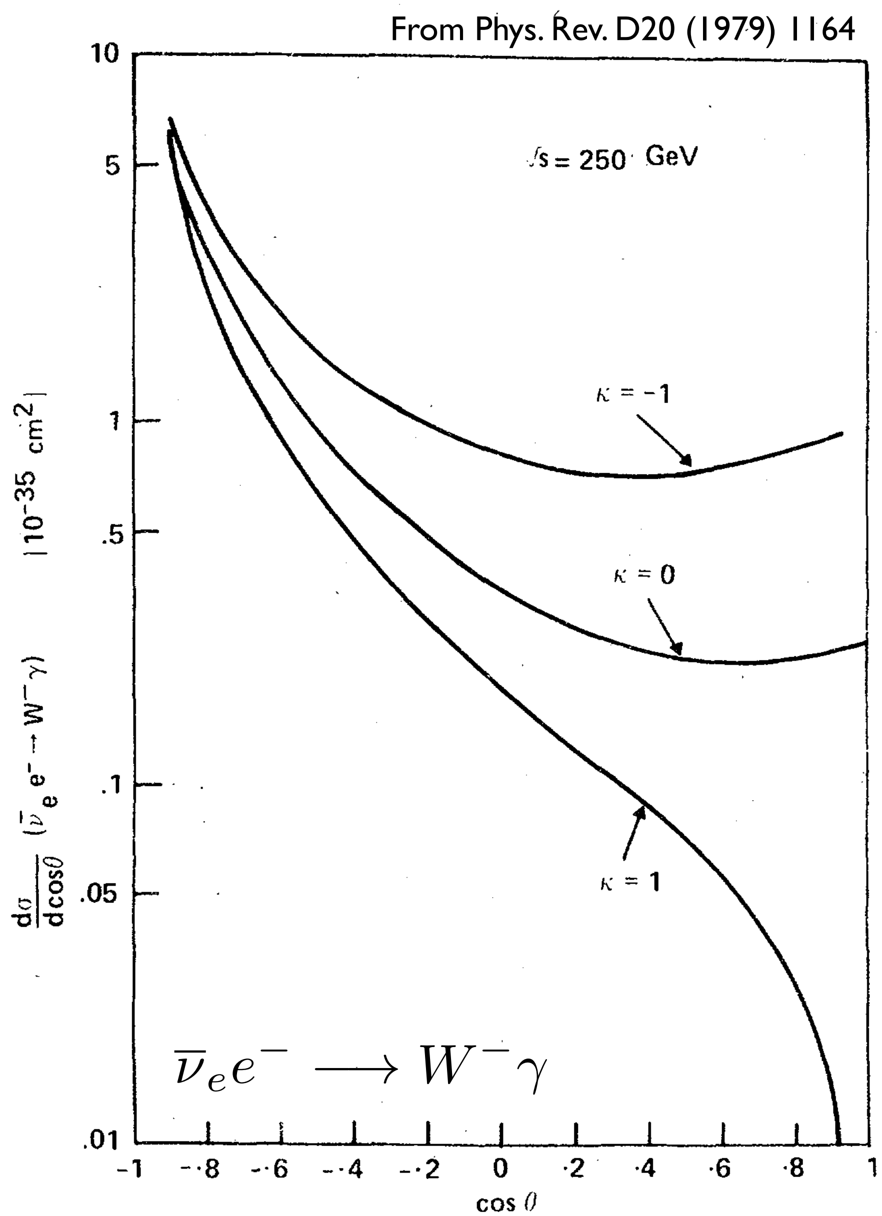
東京大学国際高等研究所カブリ数物連携宇宙研究機構　　２０１６年１１月２９日

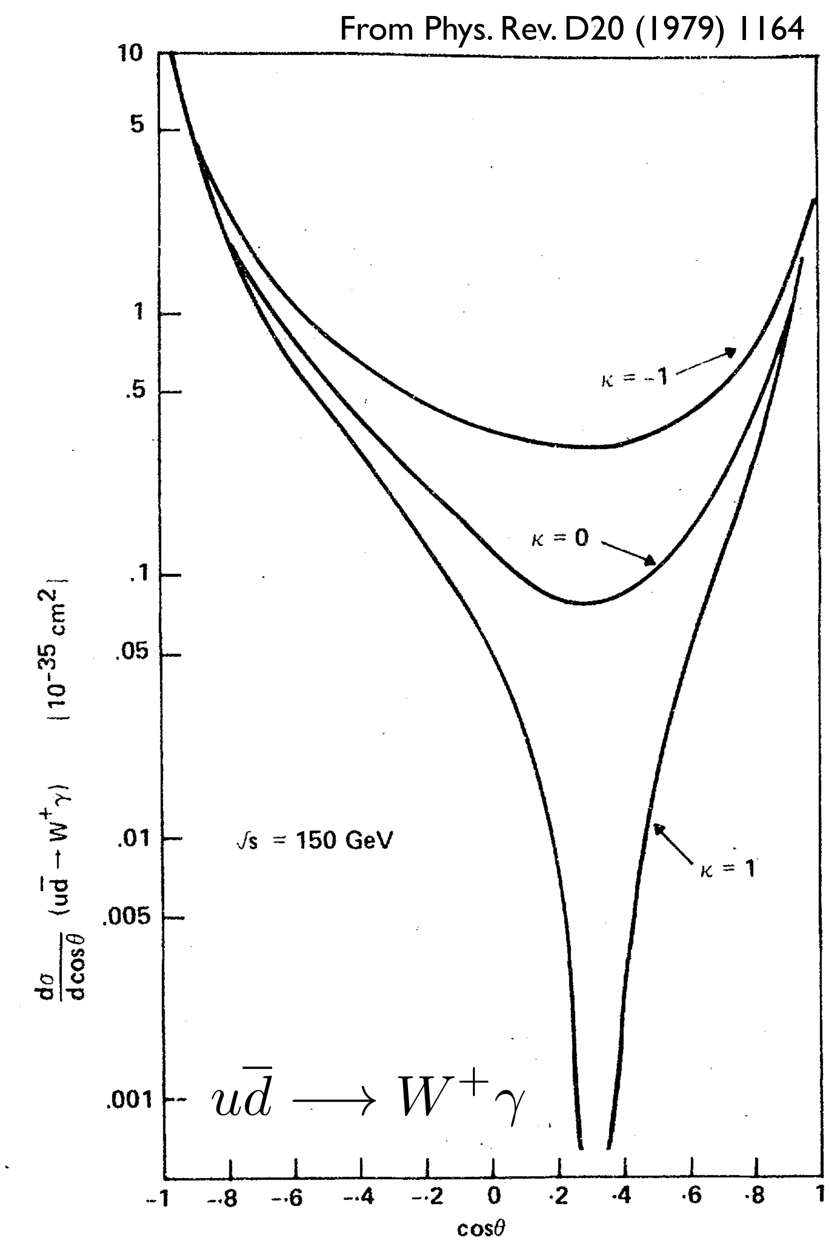
Summary

- Radiation zeros and planar zeros
- Planar zeros and projective curves
- Scalar gauge amplitudes
- Planar gravitational scattering
- Outlook

There are many zeros of scattering amplitudes, but **few** of them are in the **physical region**…

(Brown, Sahdev & Mikaelian 1979)





Here, κ is the W-boson "anomalous momentum" (i.e., $\kappa = 1$). The position of the zero provides a good **test** of gauge bosons **trilinear** couplings.

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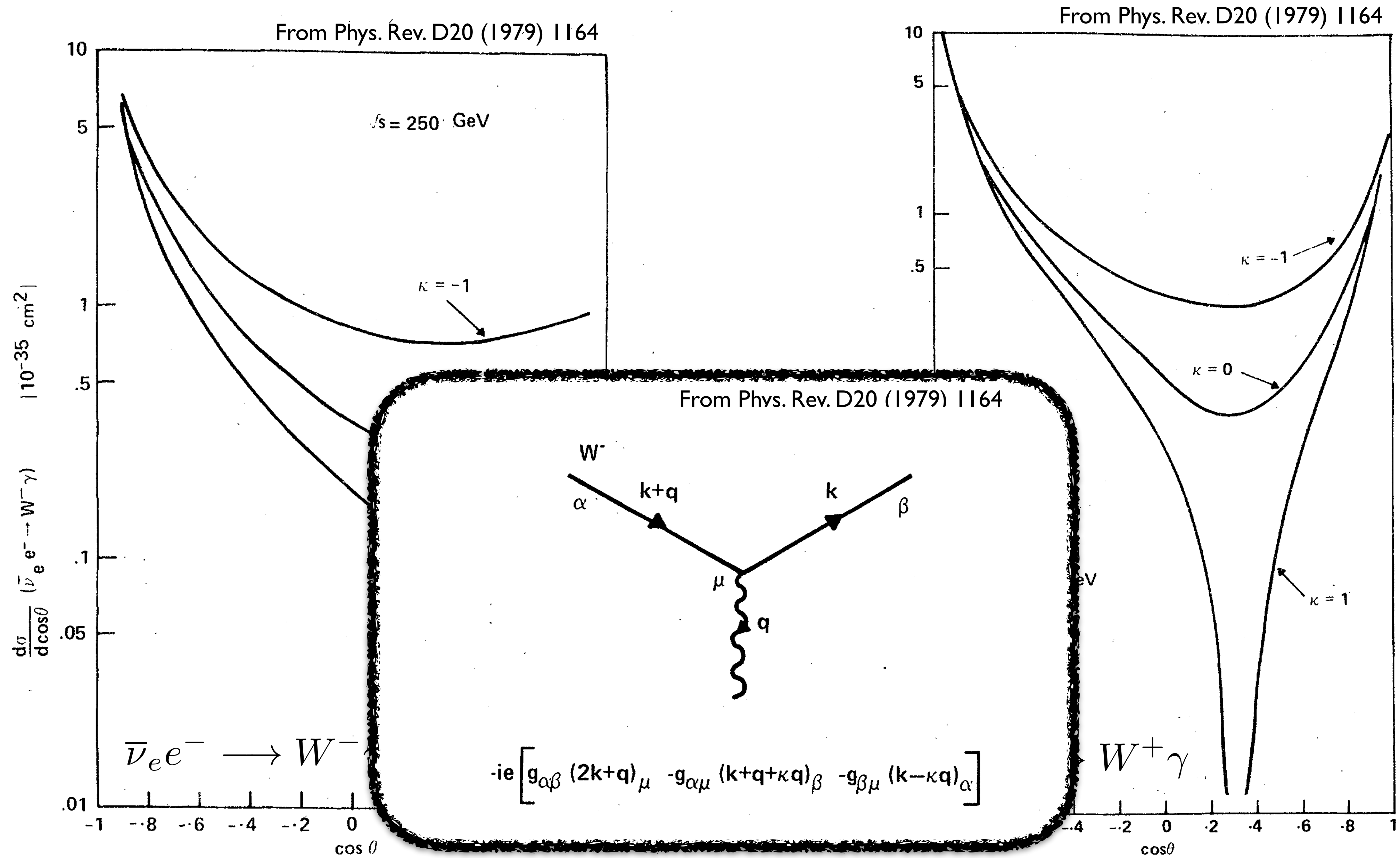

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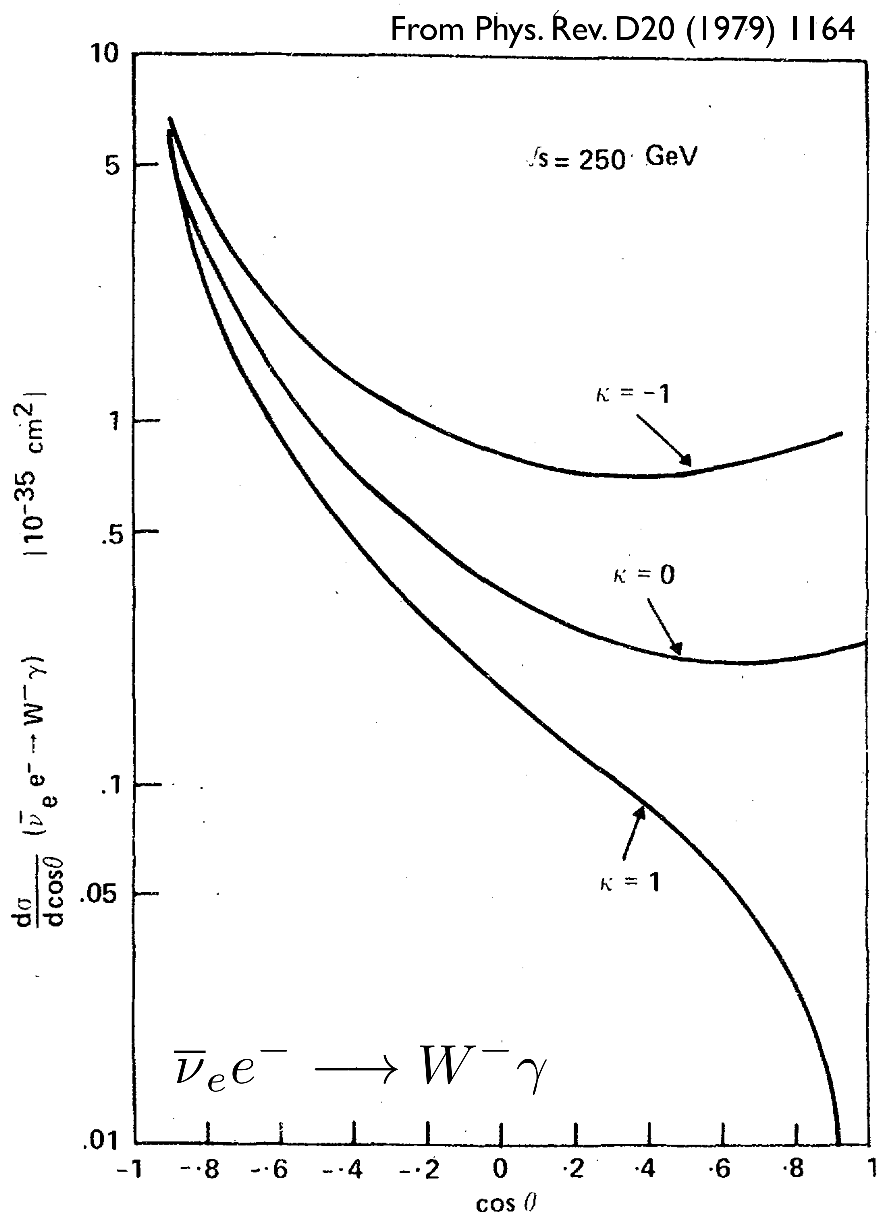
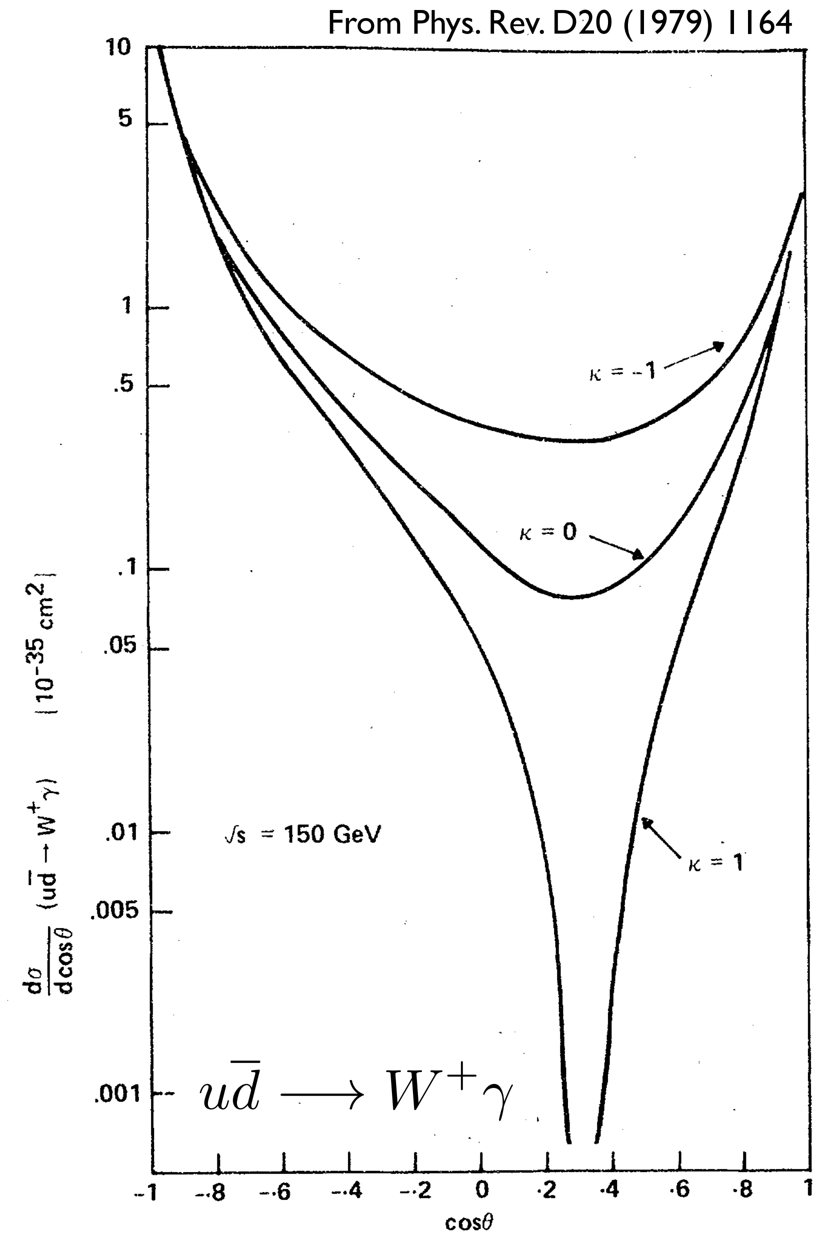




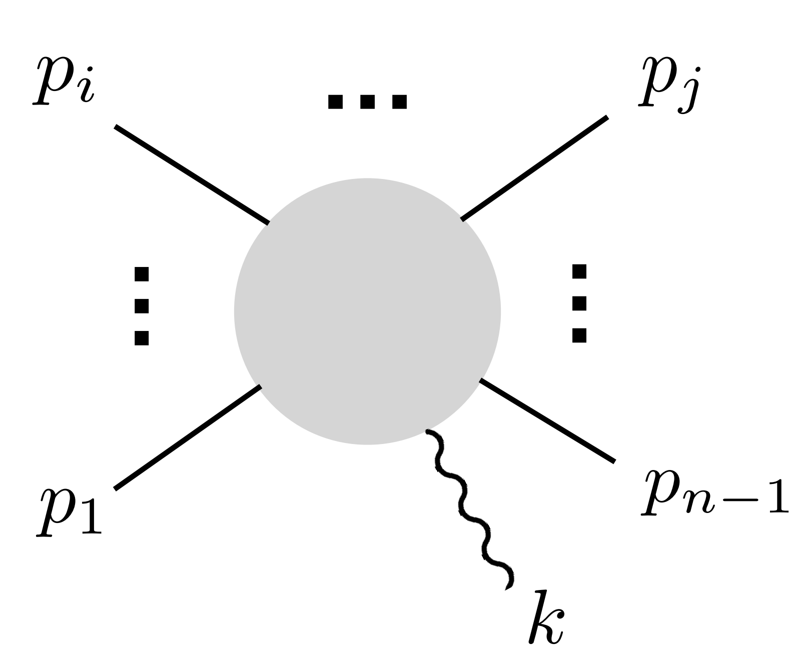
Here, κ is the W-boson "anomalous momentum" (i.e., $\kappa = 1$). The position of the zero provides a good **test** of gauge bosons **trilinear** couplings.

These zeros are **not kinematical**, but the result of **destructive interference** among different channels

Theorem: given an *n*-point, tree-level graph with the emission of a **single photon**, the amplitude **vanishes** at momenta satisfying

(Brown, Kowalski & Brodsky 1982)

$$\frac{Q_1}{p_1\cdot k}=\ldots=\frac{Q_{n-1}}{p_{n-1}\cdot k}=\text{constant}$$



Pedestrian's proof: take the limit in which the photon in **soft**

$$\mathcal{A}_n=\left[\sum_{i=1}^{n-1}Q_i\frac{p_i\cdot\epsilon(k)}{p_i\cdot k}\right]\mathcal{A}_{n-1}$$

$$\mathcal{A}_n=\text{constant}\left[\left(\sum_{i=1}^{n-1}p_i\right)\cdot\epsilon(k)\right]\mathcal{A}_{n-1}=0$$

$=-k$

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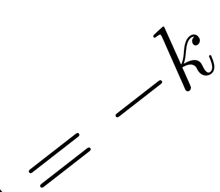

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A classical **analog**:

Take a number of particles with

$$\frac{Q_1}{m_1}=\ldots=\frac{Q_n}{m_n}=\text{constant}$$

In the **absence** of external forces, the dipolar moment of the system satisfies

$$\ddot{\mathbf{d}}=\sum_{i=1}^{n}Q_i\ddot{\mathbf{r}}=\text{constant}\sum_{i=1}^{n}\mathbf{F}_i=\mathbf{0}$$

and no dipolar radiation occurs!

$$P=\frac{2\ddot{\mathbf{d}}^2}{3c^3}=0$$

Some **properties** of radiation zeros (a.k.a. **Type-I zeros**):

- They follow from a **general factorization** of tree-level amplitudes with a radiated photon $(s \leq 1)$

$$\mathcal{A}(k;p_1,\ldots,p_{n-1}) = \sum_{i<j}^{n-1}\left(\frac{Q_i}{p_i\cdot k}-\frac{Q_j}{p_j\cdot k}\right)F_{ij}(k;p_1,\ldots,p_{n-1})$$

Very sensitive to the form of the couplings!

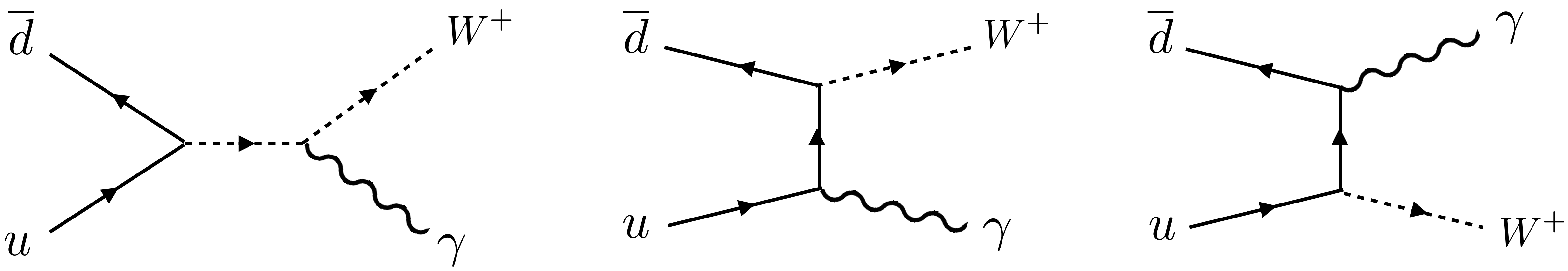
- They only occur when the **charges** of all particles involved have the **same sign**

in the physical region $p_i \cdot k \geq 0$ $\longrightarrow$ $\mathrm{sign}(Q_i) = \mathrm{sign}(Q_j)$

- They are **corrected** by **loops** and higher order **emissions**. (zero → dip)

(Laursen, Samuel, Sen & Tupper 1983; Laursen, Samuel & Sen 1983; Tupper 1985; Baur, Han & Ohnemus 1993; Ohnemus 1994))

It is instructive to see how the theorem works for the four-point function:



s-channel t-channel u-channel

The amplitude takes the form

$$\mathcal{A} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$$c_s = eg \qquad c_t = \frac{2}{3}eg \qquad c_u = \frac{1}{3}eg$$

These color factors satisfy the "Jacobi identity"

$$c_s = c_t + c_u$$

$$\mathcal{A} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \qquad\qquad c_s = c_t + c_u$$

The numerators can be **chosen** satisfying

$$n_s = n_t + n_u$$

("color-kinematics duality")

we write

$$\mathcal{A} = \frac{(c_t + c_u)(n_t + n_u)}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} = c_t \left[n_t \left(\frac{1}{s} + \frac{1}{t} \right) + \frac{n_u}{s} \right] + c_u \left[n_u \left(\frac{1}{s} + \frac{1}{u} \right) + \frac{n_t}{s} \right]$$

Assuming **high energies** (i.e., $s + t + u \approx 0$)

$$\mathcal{A} = \frac{c_t u}{s} \left(-\frac{n_t}{t} + \frac{n_u}{u} \right) + \frac{c_u t}{s} \left(\frac{n_t}{t} - \frac{n_u}{u} \right)$$

↓

$$\mathcal{A} = -\frac{c_t u - c_u t}{s} \left(\frac{n_t}{t} - \frac{n_u}{u} \right)$$

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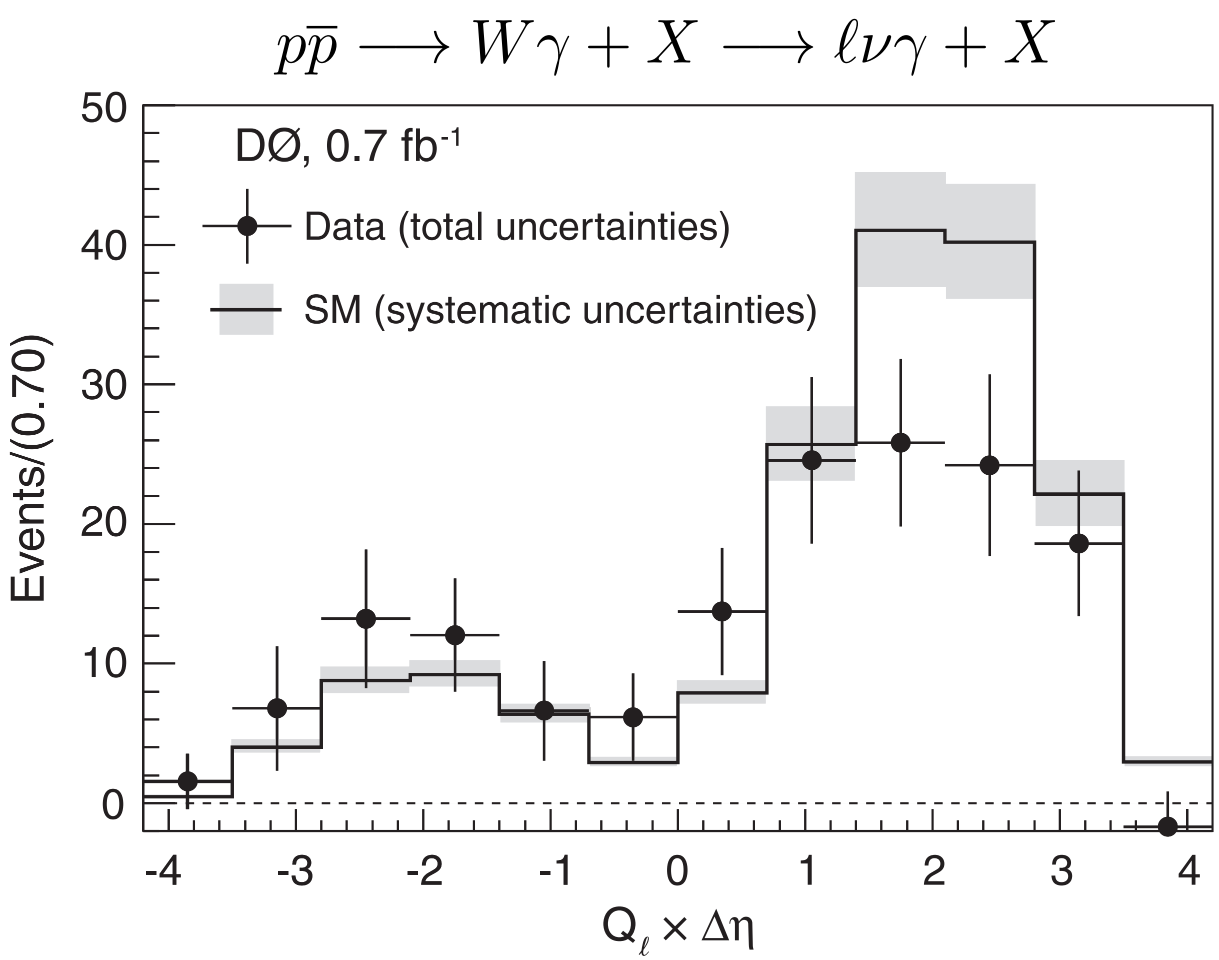
radiation zero

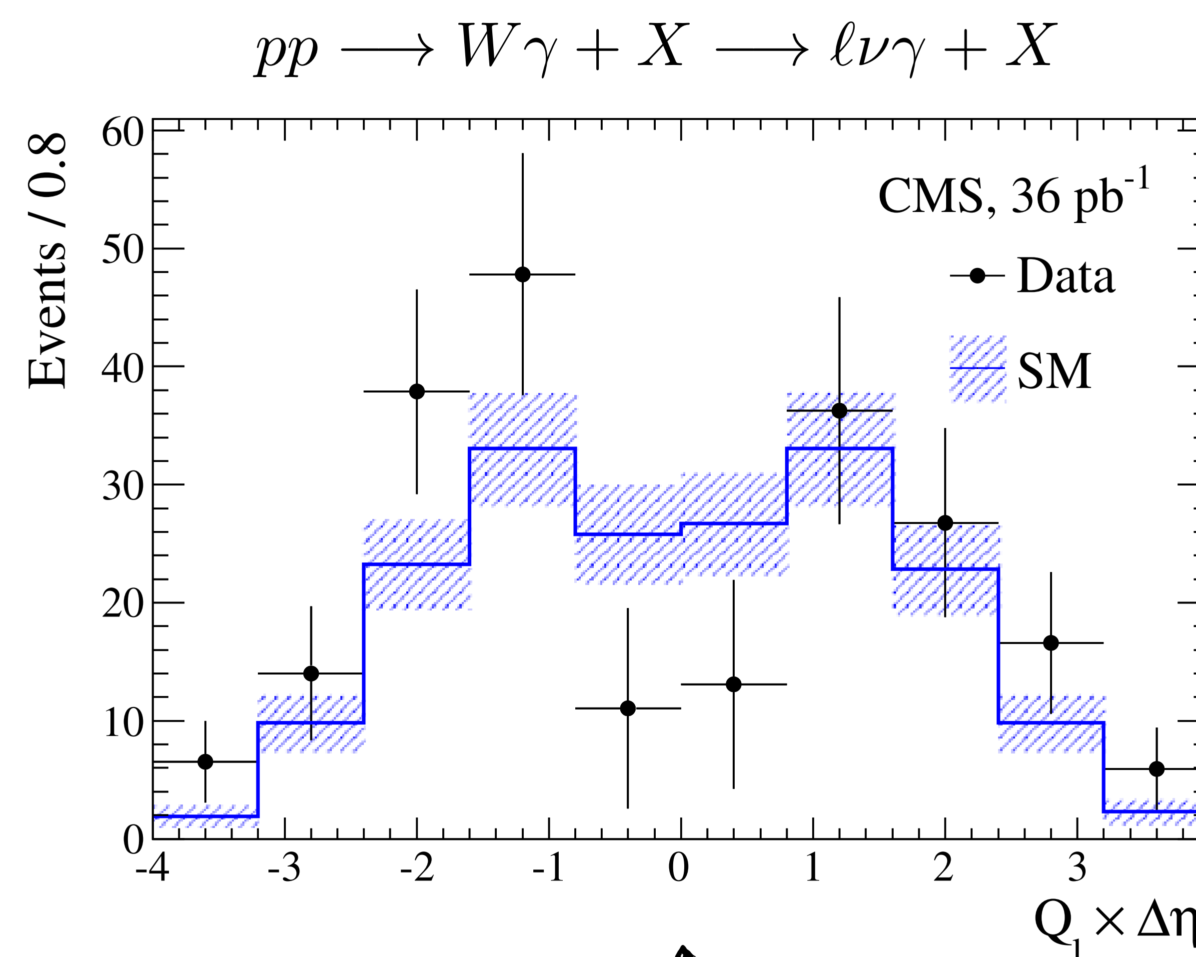
$$\frac{t}{u} = \frac{c_t}{c_u}$$

$$\mathcal{A} = -\frac{c_t u - c_u t}{s} \left(\frac{n_t}{t} - \frac{n_u}{u} \right)$$

Radiation zeros have been **observed** at both the Tevatron and the LHC

(D0 Collaboration 2008, CMS Collaboration 2011)





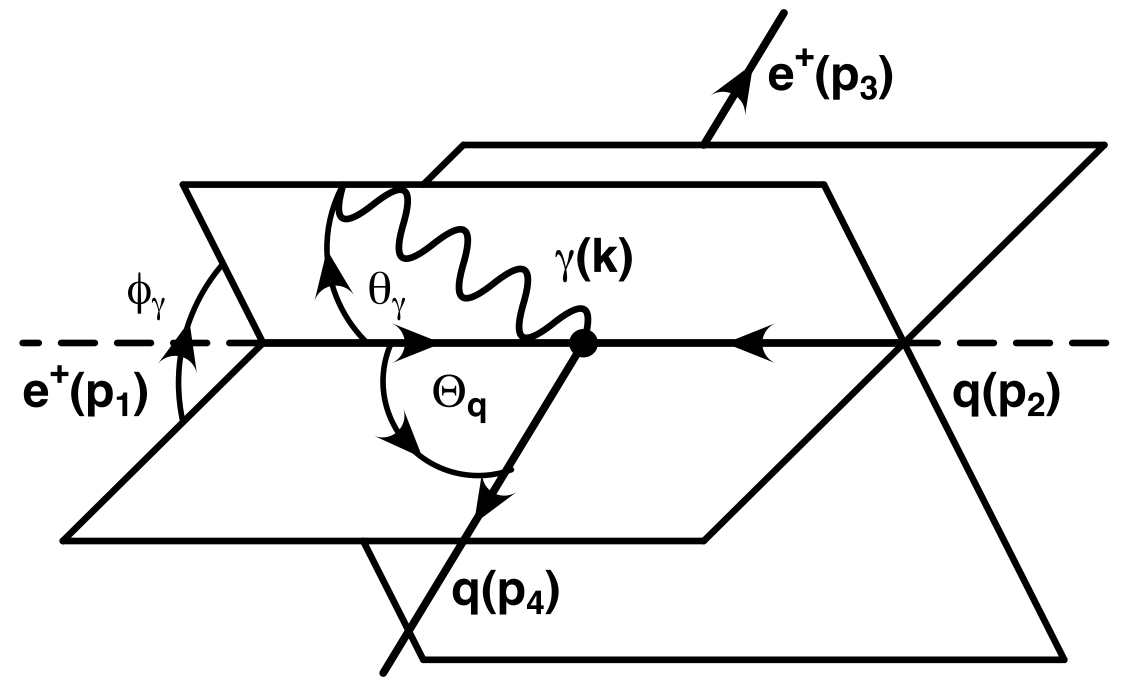
radiation zero

$\Delta\eta$ denotes the rapidity difference between the photon and the charged lepton

But there are also a **second class** of amplitude zeros (a.k.a. **type-II zeros**)

(Heyssler & Stirling 1997)

$$e^+q \longrightarrow e^+q\gamma$$



From Eur. Phys. J. C4 (1998) 289

The tree-level amplitude has "non type-I" zeros when the scattering is planar

$$\phi_\gamma = 0, \pi$$

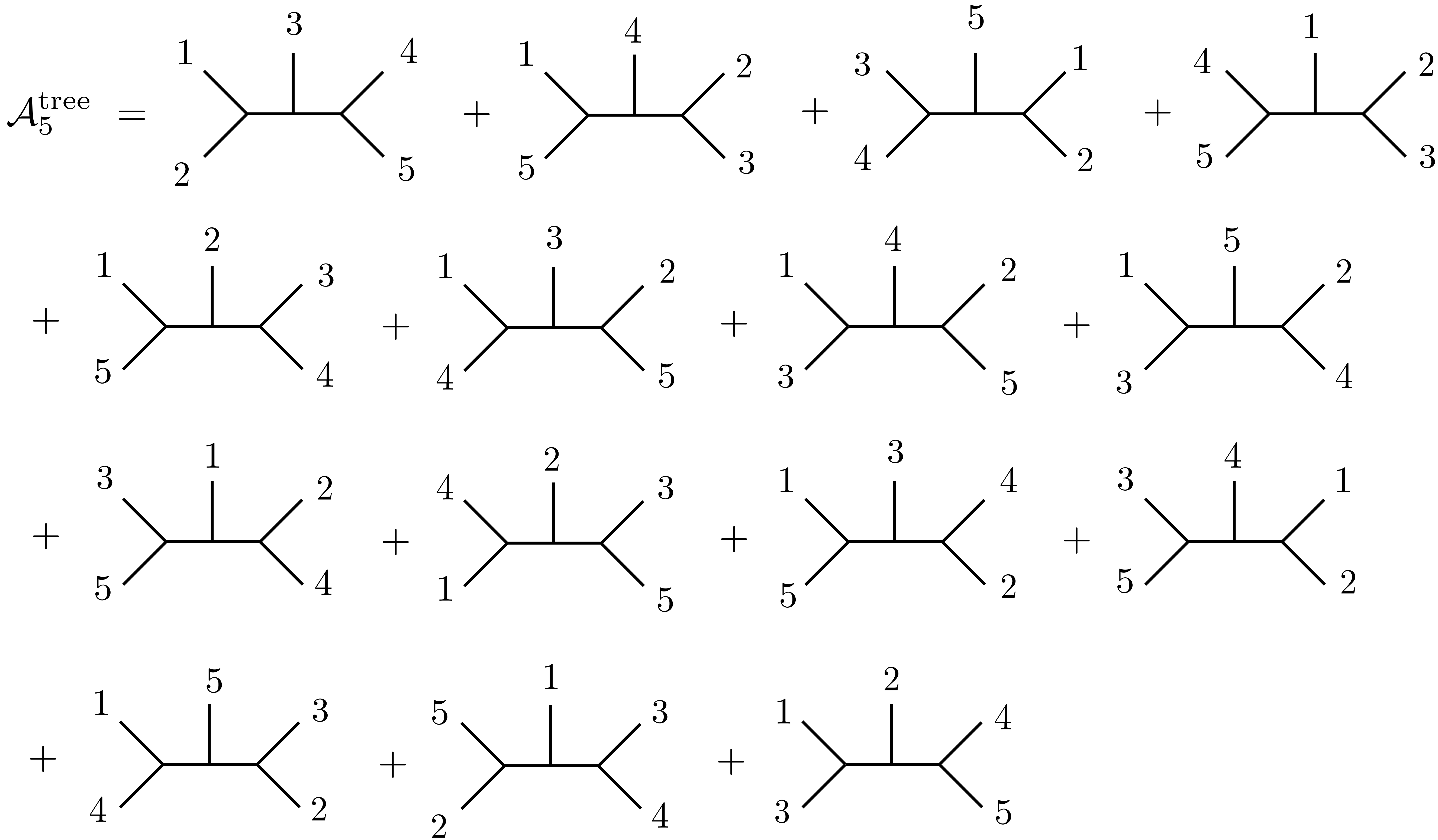
whenever a (soft) photon is emitted with an angle

$$\cos\theta_\gamma = \frac{(1-Q_q^2)(1+\cos\Theta_q) \pm \sqrt{\Delta_\gamma}}{(1-Q_q)^2}$$

where

$$\Delta_\gamma = \Big[(Q_q^2-1)(1+\cos\Theta_q)\Big]^2 - 4(1-Q_q)^2\Big(Q_q^2\cos\Theta_q + 2Q_q + \cos\Theta_q\Big)$$

Let us study **planar zeros** in nonabelian gauge theories, beginning with the **five-gluon tree amplitude** (Harland-Lang 2015)



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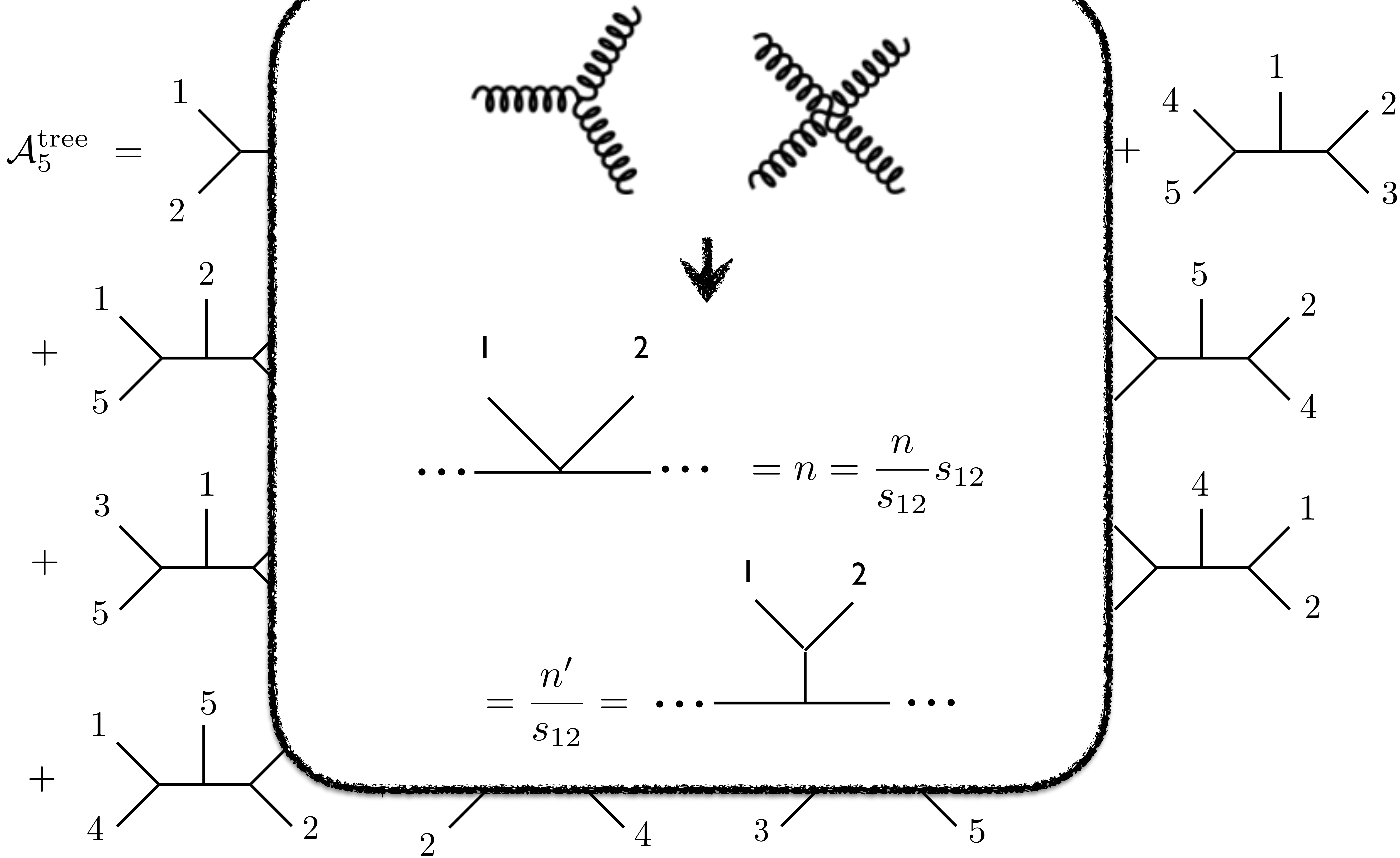

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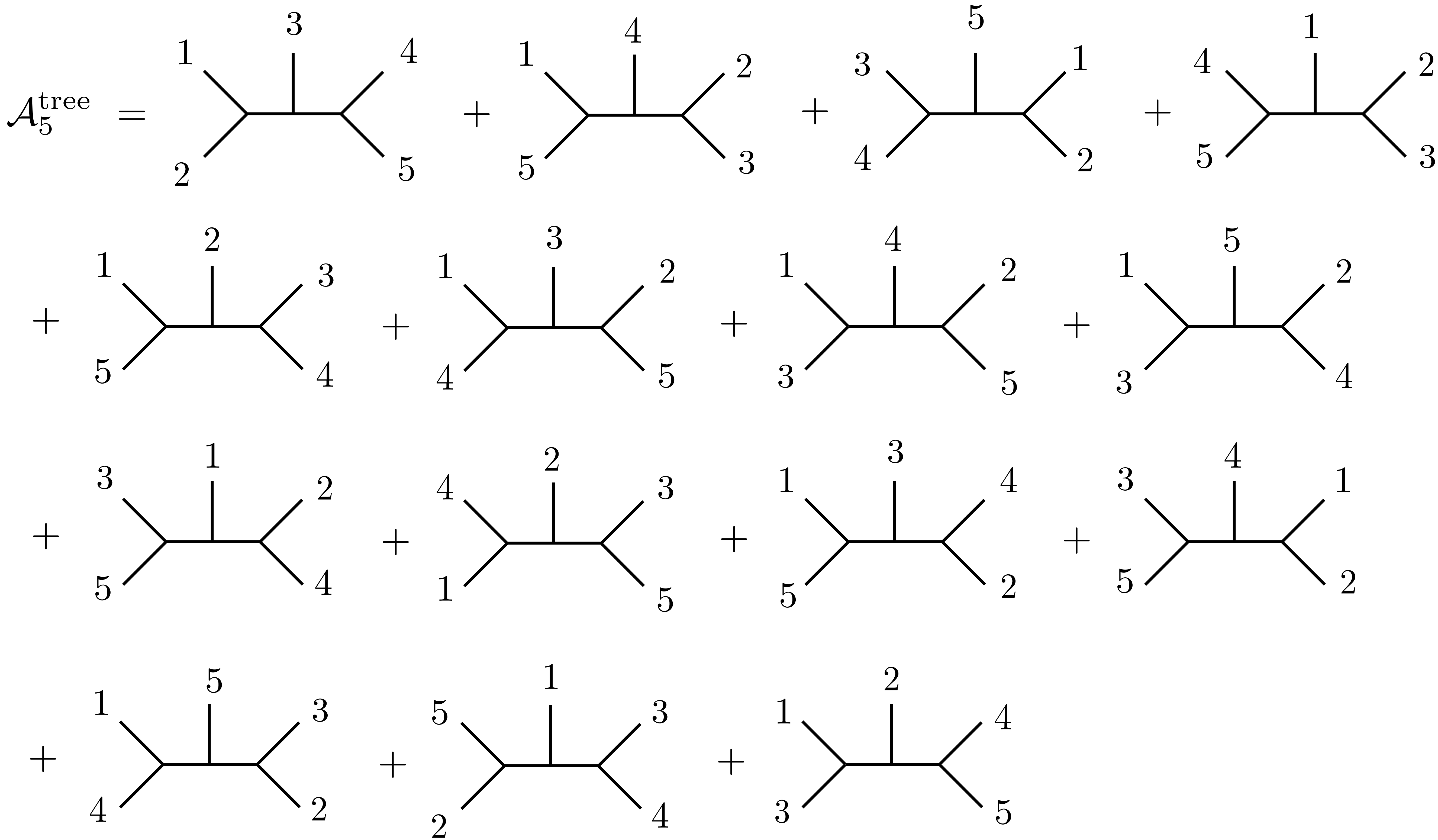
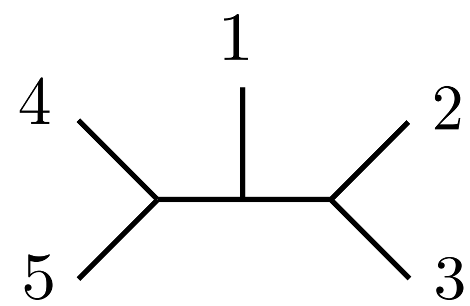


The amplitude takes the form

$$\mathcal{A}_5^{\text{tree}} = g^3 \sum_{i=1}^{15} \frac{c_i n_i}{\prod_\alpha s_{i,\alpha}}$$

$$s_{ij} = (p_i + p_j)^2 = 2 p_i \cdot p_j$$

with



$(i=4)$

$$\longrightarrow \quad \prod_\alpha s_{i,\alpha} = s_{45} s_{23}$$

and

$$c_1 = f^{a_1a_2b} f^{ba_3c} f^{ca_4a_5}, \qquad c_6 = f^{a_1a_4b} f^{ba_3c} f^{ca_2a_5}, \qquad c_{11} = f^{a_5a_1b} f^{ba_3c} f^{ca_4a_2},$$

$$c_2 = f^{a_2a_3b} f^{ba_4c} f^{ca_5a_1}, \qquad c_7 = f^{a_3a_2b} f^{ba_5c} f^{ca_1a_4}, \qquad c_{12} = f^{a_1a_2b} f^{ba_4c} f^{ca_3a_5},$$

$$c_3 = f^{a_3a_4b} f^{ba_5c} f^{ca_1a_2}, \qquad c_8 = f^{a_2a_5b} f^{ba_1c} f^{ca_4a_3}, \qquad c_{13} = f^{a_3a_5b} f^{ba_1c} f^{ca_2a_4},$$

$$c_4 = f^{a_4a_5b} f^{ba_1c} f^{ca_2a_3}, \qquad c_9 = f^{a_1a_3b} f^{ba_4c} f^{ca_2a_5}, \qquad c_{14} = f^{a_1a_4b} f^{ba_2c} f^{ca_3a_5},$$

$$c_5 = f^{a_5a_1b} f^{ba_2c} f^{ca_3a_4}, \qquad c_{10} = f^{a_4a_2b} f^{ba_5c} f^{ca_1a_3}, \qquad c_{15} = f^{a_1a_3b} f^{ba_2c} f^{ca_4a_5},$$

However, the 15 color factors are not independent:

$$c_3 - c_5 + c_8 = 0, \quad c_8 - c_6 + c_9 = 0, \quad c_4 - c_2 + c_7 = 0,$$

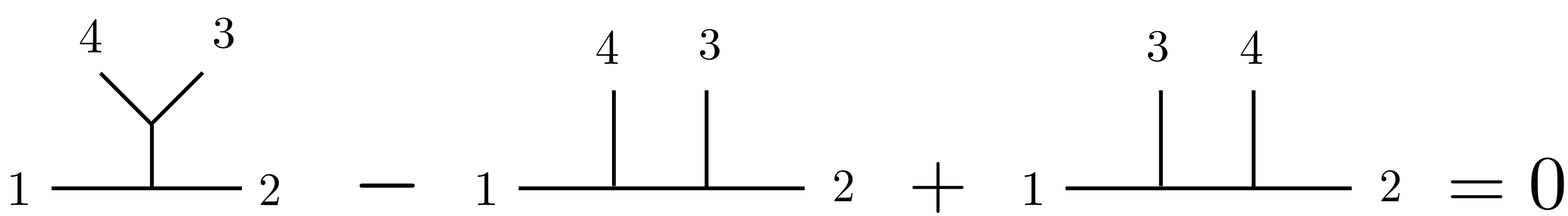
$$c_4 - c_1 + c_{15} = 0, \quad c_{10} - c_{11} + c_{13} = 0, \quad c_7 - c_6 + c_{14} = 0,$$

$$c_5 - c_2 + c_{11} = 0, \quad c_3 - c_1 + c_{12} = 0, \quad c_{10} - c_9 + c_{15} = 0,$$

$$(c_{13} - c_{12} + c_{14} = 0)$$

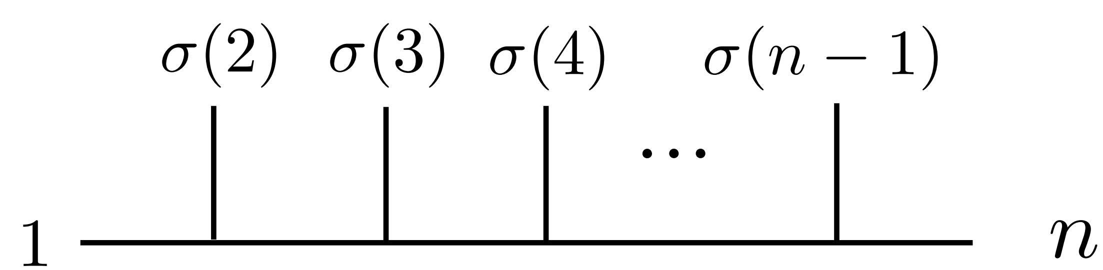
These Jacobi identities can be written **diagrammatically** as

$$f^{a_1a_2b}f^{a_3a_4b} + f^{a_2a_3b}f^{a_1a_4b} + f^{a_3a_1b}f^{a_2a_4b} = 0$$



so all topologies can be converted into **muti-peripheral** diagrams

$$\mathcal{A}_n^{\text{tree}}(k_1, \ldots, k_n) = \sum_{\sigma \in S_{n-2}}$$



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$$c_5 - c_2 + c_{11} = 0,$$

$$f^{a_1a_2b}f^{a_3a_4b} + f^{a_2a_3b}f^{a_1a_4b} + f^{a_3a_1b}f^{a_2a_4b} = 0$$

$$c_1 = f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5} = -f^{a_2a_3b}f^{a_1cb}f^{ca_4a_5} - f^{a_3a_1b}f^{a_2cb}f^{ca_4a_5}$$

$$= -f^{a_2a_3b}f^{ba_1c}f^{ca_4a_5} - f^{a_3a_1b}f^{ba_2c}f^{ca_4a_5} = c_4 + c_{15}$$

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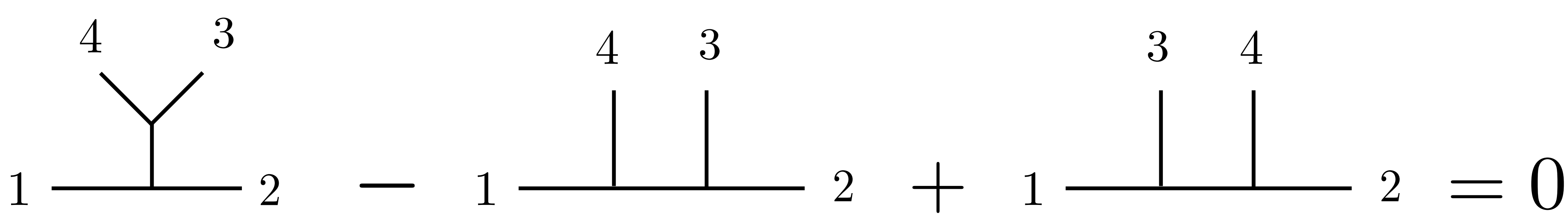

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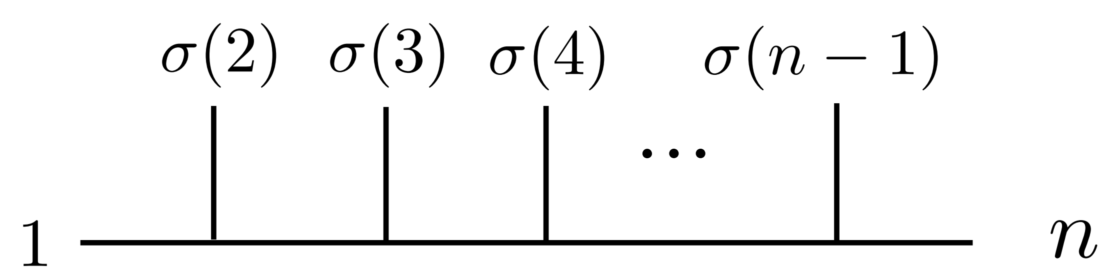

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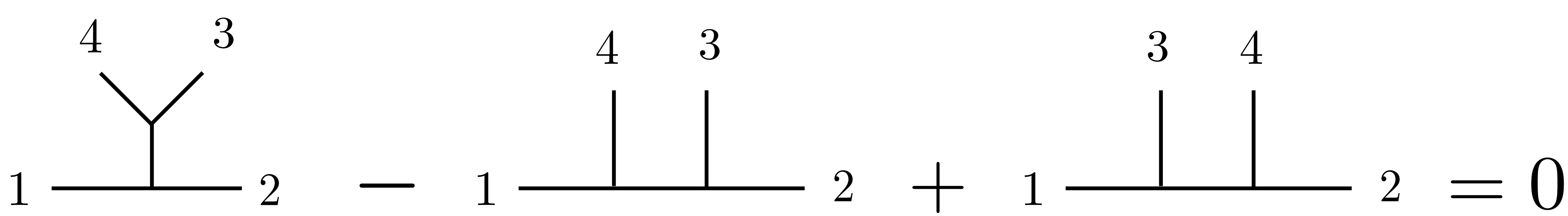
$$f^{a_1 a_2 b} f^{a_3 a_4 b} + f^{a_2 a_3 b} f^{a_1 a_4 b} + f^{a_3 a_1 b} f^{a_2 a_4 b} = 0$$



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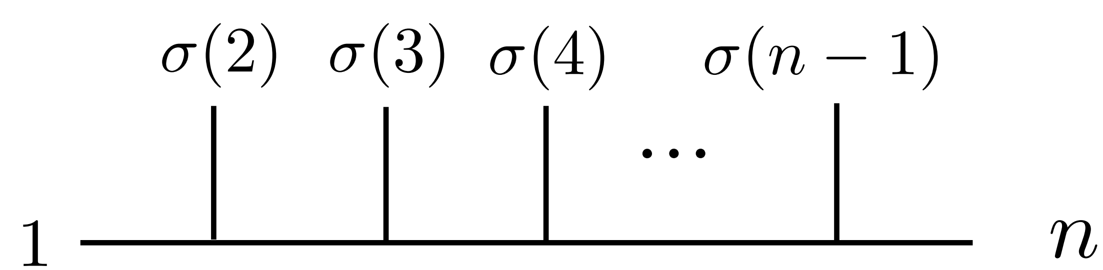

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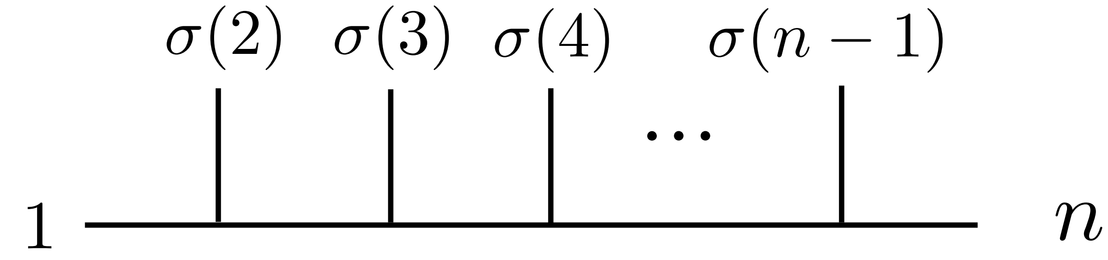


For the 5-gluon amplitude this means that it can be written in terms of 3! color ordered amplitudes

$$\mathcal{A}_5^{\text{tree}} = g^3 \sum_{\sigma \in S_3} c[1, 2, \sigma(3, 4, 5)] A_5[1, 2, \sigma(3, 4, 5)]$$

where

$$\begin{aligned}
A_5[1,2,3,4,5] &= \frac{n_1}{s_{12}s_{45}} - \frac{n_2}{s_{23}s_{15}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{15}s_{34}}, \\
A_5[1,2,3,5,4] &= -\frac{n_1}{s_{12}s_{45}} - \frac{n_{13}}{s_{23}s_{14}} + \frac{n_{12}}{s_{35}s_{12}} - \frac{n_4}{s_{45}s_{23}} + \frac{n_{10}}{s_{14}s_{35}}, \\
A_5[1,2,4,3,5] &= -\frac{n_{12}}{s_{12}s_{35}} - \frac{n_{11}}{s_{24}s_{15}} - \frac{n_3}{s_{34}s_{12}} + \frac{n_9}{s_{35}s_{24}} - \frac{n_5}{s_{15}s_{34}}, \\
A_5[1,2,4,5,3] &= \frac{n_{12}}{s_{12}s_{35}} - \frac{n_8}{s_{24}s_{13}} - \frac{n_1}{s_{45}s_{12}} - \frac{n_9}{s_{35}s_{24}} - \frac{n_{15}}{s_{13}s_{45}}, \\
A_5[1,2,5,3,4] &= -\frac{n_3}{s_{12}s_{34}} - \frac{n_6}{s_{25}s_{14}} - \frac{n_{12}}{s_{35}s_{12}} + \frac{n_{14}}{s_{34}s_{25}} - \frac{n_{10}}{s_{14}s_{35}}, \\
A_5[1,2,5,4,3] &= \frac{n_3}{s_{12}s_{34}} - \frac{n_7}{s_{25}s_{13}} + \frac{n_1}{s_{12}s_{45}} - \frac{n_{14}}{s_{34}s_{25}} + \frac{n_{15}}{s_{13}s_{45}}.
\end{aligned}$$

To make things simpler, we implement **color-kinematics duality**

$$j_A \equiv c_i + c_i - c_k = 0 \longrightarrow n'_i + n'_j - n'_k = 0$$

and solve for the numerators

$$\begin{aligned}
n_1 &= -n_{12} = n_{15} = s_{12}s_{45}A_5[1,2,3,4,5], \\
n_2 &= n_3 = n_4 = n_5 = n_{11} = n_{13} = n_{14} = 0, \\
n_6 &= n_7 = n_{10} = s_{14}s_{35}A_5[1,2,3,5,4] + s_{14}(s_{35}+s_{45})A_5[1,2,3,4,5], \\
n_8 &= n_9 = s_{14}s_{35}A_5[1,2,3,5,4] + (s_{14}s_{35} + s_{14}s_{45} + s_{12}s_{45})A_5[1,2,3,4,5].
\end{aligned}$$

We further consider **MHV** amplitudes and use the **Parke-Taylor formula**:

$$A_5[1^-, 2^-, \sigma(3^+, 4^+, 5^+)] = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2\sigma(3)\rangle\langle\sigma(3)\sigma(4)\rangle\langle\sigma(4)\sigma(5)\rangle\langle\sigma(5)1\rangle}$$

To make thi

and solve

n_1

n_2

n_6

n_8

We further ... a:

A **reminder** of the spinor helicity notation

Since (½,½) = (½,**0**) ⊗ (**0**,½), we can write $p_\mu \longrightarrow p_{a\dot{b}} = p_\mu \sigma^\mu_{a\dot{b}}$. Moreover,

$$p^2 = 0 \quad \rightarrow \quad \det p_{a\dot{b}} = 0 \quad \rightarrow \quad p_{a\dot{b}} = \lambda_a \widetilde{\lambda}_{\dot{b}}$$

The standard notation is to write $\lambda_a \to |p]_a$ and $\widetilde{\lambda}_{\dot{a}} = \langle p|_{\dot{a}}$

Introducing the **ϵ-tensor**

$$[p|^a = \epsilon^{ab}|p]_b, \qquad |p\rangle^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\langle p|_{\dot{b}}$$

we have the products

$$\langle ij\rangle \equiv \langle p_i|^{\dot{a}}|p_j\rangle_{\dot{a}}, \quad [ij] = [i|^a|j]_a$$

satisfying $\langle ij\rangle[ij] = 2p_i \cdot p_j = s_{ij}$

$$A_5[1^-, 2^-, \sigma(3^+, 4^+, 5^+)] = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2\sigma(3)\rangle\langle\sigma(3)\sigma(4)\rangle\langle\sigma(4)\sigma(5)\rangle\langle\sigma(5)1\rangle}$$

To make things simpler, we implement **color-kinematics duality**

$$j_A \equiv c_i + c_i - c_k = 0 \longrightarrow n'_i + n'_j - n'_k = 0$$

and solve for the numerators

$$n_1 = -n_{12} = n_{15} = s_{12}s_{45}A_5[1,2,3,4,5],$$
$$n_2 = n_3 = n_4 = n_5 = n_{11} = n_{13} = n_{14} = 0,$$
$$n_6 = n_7 = n_{10} = s_{14}s_{35}A_5[1,2,3,5,4] + s_{14}(s_{35}+s_{45})A_5[1,2,3,4,5],$$
$$n_8 = n_9 = s_{14}s_{35}A_5[1,2,3,5,4] + (s_{14}s_{35} + s_{14}s_{45} + s_{12}s_{45})A_5[1,2,3,4,5].$$

We further consider **MHV** amplitudes and use the **Parke-Taylor formula**:

$$A_5[1^-, 2^-, \sigma(3^+, 4^+, 5^+)] = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2\sigma(3)\rangle\langle\sigma(3)\sigma(4)\rangle\langle\sigma(4)\sigma(5)\rangle\langle\sigma(5)1\rangle}$$

We get the solution

$$n_1 = -n_{12} = n_{15} = i\frac{\langle 12\rangle^4[21][54]}{\langle 23\rangle\langle 34\rangle\langle 51\rangle},$$

$$n_6 = n_7 = n_{10} = i\frac{\langle 12\rangle^4[14][52]}{\langle 23\rangle\langle 34\rangle\langle 51\rangle},$$

$$n_8 = n_9 = i\frac{\langle 12\rangle^4[24][51]}{\langle 23\rangle\langle 34\rangle\langle 51\rangle},$$

$$n_2 = n_3 = n_4 = n_5 = n_{11} = n_{13} = n_{14} = 0$$

and write the 5-gluon amplitude in terms of the 6 independent color structures c_2, c_6, c_7, c_8, c_{11}, and c_{13} as

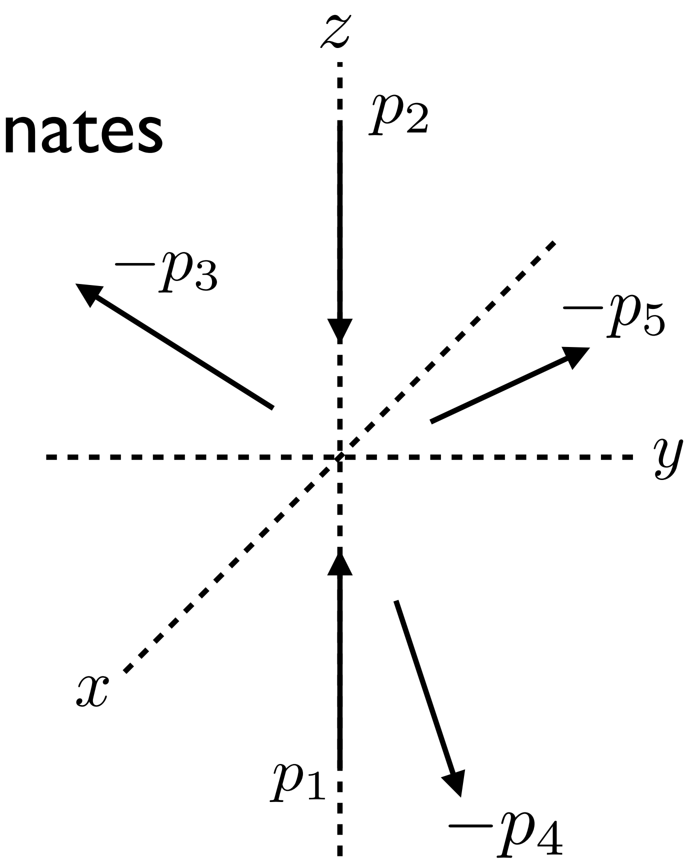
$$\mathcal{A}_5^{\text{tree}} = -ig^3\langle 12\rangle^3\left(\frac{c_2}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} + \frac{c_6}{\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle} + \frac{c_7}{\langle 25\rangle\langle 54\rangle\langle 43\rangle\langle 31\rangle} + \frac{c_8}{\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle} + \frac{c_{11}}{\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle} + \frac{c_{13}}{\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle}\right).$$

We want to parametrize momenta using stereographic coordinates

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1) \qquad (\zeta_1 = \infty)$$

$$p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1) \qquad (\zeta_2 = 0)$$

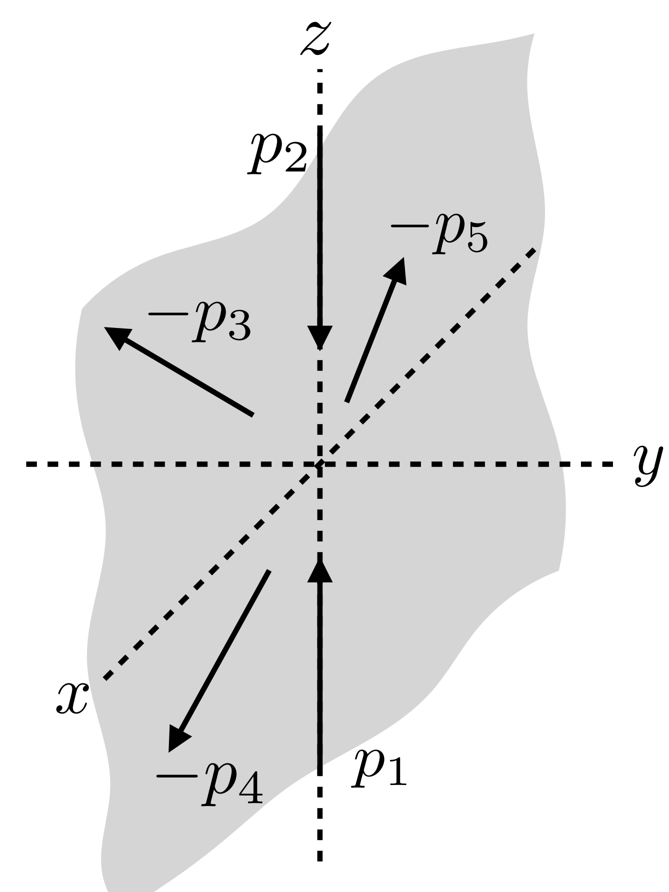
$$p_a = -\omega_a \left(1, \frac{\zeta_a + \overline{\zeta}_a}{1+\zeta_a\overline{\zeta}_a}, i\frac{\overline{\zeta}_a - \zeta_a}{1+\zeta_a\overline{\zeta}_a}, \frac{\zeta_a\overline{\zeta}_a - 1}{1+\zeta_a\overline{\zeta}_a}\right)$$



and consider processes taking place on the **plane $y = 0$**

$$\zeta_a = \overline{\zeta}_a$$

$$p_a = -\frac{\omega_a}{1+\zeta_a^2}(1+\zeta_a^2, 2\zeta_a, 0, \zeta_a^2 - 1).$$



$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1) \qquad p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1) \qquad p_a = -\frac{\omega_a}{1+\zeta_a^2}(1+\zeta_a^2, 2\zeta_a, 0, \zeta_a^2-1).$$

Using four-momentum conservation we **solve** for the energies

$$p_1 + p_2 + p_3 + p_4 + p_5 = 0 \quad \longrightarrow \quad \begin{aligned} \omega_3 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_3^2)(1+\zeta_4\zeta_5)}{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)}, \\ \omega_4 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_4^2)(1+\zeta_3\zeta_5)}{(\zeta_4-\zeta_3)(\zeta_4-\zeta_5)}, \\ \omega_5 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_5^2)(1+\zeta_3\zeta_4)}{(\zeta_5-\zeta_3)(\zeta_5-\zeta_4)}. \end{aligned}$$

Thus, the amplitude reads

$$\mathcal{A}_5^{\text{tree}} = \frac{2ig^3}{\sqrt{s}}\frac{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)}{(1+\zeta_3\zeta_4)(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)}\left(-c_2\frac{\zeta_5-\zeta_3}{\zeta_3} - c_6\frac{\zeta_4-\zeta_5}{\zeta_5}\right.$$
$$\left. +c_7\frac{\zeta_3-\zeta_5}{\zeta_5} - c_8\frac{\zeta_3-\zeta_4}{\zeta_4} + c_{11}\frac{\zeta_5-\zeta_4}{\zeta_4} + c_{13}\frac{\zeta_4-\zeta_3}{\zeta_3}\right)$$

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1) \qquad p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1) \qquad p_a = -\frac{\omega_a}{1+\zeta_a^2}(1+\zeta_a^2, 2\zeta_a, 0, \zeta_a^2 - 1).$$

Using four-momentum conservation we **solve** for the energies

$$p_1 + p_2 + p_3 + p_4 + p_5 = 0 \quad \longrightarrow \quad \begin{aligned} \omega_3 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_3^2)(1+\zeta_4\zeta_5)}{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)}, \\ \omega_4 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_4^2)(1+\zeta_3\zeta_5)}{(\zeta_4-\zeta_3)(\zeta_4-\zeta_5)}, \\ \omega_5 &= \frac{\sqrt{s}}{2}\frac{(1+\zeta_5^2)(1+\zeta_3\zeta_4)}{(\zeta_5-\zeta_3)(\zeta_5-\zeta_4)}. \end{aligned}$$

Thus, the amplitude reads

planar zeros

$$\mathcal{A}_5^{\text{tree}} = \frac{2ig^3}{\sqrt{s}}\frac{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)}{(1+\zeta_3\zeta_4)(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)}\left(-c_2\frac{\zeta_5-\zeta_3}{\zeta_3} - c_6\frac{\zeta_4-\zeta_5}{\zeta_5} + c_7\frac{\zeta_3-\zeta_5}{\zeta_5} - c_8\frac{\zeta_3-\zeta_4}{\zeta_4} + c_{11}\frac{\zeta_5-\zeta_4}{\zeta_4} + c_{13}\frac{\zeta_4-\zeta_3}{\zeta_3}\right) = 0$$

$$c_2\frac{\zeta_5-\zeta_3}{\zeta_3}+c_6\frac{\zeta_4-\zeta_5}{\zeta_5}-c_7\frac{\zeta_3-\zeta_5}{\zeta_5}$$
$$+c_8\frac{\zeta_3-\zeta_4}{\zeta_4}-c_{11}\frac{\zeta_5-\zeta_4}{\zeta_4}-c_{13}\frac{\zeta_4-\zeta_3}{\zeta_3}=0$$

This is **homogeneous equation** of degree zero. Multiplying by $\zeta_3\zeta_4\zeta_5$ we get

$$c_7\zeta_3^2\zeta_4-c_8\zeta_3^2\zeta_5-c_6\zeta_3\zeta_4^2+c_{11}\zeta_3\zeta_5^2$$
$$+(c_2+c_6-c_7+c_8-c_{11}-c_{13})\zeta_3\zeta_4\zeta_5+c_{13}\zeta_4^2\zeta_5-c_2\zeta_4\zeta_5^2=0$$

The loci of planar zeros defines an **integer cubic curve** in the **projective plane**. Working in the **patch** centered at (1,0,0)

$$(\zeta_3,\zeta_4,\zeta_5)=\lambda(1,U,V), \qquad \lambda,U,V\neq 0$$

the cubic is

$$c_7U-c_8V-c_6U^2+c_{11}V^2+(c_2+c_6-c_7+c_8-c_{11}-c_{13})UV+c_{13}U^2V-c_2UV^2=0.$$

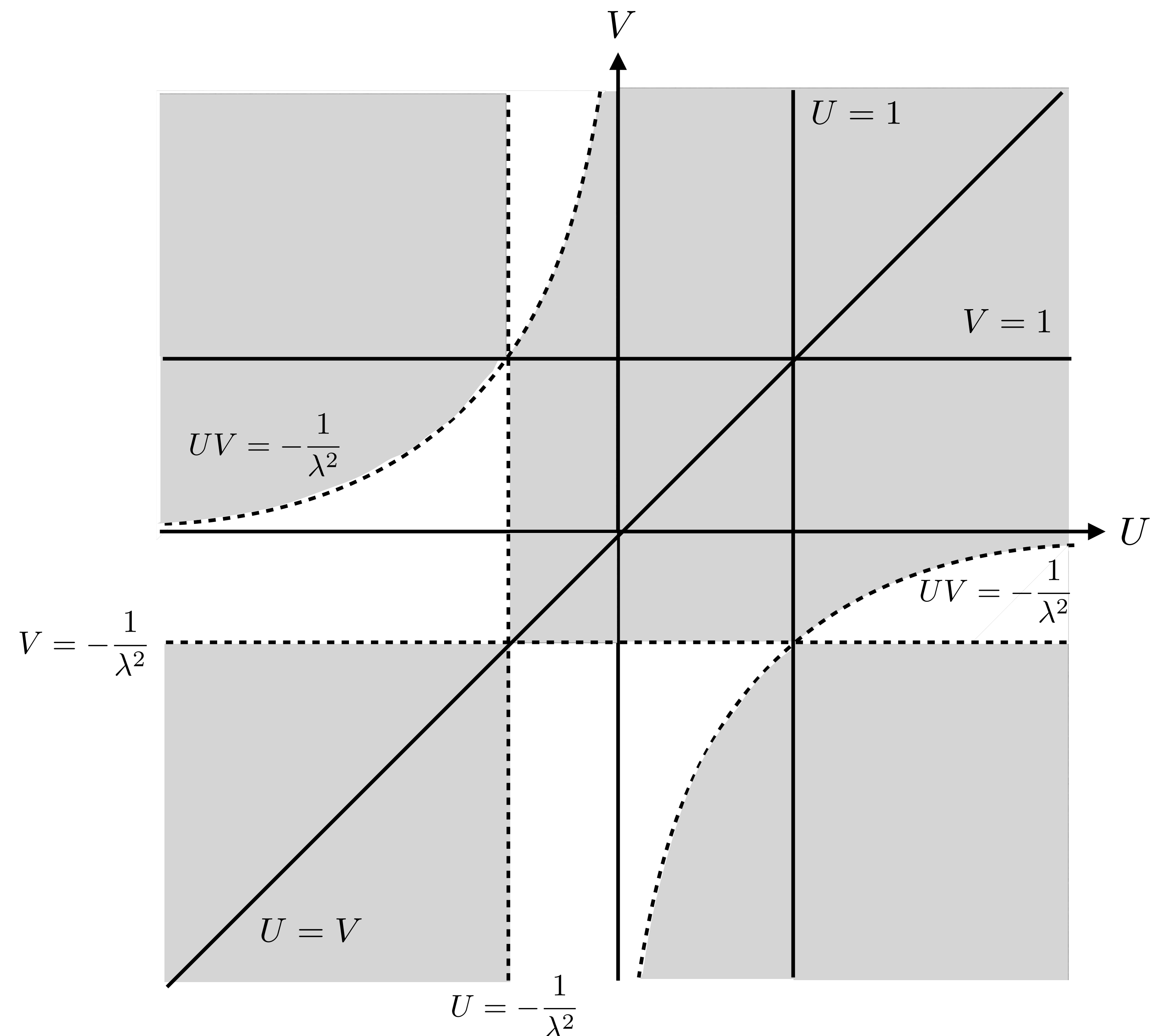
$$c_7U - c_8V - c_6U^2 + c_{11}V^2 + (c_2 + c_6 - c_7 + c_8 - c_{11} - c_{13})UV + c_{13}U^2V - c_2UV^2 = 0.$$

Let us revisit the **energies**…

$$\omega_3 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2)(1+\lambda^2UV)}{\lambda^2(1-U)(1-V)},$$

$$\omega_4 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2U^2)(1+\lambda^2V)}{\lambda^2(U-1)(U-V)},$$

$$\omega_5 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2V^2)(1+\lambda^2U)}{\lambda^2(V-1)(V-U)}.$$



The physical region is determined by requiring

$$0 \leq \omega_3 < \infty$$

$$0 \leq \omega_4 < \infty$$

$$0 \leq \omega_5 < \infty$$

What about **soft limits**?

$$\omega_3 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2)(1+\lambda^2UV)}{\lambda^2(1-U)(1-V)},$$

$$\omega_4 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2U^2)(1+\lambda^2V)}{\lambda^2(U-1)(U-V)},$$

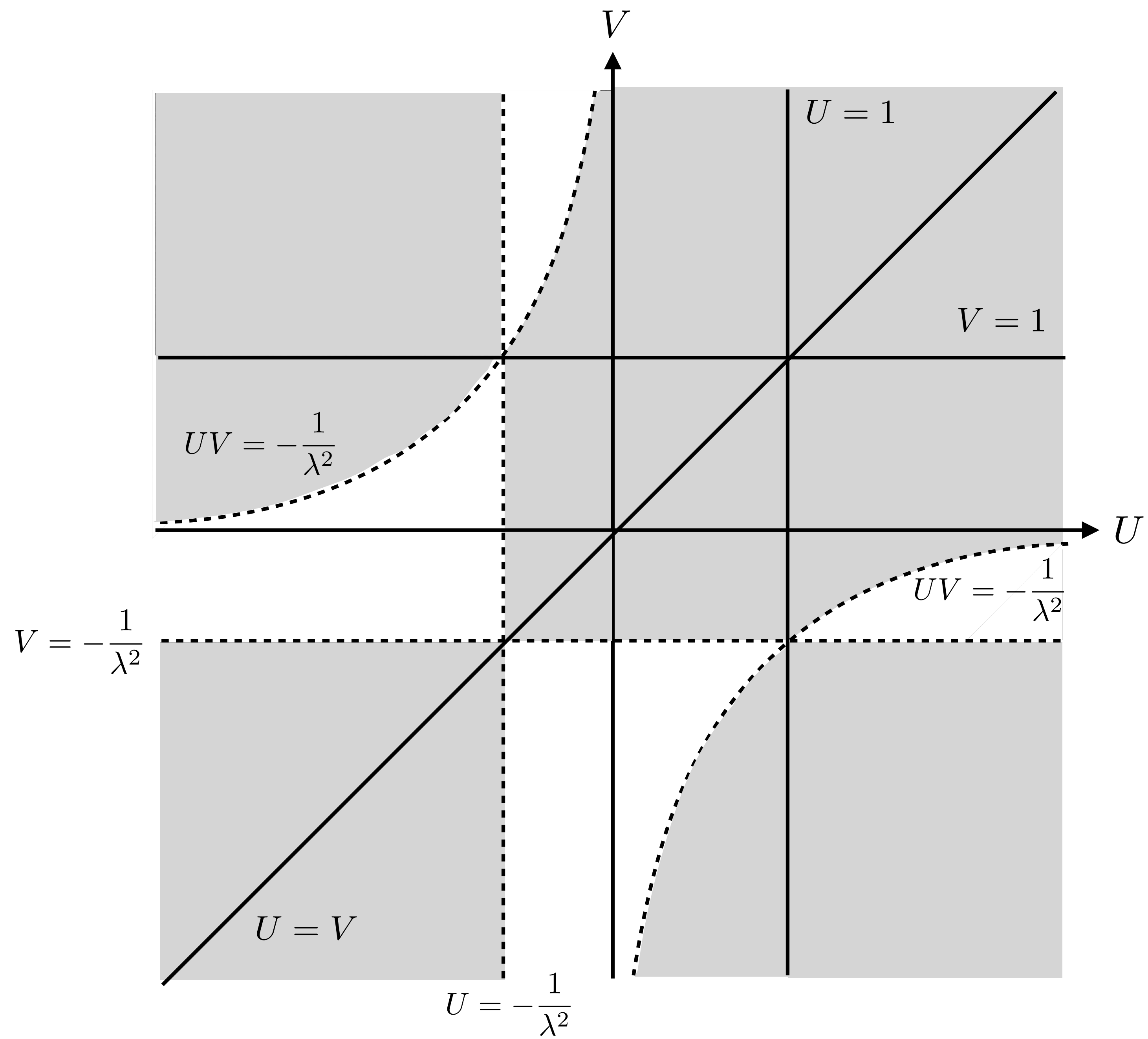
$$\omega_5 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2V^2)(1+\lambda^2U)}{\lambda^2(V-1)(V-U)}.$$

$$\Downarrow$$

$$UV = -\frac{1}{\lambda^2} \qquad (\omega_3 = 0),$$

$$V = -\frac{1}{\lambda^2} \qquad (\omega_4 = 0),$$

$$U = -\frac{1}{\lambda^2} \qquad (\omega_5 = 0),$$



What about **soft limits**?

$$\omega_3 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2)(1+\lambda^2 UV)}{\lambda^2(1-U)(1-V)},$$

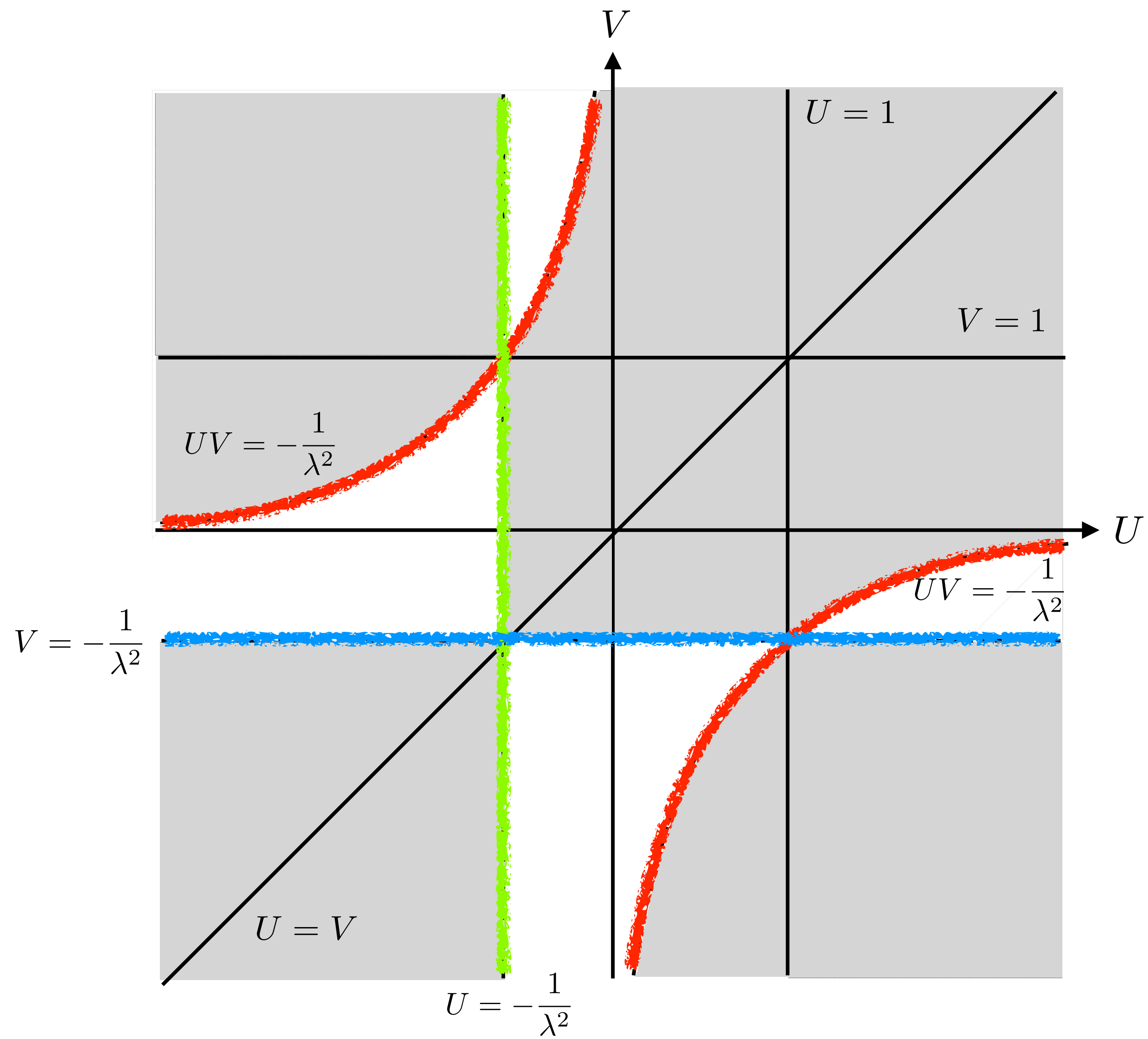
$$\omega_4 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2 U^2)(1+\lambda^2 V)}{\lambda^2(U-1)(U-V)},$$

$$\omega_5 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2 V^2)(1+\lambda^2 U)}{\lambda^2(V-1)(V-U)}.$$

$$UV = -\frac{1}{\lambda^2} \qquad (\omega_3 = 0),$$

$$V = -\frac{1}{\lambda^2} \qquad (\omega_4 = 0),$$

$$U = -\frac{1}{\lambda^2} \qquad (\omega_5 = 0),$$



What about **soft limits**?

$$\omega_3 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2)(1+\lambda^2 UV)}{\lambda^2(1-U)(1-V)},$$

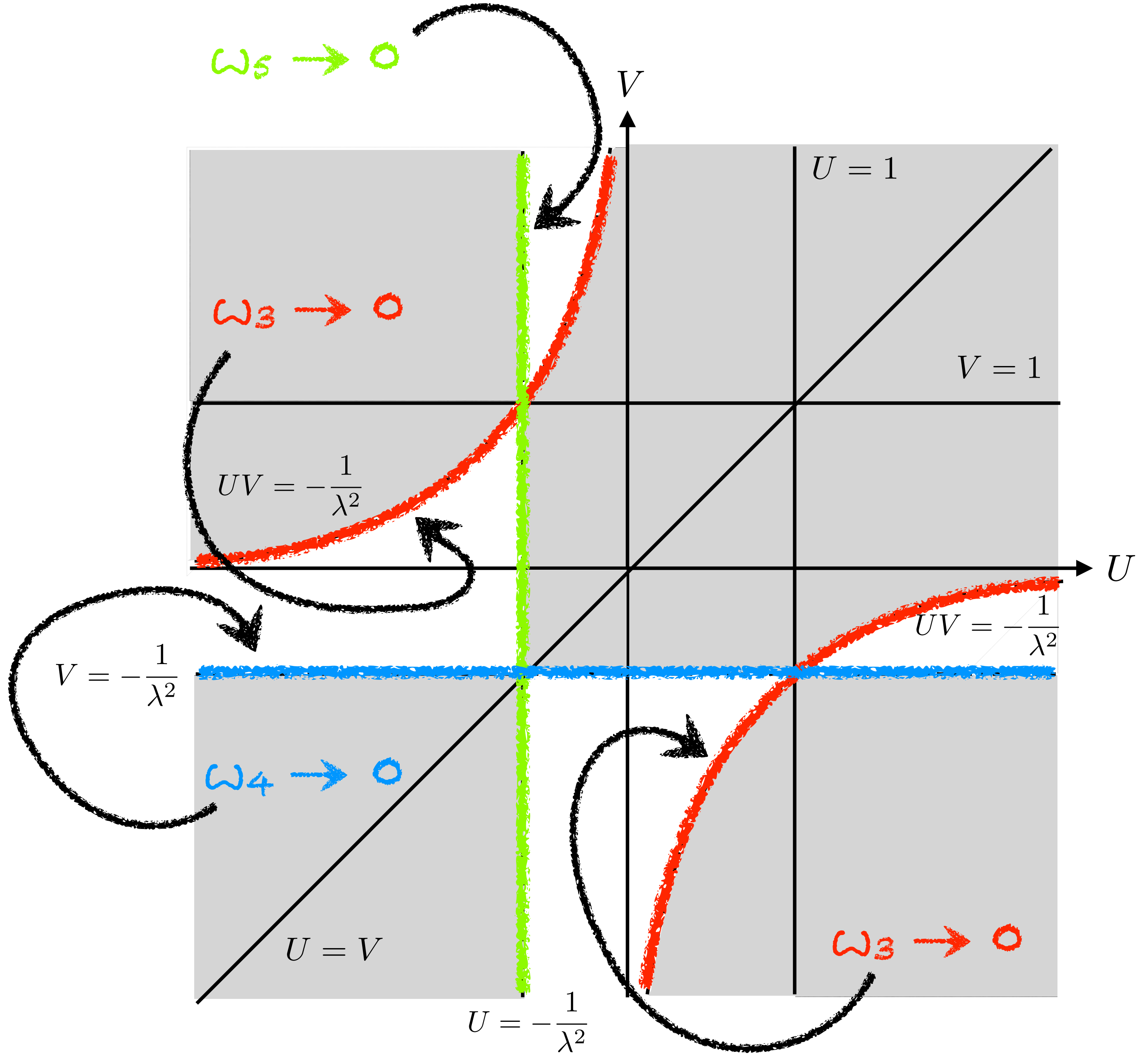
$$\omega_4 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2 U^2)(1+\lambda^2 V)}{\lambda^2(U-1)(U-V)},$$

$$\omega_5 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2 V^2)(1+\lambda^2 U)}{\lambda^2(V-1)(V-U)}.$$

$$UV = -\frac{1}{\lambda^2} \qquad (\omega_3 = 0),$$

$$V = -\frac{1}{\lambda^2} \qquad (\omega_4 = 0),$$

$$U = -\frac{1}{\lambda^2} \qquad (\omega_5 = 0),$$



What about **soft limits**?

$$\omega_3 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2)(1+\lambda^2 UV)}{\lambda^2(1-U)(1-V)},$$

$$\omega_4 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2 U^2)(1+\lambda^2 V)}{\lambda^2(U-1)(U-V)},$$

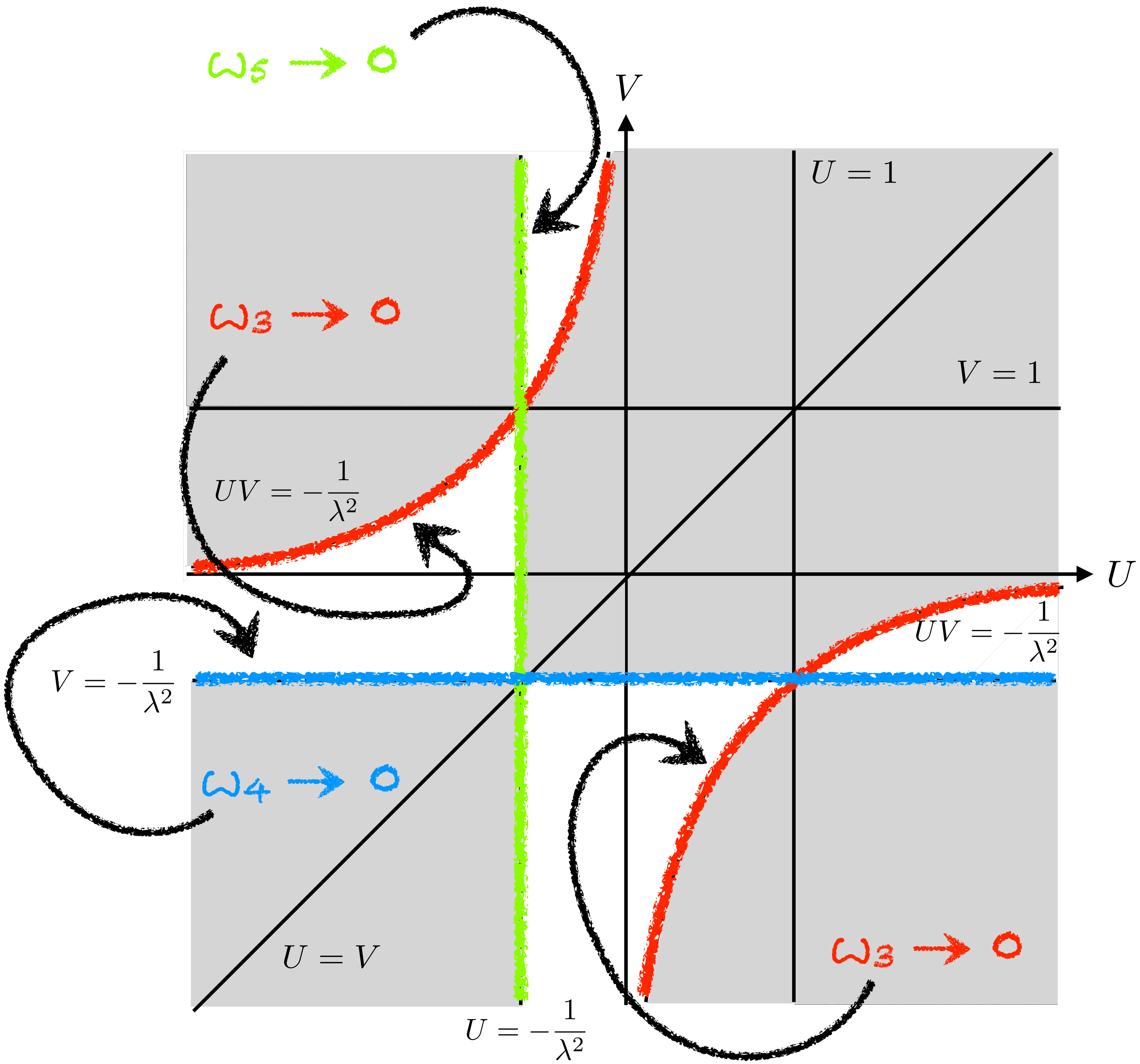
$$\omega_5 = \frac{\sqrt{s}}{2}\frac{(1+\lambda^2 V^2)(1+\lambda^2 U)}{\lambda^2(V-1)(V-U)}.$$

↓

$$UV = -\frac{1}{\lambda^2} \qquad (\omega_3 = 0),$$

$$V = -\frac{1}{\lambda^2} \qquad (\omega_4 = 0),$$

$$U = -\frac{1}{\lambda^2} \qquad (\omega_5 = 0),$$



For appropriate λ, **all** planar zeros can be captured in **a soft limit**.

$$c_7 U - c_8 V - c_6 U^2 + c_{11} V^2 + (c_2 + c_6 - c_7 + c_8 - c_{11} - c_{13}) UV + c_{13} U^2 V - c_2 UV^2 = 0.$$

Planar zeros for **“colourless” incoming gluons**:

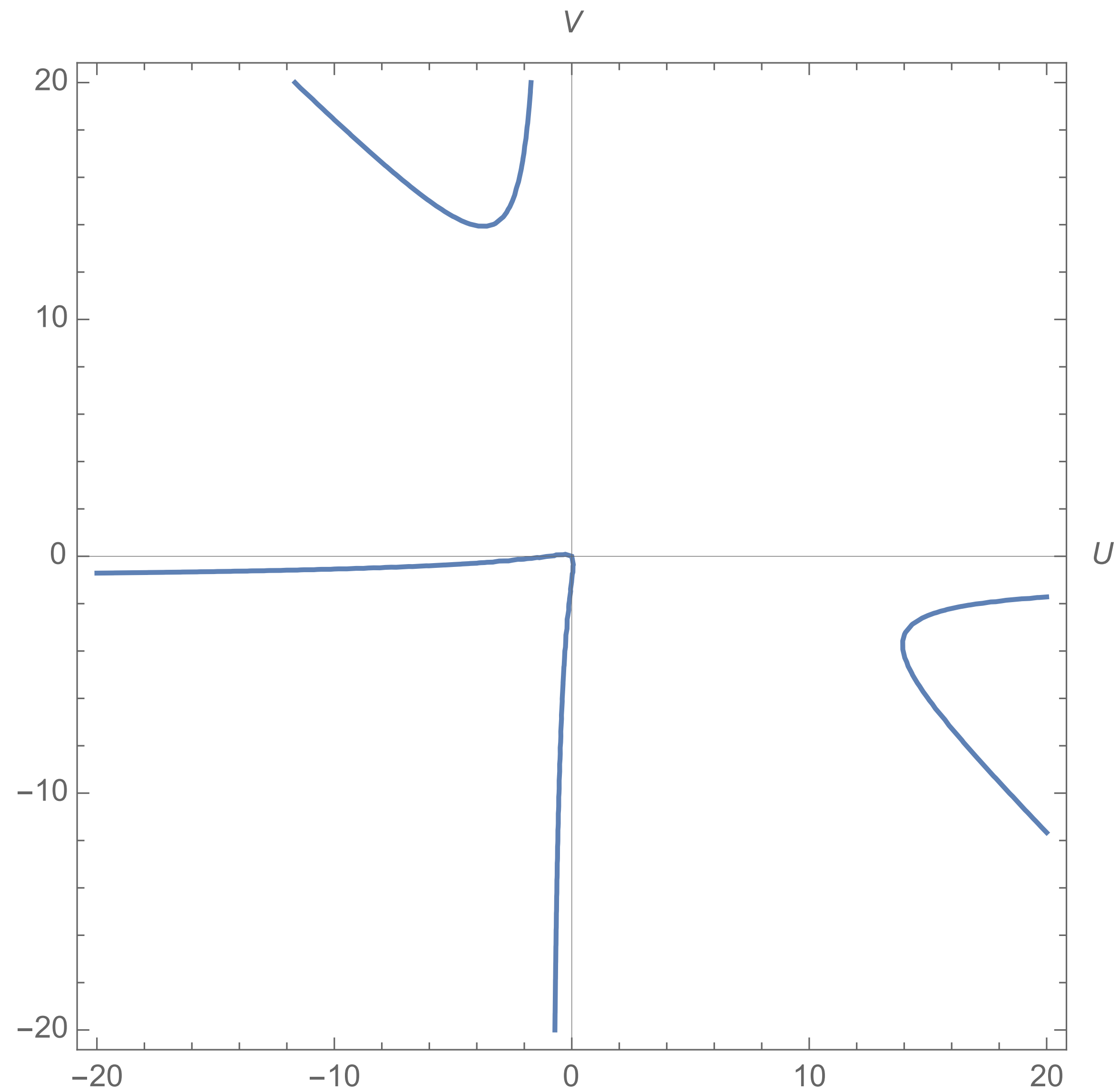
Tracing over incoming color indices,

$$f^{da_3b} f^{ba_4c} f^{ca_5d} \sim f^{a_3a_4a_5}$$

$$c_2 = c_6 = -c_7 = c_8 = -c_{11} = -c_{13} = -f^{a_3a_4a_5}$$

The resulting cubic curve is

$$U + V + U^2 + V^2 - 6UV + U^2V + UV^2 = 0$$



There **exist** “physical” planar zeros.

SU(2): the color factors take the form

$$
\begin{aligned}
c_2 &= \delta^{a_3a_4}\epsilon^{a_2a_5a_1} - \delta^{a_2a_4}\epsilon^{a_3a_5a_1}, \\
c_6 &= \delta^{a_5a_3}\epsilon^{a_2a_4a_1} - \delta^{a_2a_3}\epsilon^{a_5a_4a_1}, \\
c_7 &= \delta^{a_1a_4}\epsilon^{a_3a_5a_2} - \delta^{a_3a_4}\epsilon^{a_1a_5a_2}, \\
c_8 &= \delta^{a_2a_5}\epsilon^{a_4a_1a_3} - \delta^{a_4a_5}\epsilon^{a_2a_1a_3}, \\
c_{11} &= \delta^{a_4a_3}\epsilon^{a_2a_5a_1} - \delta^{a_2a_3}\epsilon^{a_4a_5a_1}, \\
c_{13} &= \delta^{a_4a_5}\epsilon^{a_1a_2a_3} - \delta^{a_1a_5}\epsilon^{a_4a_2a_3},
\end{aligned}
\quad \longrightarrow \quad c_i = 0, \pm 1
$$

There are **no physical planar zeros**.

E.g.:

$$(a_1, a_2, a_3, a_4, a_5) = (2, 3, 1, 1, 1)$$
$$c_2 = c_6 = c_7 = c_8 = c_{11} = c_{13} = 1 \quad \longrightarrow \quad (U-1)(V-1)(U-V) = 0$$

$$(a_1, a_2, a_3, a_4, a_5) = (2, 2, 2, 1, 3)$$
$$c_2 = c_7 = c_8 = c_{13} = 0, \qquad c_6 = -c_{11} = 1 \quad \longrightarrow \quad (U-V)^2 = 0$$

$$(a_1, a_2, a_3, a_4, a_5) = (1, 2, 2, 2, 3)$$
$$c_2 = c_8 = c_{11} = c_{13} = 0, \qquad c_6 = c_7 = 1 \quad \longrightarrow \quad U(U-1) = 0$$

$$\vdots$$

SU(3): for all color configurations, the cubic equation **factorizes**. E.g.,

$$(a_1, a_2, a_3, a_4, a_5) = (7, 7, 6, 1, 5)$$

$$c_2 = -c_7 = c_8 = -c_{13} = 2, \qquad c_6 = -c_{11} = -1$$

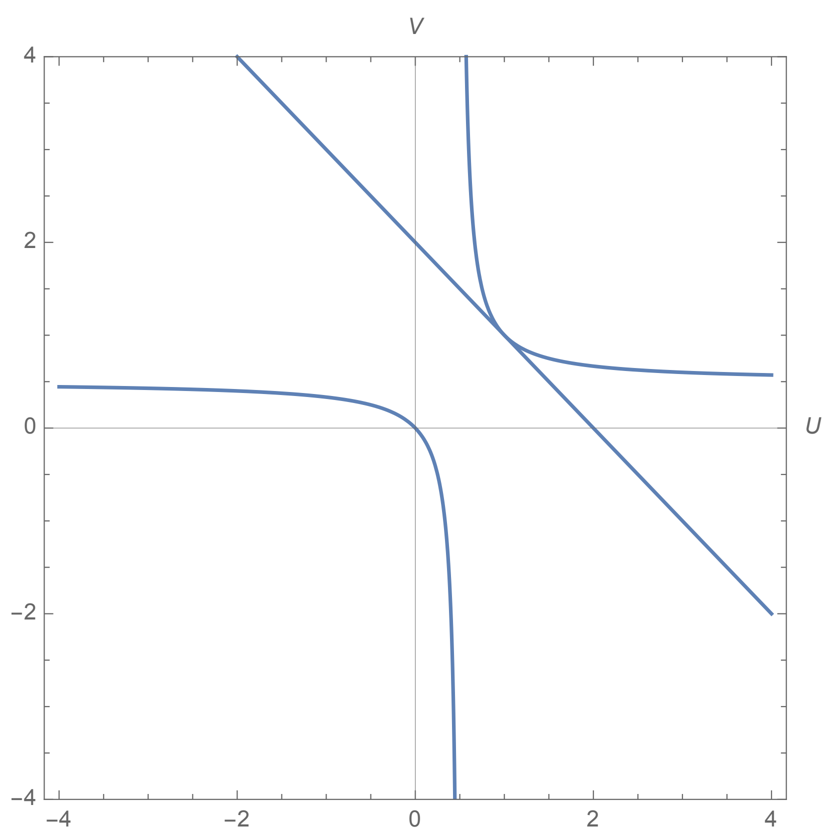
$$\longrightarrow \quad (U + V - 2)(U + V - 2UV) = 0.$$

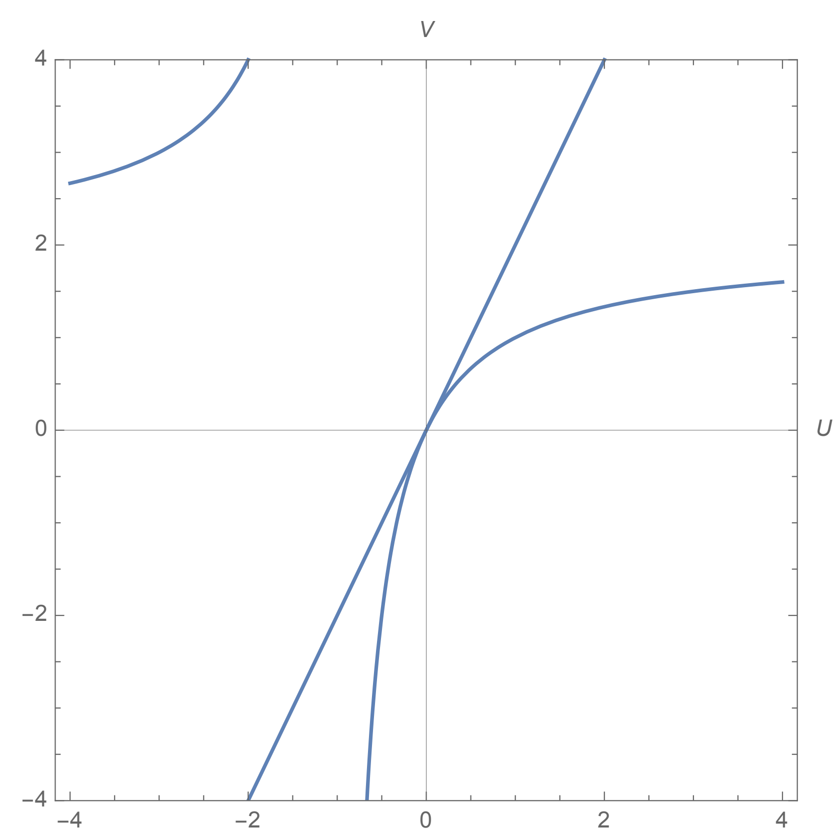
$$(a_1, a_2, a_3, a_4, a_5) = (1, 4, 1, 2, 6)$$

$$c_2 = -c_{11} = -1, \qquad c_6 = -4,$$

$$c_7 = c_8 = 0, \qquad c_{13} = -2$$

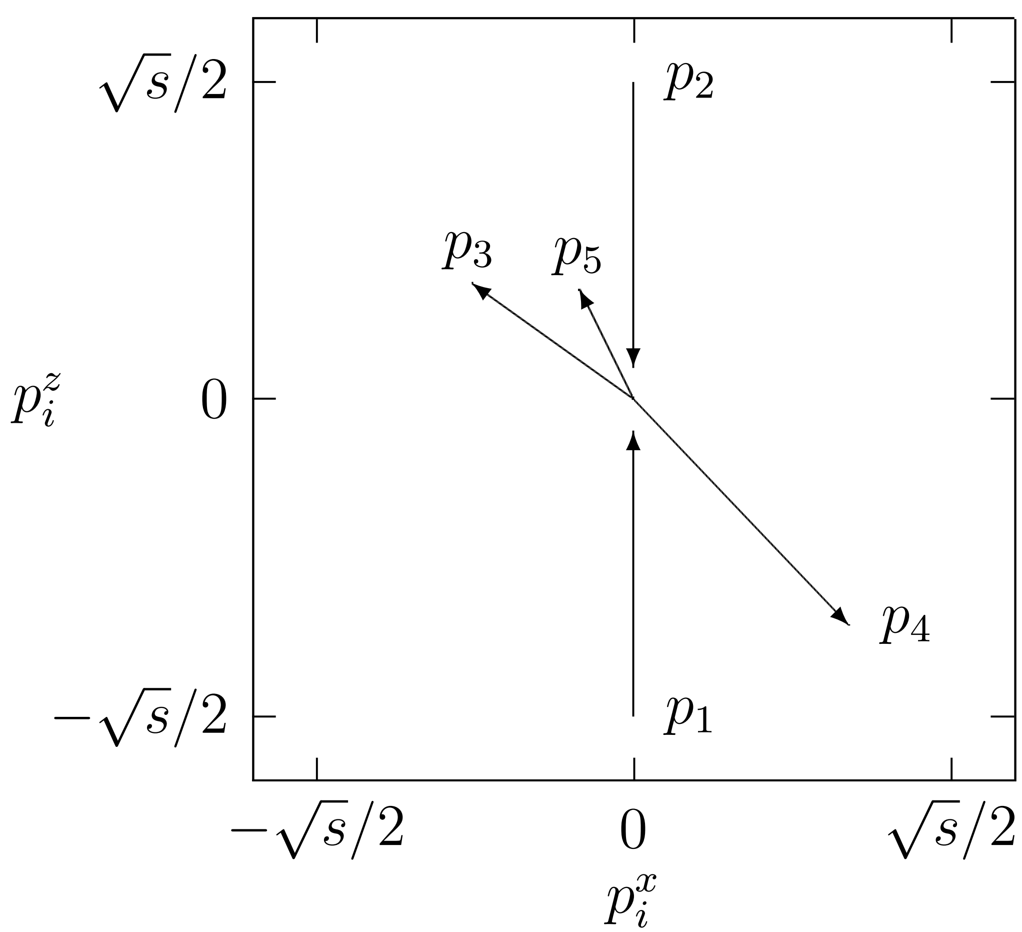
$$\longrightarrow \quad (2U - V)(-2U + V + UV) = 0$$

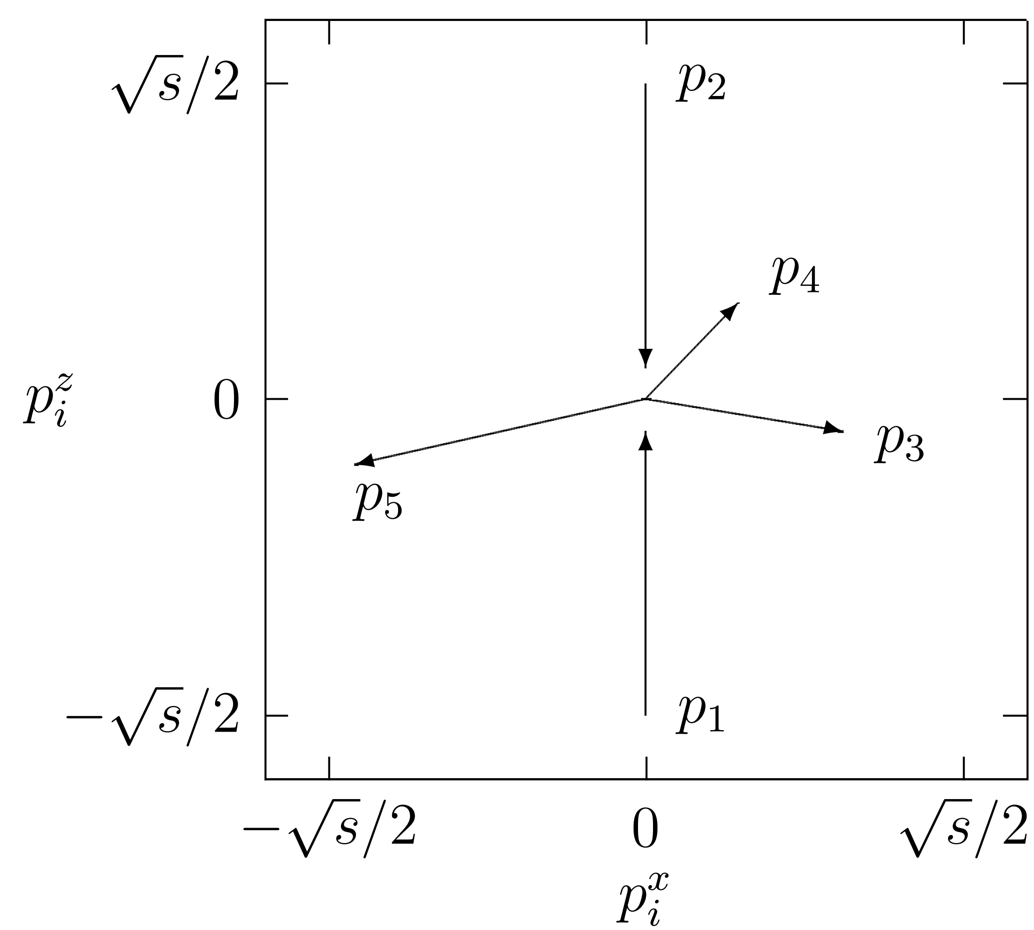




There are physical planar zeros. E.g.,

$$(a_1, a_2, a_3, a_4, a_5) = (7, 7, 6, 1, 5)$$





$$(\zeta_3, \zeta_4, \zeta_5) = (-1.95, 0.4, -4.3) \qquad (\zeta_3, \zeta_4, \zeta_5) = (0.85, 2.5, -0.8)$$

SU(5): in this case there are also **non-factorizable** curves. E.g.,

$$(a_1, a_2, a_3, a_4, a_5) = (19, 18, 23, 17, 19)$$

$$c_2 = c_{11} = 0, \qquad c_6 = c_8 = 2,$$

$$c_7 = c_{13} = 1$$

$\longrightarrow$

$$U - 2U^2 - 2V + 2UV + U^2V = 0$$

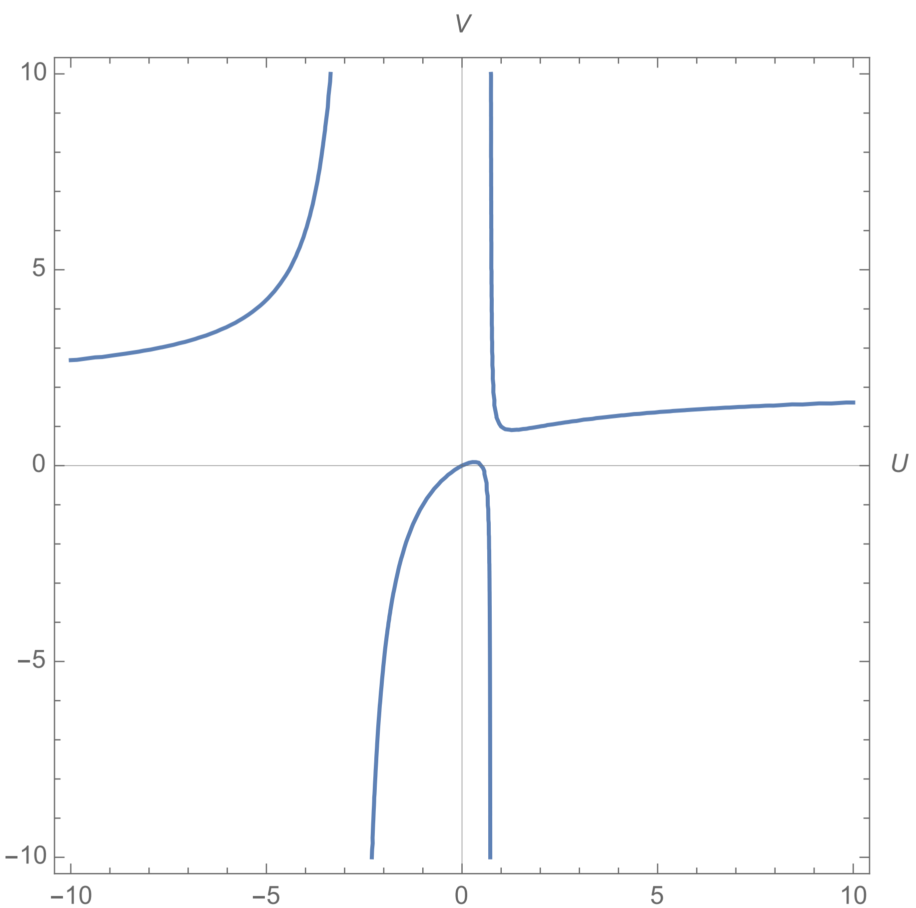
$$(a_1, a_2, a_3, a_4, a_5) = (19, 19, 18, 23, 17)$$

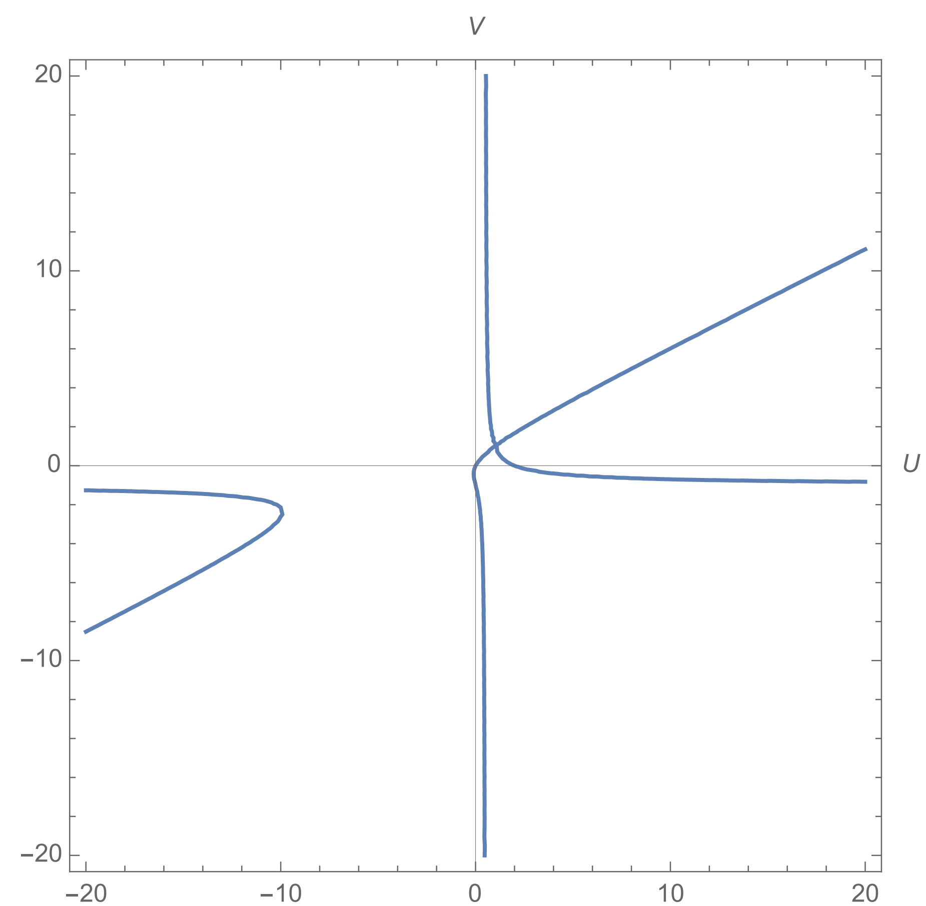
$$c_2 = -c_7 = -2,$$

$$c_6 = c_8 = -c_{11} = -c_{13} = 1$$

$\longrightarrow$

$$2U - U^2 - V - U^2V - V^2 + 2UV^2 = 0$$





We can study the transformation of the curves under **permutations** of the color indices. The space of color structures is

$$\mathrm{TCS}_5 = \mathrm{Lie}((5))$$

(Kol & Shir 2014)

First we look at S_3 acting on the **outgoing** color indices

active | passive

$$S_3 : (a_3, a_4, a_5) \quad \Longleftrightarrow \quad S_3 : (\zeta_3, \zeta_4, \zeta_5)$$

The color factors transform in the six-dimensional, **regular** representation of S_3 acting on $\boldsymbol{C} = (c_2, c_6, c_7, c_8, c_{11}, c_{13})^T$

$$(1)(2)(3) = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad (123) = \begin{pmatrix} 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0 \end{pmatrix}, \quad (132) = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix}, \quad (12)(3) = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix},$$

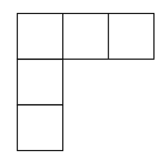
$$(13)(2) = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0 \end{pmatrix}, \quad (1)(23) = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}.$$

S_3 can be decomposed into cosets of S_2

$$S_3 = (1)(2)(3)S_2 \uplus (123)S_2 \uplus (132)S_2$$

interchanges the two coordinates on each patch

permutes the three affine patches

General permutations acts on TCS_5 as  of S_5. For example,

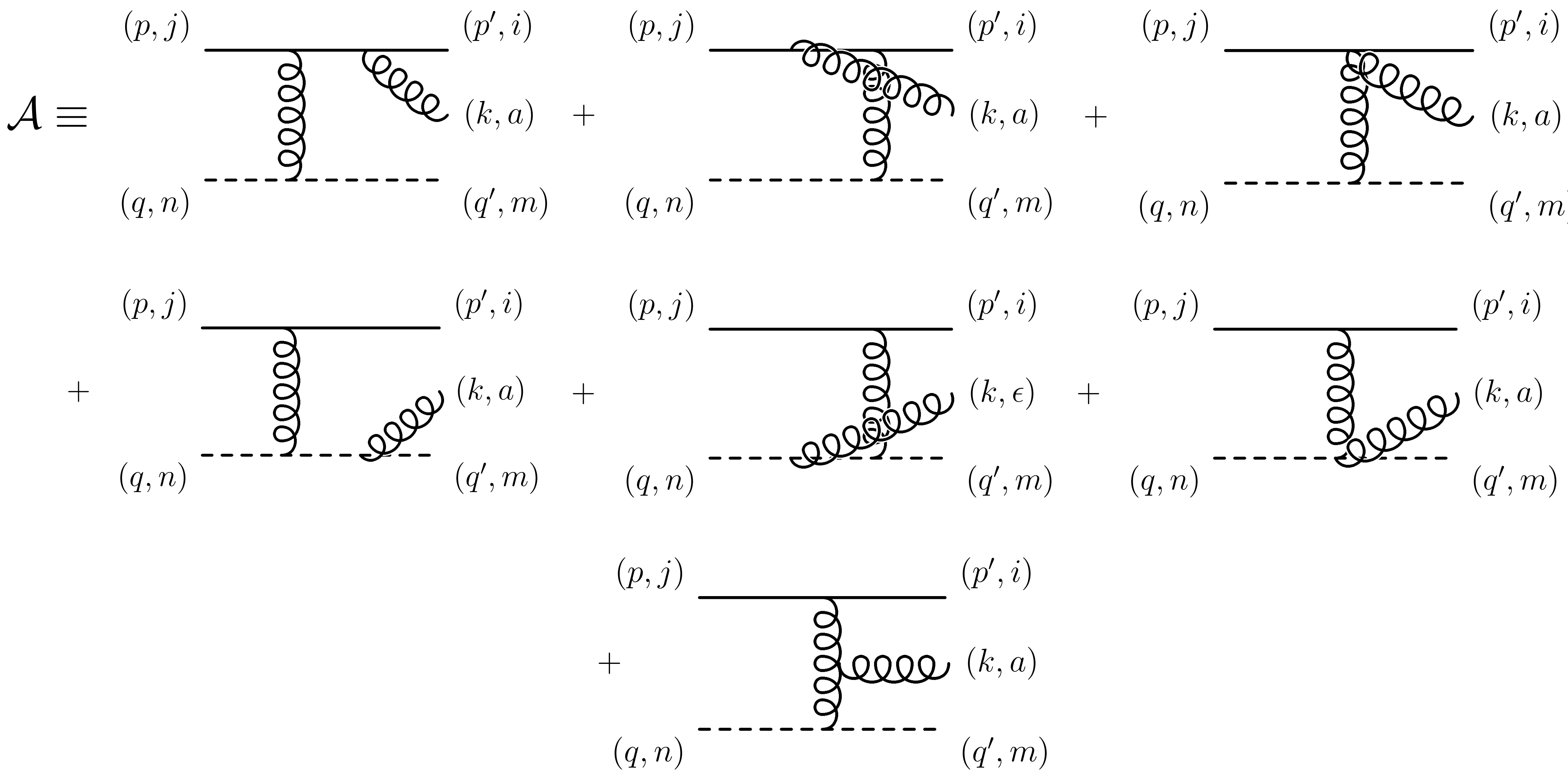
$$(12)(3)(4)(5) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad (134)(25) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}, \quad (1245)(3) = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

This can be used to generate the whole **orbit** from a given color configuration.

What about **scalars**?

$$\Phi(p,j) + \Phi'(q,n) \longrightarrow \Phi(p',i) + \Phi'(q',m) + g(k,a,\epsilon)$$

The diagrams to compute are



The amplitude has the structure

$$\mathcal{A} = g^3 \left(\frac{C_1 n_1}{s_{24}s_{35}} + \frac{C_2 n_2}{s_{24}s_{15}} + \frac{C_3 n_3}{s_{24}} + \frac{C_4 n_4}{s_{13}s_{45}} + \frac{C_5 n_5}{s_{13}s_{25}} + \frac{C_6 n_6}{s_{13}} + \frac{C_7 n_7}{s_{13}s_{24}} \right)$$

where

$$\begin{aligned} C_1 &= T^a_{ik}T^b_{kj}\overline{T}^b_{mn}, \\ C_2 &= T^b_{ik}T^a_{kj}\overline{T}^b_{mn}, \\ C_3 &= T^a_{ik}T^b_{kj}\overline{T}^b_{mn} + T^b_{ik}T^a_{kj}\overline{T}^b_{mn}, \end{aligned}$$

$$\begin{aligned} C_4 &= T^b_{ij}\overline{T}^a_{mk}\overline{T}^b_{kn}. \\ C_5 &= T^b_{ij}\overline{T}^b_{mk}\overline{T}^a_{kn}, \\ C_6 &= T^b_{ij}\overline{T}^a_{mk}\overline{T}^b_{kn} + T^b_{ij}\overline{T}^b_{mk}\overline{T}^a_{kn}, \end{aligned}$$

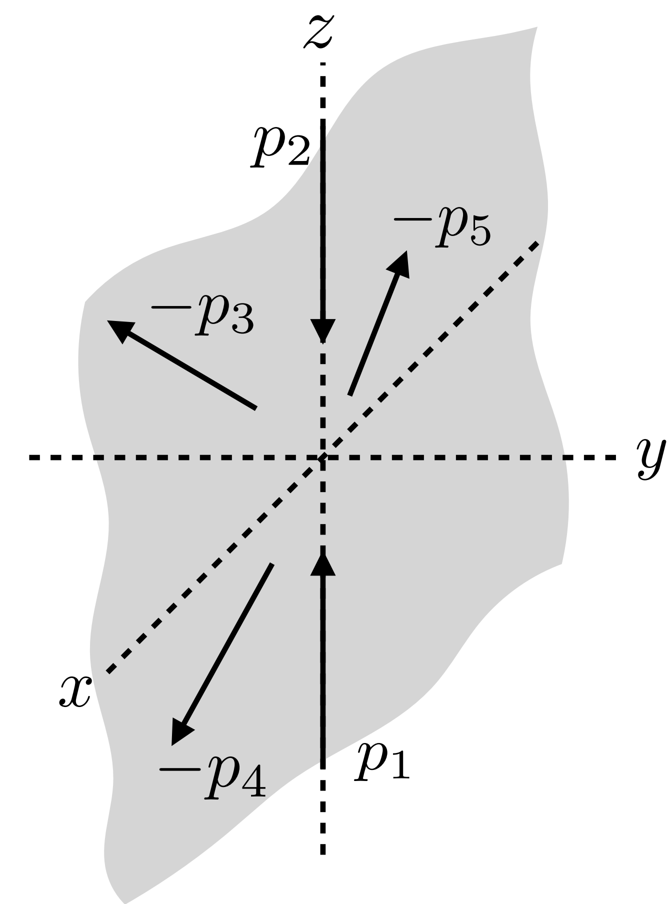
$$C_7 = if^{abc}T^b_{ij}\overline{T}^c_{mn}$$

The color factors satisfy the **Jacobi identities**

$$\begin{aligned} C_1 - C_2 + C_7 &= 0, \\ C_1 + C_2 - C_3 &= 0, \\ C_4 - C_5 - C_7 &= 0, \\ C_4 + C_5 - C_6 &= 0. \end{aligned}$$

→ We use them to eliminate C_3, C_5, C_6, and C_7.

Using the same kinematic setup as for the gluon amplitude and



$$\epsilon_{\pm} = \pm\frac{1}{\sqrt{2}}\left(0, \frac{\zeta_5^2 - 1}{1+\zeta_5^2}, \mp i, -\frac{2\zeta_5}{1+\zeta_5^2}\right)$$

the planar amplitude reads

$$\mathcal{A} = \frac{i\sqrt{2}g^3(2\zeta_3 - \zeta_4)}{\sqrt{s}\zeta_4\zeta_5(1+\zeta_3\zeta_4)}\Big[(C_1 - C_2 + C_4)\zeta_3\zeta_4 - (C_1 - C_2)\zeta_3\zeta_5 + (C_2 - C_4)\zeta_4\zeta_5 - C_2\zeta_5^2\Big]$$

There are two branches of zeros:

$$2\zeta_3 - \zeta_4 = 0$$

$$(C_1 - C_2 + C_4)\zeta_3\zeta_4 - (C_1 - C_2)\zeta_3\zeta_5 + (C_2 - C_4)\zeta_4\zeta_5 - C_2\zeta_5^2 = 0$$

$$(C_1 - C_2 + C_4)\zeta_3\zeta_4 - (C_1 - C_2)\zeta_3\zeta_5 + (C_2 - C_4)\zeta_4\zeta_5 - C_2\zeta_5^2 = 0$$

Choosing the **patch** around $(\zeta_3, \zeta_4, \zeta_5) = (0, 0, 1)$

$$(\zeta_3, \zeta_4, \zeta_5) = \lambda(U, V, 1)$$

the two loci of zeros are

$$2U - V = 0, \qquad \longrightarrow \qquad \text{physical!}$$

$$(C_1 - C_2 + C_4)UV - (C_1 - C_2)U + (C_2 - C_4)V - C_2 = 0.$$

Analyzing the **invariants** of the quadratic curve

$$\Delta = \frac{1}{4}C_1C_4(C_1 - C_2 + C_4), \qquad \delta = -\frac{1}{4}(C_1 - C_2 + C_4)^2, \qquad I = 0,$$

$$\sigma = -\frac{1}{4}(C_1 - C_2)^2 - \frac{1}{4}(C_2 - C_4)^2$$

the only possibilities for **all gauge groups** and **representations** are a hyperbola $(\Delta \neq 0, \delta < 0)$, two parallel lines $(\Delta = 0, \delta < 0)$, and two intersecting lines $(\Delta = \delta = 0, \sigma < 0)$.

The **color-dependent** zeros are fully **captured** in the soft gluon limit:

$$\mathcal{A}_{\rm soft} = 2g\left(C_1\frac{p_3\cdot\epsilon_\pm}{s_{35}} - C_2\frac{p_1\cdot\epsilon_\pm}{s_{15}} + C_4\frac{p_4\cdot\epsilon_\pm}{s_{45}} - C_5\frac{p_2\cdot\epsilon_\pm}{s_{25}}\right)\mathcal{A}_4$$

$$= 2g\left[C_1\left(\frac{p_3\cdot\epsilon_\pm}{s_{35}} - \frac{p_2\cdot\epsilon_\pm}{s_{25}}\right) + C_2\left(\frac{p_2\cdot\epsilon_\pm}{s_{25}} - \frac{p_1\cdot\epsilon_\pm}{s_{15}}\right) + C_4\left(\frac{p_4\cdot\epsilon_\pm}{s_{45}} - \frac{p_2\cdot\epsilon_\pm}{s_{25}}\right)\right]\mathcal{A}_4$$

and in terms of stereographic coordinates

$$\mathcal{A}_{\rm soft} = \mp\frac{g\sqrt{2}}{\sqrt{s}\zeta_5(1+\zeta_3\zeta_4)}\Big[(C_1-C_2+C_4)\zeta_3\zeta_4 - (C_1-C_2)\zeta_3\zeta_5 + C_2 - C_4)\zeta_4\zeta_5 - C_2\zeta_5^2\Big]\mathcal{A}_4$$

$$\Downarrow$$

$$(C_1-C_2+C_4)\zeta_3\zeta_4 - (C_1-C_2)\zeta_3\zeta_5 + (C_2-C_4)\zeta_4\zeta_5 - C_2\zeta_5^2 = 0$$

The "trivial" branch is **far away** from the soft gluon limit $\omega_5 \to 0$

$$1+\zeta_3\zeta_4 \longrightarrow 0 \qquad \Longrightarrow \qquad 2\zeta_3 - \zeta_4 \to \frac{2\zeta_3^2+1}{\zeta_3} \neq 0$$

Finally, we look at planar zeros in **gravity** studying the **five-graviton** tree-level amplitude.

To compute it we use the **BCJ double copy prescription**

(Bern, Carrasco & Johansson 2008)

$$\mathcal{A}_n^{\rm tree} = g^{n-2}\sum_{i\in\Gamma}\frac{c_i n_i}{\prod_\alpha s_{i,\alpha}} \quad \xrightarrow{c_i \to n_i'} \quad \mathcal{M}_n^{\rm tree} = i\left(\frac{\kappa}{2}\right)^{n-2}\sum_{i\in\Gamma}\frac{n_i' n_i}{\prod_\alpha s_{\alpha,i}}$$

provided n_i' satisfies **color-kinematics duality**

$$c_i + c_j - c_k = 0 \quad \longrightarrow \quad n_i' + n_j' - n_k' = 0$$

Exploiting our result for the five-gluon scattering

$$-i\mathcal{M}_5 = \left(\frac{\kappa}{2}\right)^3\left(\frac{n_1^2}{s_{12}s_{45}} + \frac{n_2^2}{s_{23}s_{15}} + \frac{n_3^2}{s_{34}s_{12}} + \frac{n_4^2}{s_{45}s_{23}} + \frac{n_5^2}{s_{15}s_{34}} + \frac{n_6^2}{s_{14}s_{25}} + \frac{n_7^2}{s_{13}s_{25}} + \frac{n_8^2}{s_{24}s_{13}}\right.$$

$$\left. + \frac{n_9^2}{s_{35}s_{24}} + \frac{n_{10}^2}{s_{14}s_{35}} + \frac{n_{11}^2}{s_{15}s_{24}} + \frac{n_{12}^2}{s_{12}s_{35}} + \frac{n_{13}^2}{s_{23}s_{14}} + \frac{n_{14}^2}{s_{25}s_{34}} + \frac{n_{15}^2}{s_{13}s_{45}}\right)$$

$$-i\mathcal{M}_5 = \left(\frac{\kappa}{2}\right)^3 \left(\frac{n_1^2}{s_{12}s_{45}} + \frac{n_2^2}{s_{23}s_{15}} + \frac{n_3^2}{s_{34}s_{12}} + \frac{n_4^2}{s_{45}s_{23}} + \frac{n_5^2}{s_{15}s_{34}} + \frac{n_6^2}{s_{14}s_{25}} + \frac{n_7^2}{s_{13}s_{25}} + \frac{n_8^2}{s_{24}s_{13}} \right.$$
$$\left. + \frac{n_9^2}{s_{35}s_{24}} + \frac{n_{10}^2}{s_{14}s_{35}} + \frac{n_{11}^2}{s_{15}s_{24}} + \frac{n_{12}^2}{s_{12}s_{35}} + \frac{n_{13}^2}{s_{23}s_{14}} + \frac{n_{14}^2}{s_{25}s_{34}} + \frac{n_{15}^2}{s_{13}s_{45}}\right)$$

Substituting the computed **numerators**

$$-i\mathcal{M}_5 = -i\left(\frac{\kappa}{2}\right)^3 \langle 12\rangle^3 \left(\frac{n_2}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} + \frac{n_6}{\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle} + \frac{n_7}{\langle 25\rangle\langle 53\rangle\langle 43\rangle\langle 31\rangle} \right.$$
$$\left. + \frac{n_8}{\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle} + \frac{n_{11}}{\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle} + \frac{n_{13}}{\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle}\right)$$

We compute the numerators using stereographic coordinates and in the planar case (i.e., $\zeta_a \in \mathbb{R}$)

$$n_6 = n_7 = is^{\frac{3}{2}}\frac{(\zeta_3-\zeta_5)\zeta_5}{\zeta_3(1+\zeta_4\zeta_5)}, \qquad n_8 = is^{\frac{3}{2}}\frac{(\zeta_3-\zeta_5)\zeta_4}{\zeta_3(1+\zeta_4\zeta_5)}, \qquad n_2 = n_{11} = n_{13} = 0.$$

$$-i\mathcal{M}_5 = \left(\frac{\kappa}{2}\right)^3 \left(\frac{n_1^2}{s_{12}s_{45}} + \frac{n_2^2}{s_{23}s_{15}} + \frac{n_3^2}{s_{34}s_{12}} + \frac{n_4^2}{s_{45}s_{23}} + \frac{n_5^2}{s_{15}s_{34}} + \frac{n_6^2}{s_{14}s_{25}} + \frac{n_7^2}{s_{13}s_{25}} + \frac{n_8^2}{s_{24}s_{13}}\right.$$

$$+ \frac{n_9^2}{s_{35}s_{24}} + \frac{n_{10}^2}{s_{14}s_{35}} + \frac{n_{11}^2}{s_{15}s_{24}} + \frac{n_{12}^2}{s_{12}\ldots}$$

Substituting the computed **numer**

$$-i\mathcal{M}_5 = -i\left(\frac{\kappa}{2}\right)^3 \langle 12\rangle^3 \left(\frac{n_2}{\langle 23\rangle\langle 34\rangle\langle 4\ldots}\right.$$

$$+ \frac{n_8}{\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle} + \frac{n\ldots}{\langle 24\rangle\langle 43\rangle\ldots}$$

$$n_1 = -n_{12} = n_{15} = i\frac{\langle 12\rangle^4[21][54]}{\langle 23\rangle\langle 34\rangle\langle 51\rangle},$$

$$n_6 = n_7 = n_{10} = i\frac{\langle 12\rangle^4[14][52]}{\langle 23\rangle\langle 34\rangle\langle 51\rangle},$$

$$n_8 = n_9 = i\frac{\langle 12\rangle^4[24][51]}{\langle 23\rangle\langle 34\rangle\langle 51\rangle},$$

$$n_2 = n_3 = n_4 = n_5 = n_{11} = n_{13} = n_{14} = 0$$

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$$\left. + \frac{n_9^2}{s_{35}s_{24}} + \frac{n_{10}^2}{s_{14}s_{35}} + \frac{n_{11}^2}{s_{15}s_{24}} + \frac{n_{12}^2}{s_{12}s_{35}} + \frac{n_{13}^2}{s_{23}s_{14}} + \frac{n_{14}^2}{s_{25}s_{34}} + \frac{n_{15}^2}{s_{13}s_{45}}\right)$$

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$$\left. + \frac{n_8}{\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle} + \frac{n_{11}}{\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle} + \frac{n_{13}}{\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle}\right)$$

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The gravitational planar amplitude then gives

$$-i\mathcal{M}_5 = \frac{2i}{\sqrt{s}}\left(\frac{\kappa}{2}\right)^2 \frac{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)}{(1+\zeta_3\zeta_4)(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)} \left(-n_2\frac{\zeta_5-\zeta_3}{\zeta_3} - n_6\frac{\zeta_4-\zeta_5}{\zeta_5}\right.$$
$$\left. + n_7\frac{\zeta_3-\zeta_5}{\zeta_5} - n_8\frac{\zeta_3-\zeta_4}{\zeta_4} + n_{11}\frac{\zeta_5-\zeta_4}{\zeta_4} + n_{13}\frac{\zeta_4-\zeta_3}{\zeta_3}\right)$$
$$= -2s\left(\frac{\kappa}{2}\right)^2 \frac{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)^2(\zeta_4-\zeta_5)}{\zeta_3(1+\zeta_3\zeta_4)(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)^2}\Big(-\zeta_4+\zeta_5+\zeta_3-\zeta_5-\zeta_3+\zeta_4\Big) = 0$$

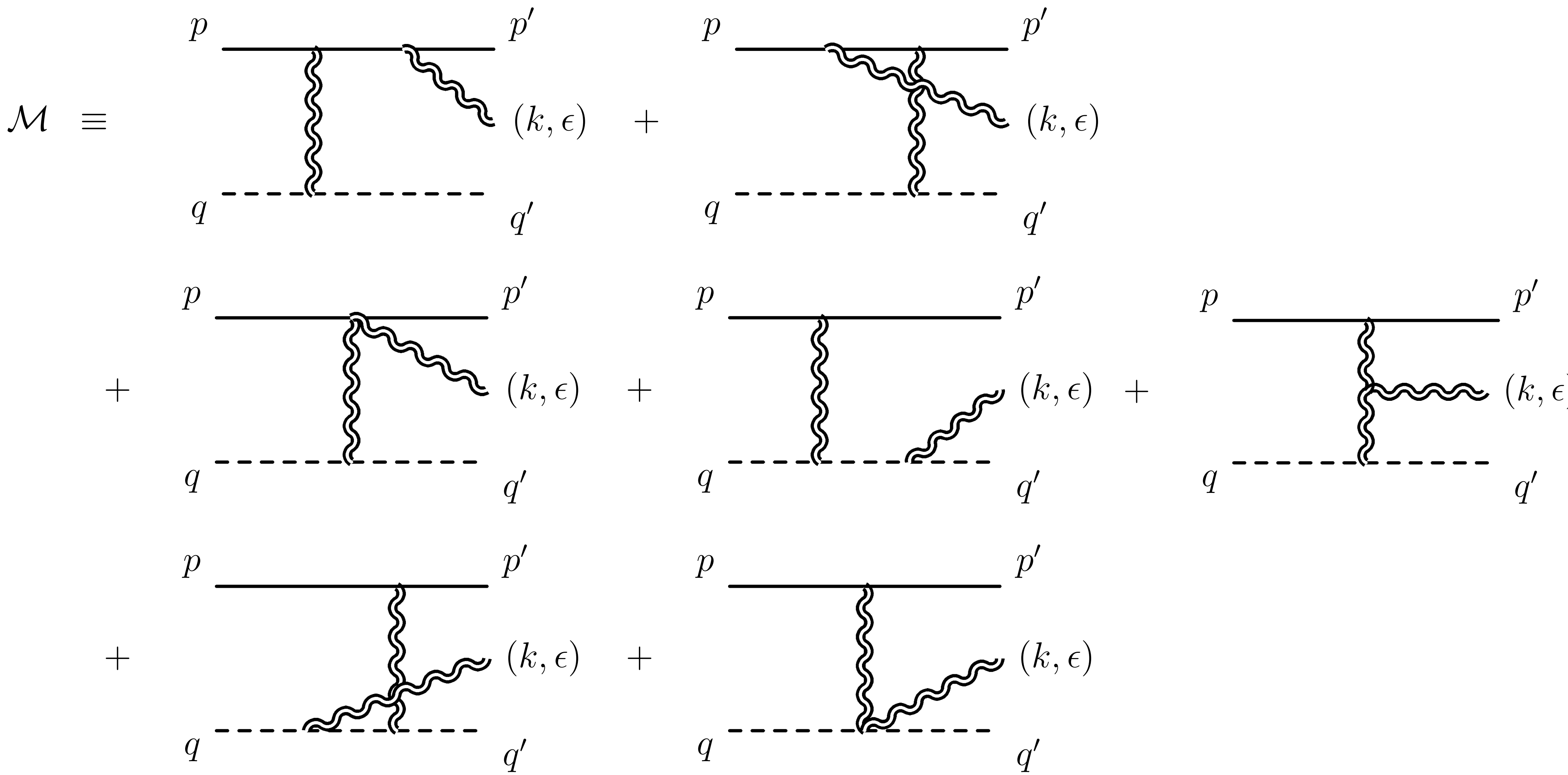
$$\Downarrow$$

$$\mathcal{M}_5\Big|_{\text{planar}} = 0$$

The five-gluon scattering amplitude is **trivial** in the **planar limit**.

Something similar happens for **scalar gravitational scattering**

$$\Phi(p) + \Phi'(q) \longrightarrow \Phi(p') + \Phi'(q') + G(k,\varepsilon)$$



$$\mathcal{M}\Big|_{\text{planar}} = 0$$

Outlook

- For **gauge theories** (and **scalar gauge** theories), planar zeros are determined by **projective** properties of the amplitude.

  For non-gauge scalar theories (e.g., massless φ^3 theories) the planar zeros are **not** determined by **homogeneous** polynomials.

- Since planar zeros are captured in the **soft limit**, they might play a role for the **asymptotic symmetries** of gauge theories.

- Planar zeros **are corrected** by string α'**-effects**:

$$\mathcal{A}_5 = \frac{i(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)}{\sqrt{s}\zeta_3\zeta_4\zeta_5(1+\zeta_3\zeta_4)(1+\zeta_3\zeta_5)(1+\zeta_4\zeta_5)}$$

$$\times\left[\mathcal{A}_5^{(0)} + \frac{\alpha'^2\zeta(2)s^2\mathcal{A}_5^{(2)}}{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)} + \frac{\alpha'^3\zeta(3)s^3\mathcal{A}_5^{(3)}}{(\zeta_3-\zeta_4)^2(\zeta_3-\zeta_5)^2(\zeta_4-\zeta_5)^2} + \mathcal{O}(\alpha'^4)\right]$$

Outlook

- For
  dete

$$
\begin{aligned}
P_{10}(\zeta_3,\zeta_4,\zeta_5) = {} & \zeta_3^6\zeta_4^3\zeta_5 - \zeta_3^6\zeta_4^2\zeta_5^2 + \zeta_3^6\zeta_4\zeta_5^3 - \zeta_3^5\zeta_4^4\zeta_5 - \zeta_3^5\zeta_4^3\zeta_5^2 + \zeta_3^5\zeta_4^3 \\
& - \zeta_3^5\zeta_4^2\zeta_5^3 - \zeta_3^5\zeta_4^2\zeta_5 - \zeta_3^5\zeta_4\zeta_5^4 - \zeta_3^5\zeta_4\zeta_5^2 + \zeta_3^5\zeta_5^3 - \zeta_3^4\zeta_4^5\zeta_5 \\
& + 4\zeta_3^4\zeta_4^4\zeta_5^2 - \zeta_3^4\zeta_4^4 - \zeta_3^4\zeta_4^3\zeta_5^3 - \zeta_3^4\zeta_4^3\zeta_5 + 4\zeta_3^4\zeta_4^2\zeta_5^4 + 4\zeta_3^4\zeta_4^2\zeta_5^2 \\
& - \zeta_3^4\zeta_4\zeta_5^5 - \zeta_3^4\zeta_4\zeta_5^3 - \zeta_3^4\zeta_5^4 + \zeta_3^3\zeta_4^6\zeta_5 - \zeta_3^3\zeta_4^5\zeta_5^2 + \zeta_3^3\zeta_4^5 \\
& - \zeta_3^3\zeta_4^4\zeta_5^3 - \zeta_3^3\zeta_4^4\zeta_5 - \zeta_3^3\zeta_4^3\zeta_5^4 - \zeta_3^3\zeta_4^3\zeta_5^2 - \zeta_3^3\zeta_4^2\zeta_5^5 - \zeta_3^3\zeta_4^2\zeta_5^3 \\
& + \zeta_3^3\zeta_4\zeta_5^6 - \zeta_3^3\zeta_4\zeta_5^4 + \zeta_3^3\zeta_5^5 - \zeta_3^2\zeta_4^6\zeta_5^2 - \zeta_3^2\zeta_4^5\zeta_5^3 - \zeta_3^2\zeta_4^5\zeta_5 \\
& + 4\zeta_3^2\zeta_4^4\zeta_5^4 + 4\zeta_3^2\zeta_4^4\zeta_5^2 - \zeta_3^2\zeta_4^3\zeta_5^5 - \zeta_3^2\zeta_4^3\zeta_5^3 - \zeta_3^2\zeta_4^2\zeta_5^6 \\
& + 4\zeta_3^2\zeta_4^2\zeta_5^4 - \zeta_3^2\zeta_4\zeta_5^5 + \zeta_3\zeta_4^6\zeta_5^3 - \zeta_3\zeta_4^5\zeta_5^4 - \zeta_3\zeta_4^5\zeta_5^2 - \zeta_3\zeta_4^4\zeta_5^5 \\
& - \zeta_3\zeta_4^4\zeta_5^3 + \zeta_3\zeta_4^3\zeta_5^6 - \zeta_3\zeta_4^3\zeta_5^4 - \zeta_3\zeta_4^2\zeta_5^5 + \zeta_4^5\zeta_5^3 - \zeta_4^4\zeta_5^4 + \zeta_4^3\zeta_5^5
\end{aligned}
$$

- Since
  the a

- Plana

$\mathcal{A}$

$$
\times\left[\mathcal{A}_5^{(0)} + \frac{\alpha'^2\zeta(2)s^2\mathcal{A}_5^{(2)}}{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)} + \frac{\alpha'^3\zeta(3)s^3\mathcal{A}_5^{(3)}}{(\zeta_3-\zeta_4)^2(\zeta_3-\zeta_5)^2(\zeta_4-\zeta_5)^2} + \mathcal{O}(\alpha'^4)\right]
$$

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$$\times\left[\mathcal{A}_5^{(0)} + \frac{\alpha'^2\zeta(2)s^2\mathcal{A}_5^{(2)}}{(\zeta_3-\zeta_4)(\zeta_3-\zeta_5)(\zeta_4-\zeta_5)} + \frac{\alpha'^3\zeta(3)s^3\mathcal{A}_5^{(3)}}{(\zeta_3-\zeta_4)^2(\zeta_3-\zeta_5)^2(\zeta_4-\zeta_5)^2} + \mathcal{O}(\alpha'^4)\right]$$

Thank you