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Modified Supergravity and Dynamical Dark Energy

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- Motivation: Dark Energy as the geometrical (curvature) effect
- Main challenges, prejudices and lessons [// Break!]
- Review of modified $f(R)$ gravity
- New: Modified $F(\mathcal{R})$ supergravity
- Massive gravitino as CDM particle in $F(\mathcal{R})$ supergravity
- Connections to String Theory and Loop Quantum Gravity
- Conclusion and Discussion

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Motivation

- Studies of large-scale distribution and evolution of galaxies and (distant) type-Ia supernovae (Perlmutter at al, Schmidt et al, 1998), have led to a discovery of DARK MATTER and DARK ENERGY in the present Universe, with

$$\rho_{total} = \rho_{\text{visible}} + \rho_{\text{dark matter}} + \rho_{\text{dark energy}}$$

in the proportion $4\% + 22\% + 74\% = 100\%$, respectively.

- The DE is needed to balance the energy budget of the present Universe and explain the accelerated rate of its expansion. The DE works against gravity to boost the expansion of the Universe. A small positive cosmological constant $\Lambda > 0$ may account for the present Universe accelerated expansion, due to the (experimentally dictated) DE-equation of state with $w = P/\rho = -0.97 \pm 0.07$

- The DM plays the key role in the formation of structure in our Universe and holds the clusters and galaxies together. It does not interact electromagnetically but it does interact gravitationally. The DM should be Cold and non-baryonic. Possible candidates for the massive CDM particle include axion, gravitino and neutralino (= WIMP in MSSM).

Experimental evidence

The experimental evidence for DM and DE comes from at least 3 independent sources:

- type Ia supernovae observations (S. Perlmutter, UC Berkeley, and B. Schmidt, ANU Canberra, 1998),
- precision measurements of CMB temperature fluctuations (BOOMERANG, MAXIMA, WMAP, 2000 and 2003)
- baryonic acoustic oscillations (Sloan Digital Skies Survey, 2005)

see Turner map (well known).

DE and DM in Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi\, G_N \left(T_{\mu\nu}^{\text{visible}} + T_{\mu\nu}^{\text{DM}} \right)$$

or

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi\, G_N \left(T_{\mu\nu}^{\text{visible}} + T_{\mu\nu}^{\text{DM}} + T_{\mu\nu}^{\text{DE}} \right)$$

- The present evolution of the Universe is phenomenologically well (and accurately) described by the standard Λ-CDM scenario within Einstein theory,

but

- exact nature of both DE and DM is unknown,
- an underying microscopic derivation is absent.

Modified Gravity and Dynamical Dark Energy

The late time acceleration of our Universe can be interpreted in two qualitatively different ways: either as

(i) a manifestation of some (new) DE component (dynamical or not), as above

or as

(ii) an indication of the breakdown of Einstein gravity at cosmological distances.

The known (popular) modifications of Einstein gravity include

- adding higher-order curvature invariants into the action,
- adding extra fields with non-minimal coupling to gravity,
- adding large extra dimensions for gravity.

Dynamical DE = quintessence, ie. replacing a cosmological constant Λ by a scalar field (or several scalars). It leads to a time- and space-dependent vacuum energy density in cosmology, which is desirable for describing a more accurate and complete history of our Universe, including inflation, etc. (perhaps, with the vanishing cosmological constant!)

Major Challenges

DE and DM are not the only challenges facing modern cosmology. In addition, one has (the 'old' ones) to

- resolve the classical cosmological singularity (Big Bang) in the Einstein-Friedmann Universe, which is accompanied by infinities in ρ and R,
- construct a consistent fundamental theory of Quantum Gravity,
- find the viable inflationary mechanism for the early Universe,
- explain the 'observed' small value of the cosmological constant Λ,

in a single package with a fundamental QFT of elementary particles beyond SM.

We would like to keep the most fundamental features of General Relativity such as (i) its diffeomorphism invariance (relativity principle), and (ii) its universality (equivalence principle), cf. Horava-Lifshitz gravity.
However, there is *a priori* no reason to restrict the gravitational Lagrangian to the EH-term linear in curvature, unless it does not contradict an experiment. The first attempt of this kind was as early as 1921 (Weyl). The main challenge here: there exist too many possibilities beyond EH.

Some prejudices

There is no doubt that any theory of Quantum Gravity is going to include the higher-order curvature terms in the UV. Those terms are expected to be relevant near curvature singularities. It may be possible that some higher-derivative gravity, subject to suitable constraints, could be the effective action to quantized theory of gravity (Sakharov, 1967), cf. String Theory.

- Objection #1: "all the higher-derivative field theories, including the higher-derivative gravity theories, have ghosts", because of Ostrogradski theorem (1850). However, the theorem does not directly apply to the degenerate (read: gauge) field theories (Woodard, 2007), and, in fact, a higher-derivative gravity does not always have ghosts (see below for some explicit examples).
- Objection #2: "all the higher-order curvature terms are suppressed by the inverse powers of M_{Pl} and thus are irrelevant in IR". However, the effective Planck scale may be brought down to TeV scale with large extra dimensions (A-HDD), there may be warp (RS) factors (eg., generated by fluxes in string theory) in a higher-dimensional metric, and there may be singular perturbations in unstable cases (see the next slide:)

Lesson from Navier-Stokes hydrodynamics

The Navier-Stokes differential equations describe classical dynamics of a non-ideal fluid with viscosity,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} P + \nu \Delta \vec{v}$$

The higher derivatives enter this equation with the kinematic viscosity parameter ν. The singular perturbation problem associated with the Navier-Stokes equation is well known both physically and mathematically: namely, a radically different behaviour of the solutions at late times, even for arbitrary small values ν of viscosity. The simplest mathematical example is given by the equation

$$-\varepsilon \ddot{y} + \dot{y} + y = 0$$

When $\varepsilon = 0$, one gets exponentially decaying solutions $y \propto e^{-t}$, whereas for small $\varepsilon > 0$, the general solution grows exponentially with time as $e^{t/\varepsilon}$. And it is not the manifestation of bad modelling or unphysical boundary conditions! One may now imagine the late time (present) Universe acceleration driven by quantum modifications of Einstein equations!

Some lessons from superstrings

- Theory of superstrings is the leading candidate for a unified Theory of all fundamental interactions, it offers a consistent perturbative Quantum Gravity, and is capable to generate the SM of elementary particles and the cosmological SM. A fully non-perturbative superstring theory (or M-theory) is unknown. The perturbative superstrings are defined on-shell (in the form of quantum amplitudes), and they imply infinitely many higher-order curvature corrections to Einstein equations, to all orders in Regge slope parameter α' and string coupling g_s (in 10 space-time dimensions). Off-shell quantum corrections are largely unknown and ambiguous.
- String theory may resolve the Big-Bang singularity (see eg., pre-Big-Bang scenario of Gasperini-Vaneziano, string gas cosmology of Brandenberger-Vafa). String theory put limits on the maximal (Hagedorn) temperature $T \leq T_H$ with $T_H = 1/\pi(8\alpha')^{1/2}$ (for type-II strings) in the (free CFT, or Matsubara) string partition function (Atick, Witten, 1988), due to the infinite tower of massive states in the string spectrum.
- String theory also put limits on the maximal values of electric and magnetic fields in Born-Infeld electrodynamics, $\left|\vec{E}\right| \leq 1/b$ and $\left|\vec{H}\right| \leq 1/b$, with $b = 2\pi\alpha'$.
- String theory gives rise to UV/IR mixing via dualities and non-commutativity.

It is natural to expect similar features in the string-generated gravity.

Effective higher-derivative supergravity from M-theory

- M-theory is a non-perturbative upgrade of superstrings (Witten, 1995).

(i) M-theory low-energy effective action is unique=supergravity in 11 dimensions;
(ii) the next quantum gravitational corrections to the 11-dimensional supergravity from M-theory are all quartic in the curvature;
(iii) Superstring/M-theory has no free parameters. After a compactification, there are many effective coupling constants given by the moduli VEVs.

The quartic bosonic terms of the M-theory corrected 11-dim supergravity are

$$
\begin{aligned}
S_{11} = & -\frac{1}{2\kappa_{11}^2}\int d^{11}x\,\sqrt{-g}\left[R-\frac{1}{2\cdot 4!}F^2-\frac{1}{6\cdot 3!\cdot (4!)^2}\varepsilon_{11}CFF\right] \\
& -\frac{T_2}{(2\pi)^4\cdot 3^2\cdot 2^{13}}\int d^{11}x\,\sqrt{-g}\left(J-\frac{1}{2}E_8\right)+T_2\int C\wedge X_8
\end{aligned}
$$

where κ_{11} is the gravitational constant (in 11 dims), T_2 is the M2-brane tension,

$$
T_2 = \left(\frac{2\pi^2}{\kappa_{11}^2}\right)^{1/3}
$$

Our notation in 11 (spacetime) dimensions

C is a 3-form gauge field of the 11-dimensional supergravity, and $F = dC$ is its four-form field strength, R is the gravitational scalar curvature, ε_{11} stands for the 11-dimensional Levi-Civita symbol in the Chern-Simons-like coupling, while (J, E_8, X_8) are the certain quartic polynomials in the 11-dimensional full (Weyl) curvature:

$$J = 3 \cdot 2^8 \left(R^{mijn} R_{pijq} R_m{}^{rsp} R^q{}_{rsn} + \frac{1}{2} R^{mnij} R_{pqij} R_m{}^{rsp} R^q{}_{rsn} \right) + \mathcal{O}(R_{mn})$$

the E_8 is the 11-dimensional extension of the 8-dimensional Euler density,

$$E_8 = \frac{1}{3!} \varepsilon^{abcm_1 n_1 \ldots m_4 n_4} \varepsilon_{abcm'_1 n'_1 \ldots m'_4 n'_4} R^{m'_1 n'_1}{}_{m_1 n_1} \cdots R^{m'_4 n'_4}{}_{m_4 n_4}$$

and the X_8 is the 8-form

$$X_8 = \frac{1}{192 \cdot (2\pi^2)^4} \left[\mathrm{tr} R^4 - \frac{1}{4} (\mathrm{tr} R^2)^2 \right]$$

- The J-contribution is defined modulo Ricci-dependent terms by its very derivation (on-shell). Let’s consider much simpler gravity theories [// Break!]

FLRW metric and cosmological acceleration

- The main Cosmological Principle of a spatially homogeneous and isotropic $(1+3)$-dimensional universe (at large scales) gives rise to the FLRW metric

$$ds^2_{\text{FLRW}} = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

where the function $a(t)$ is known as the scale factor in 'cosmic' (co-moving) co-ordinates (t, r, θ, ϕ), and k is the FLRW topology index, $k = (-1, 0, +1)$. The FLRW metric (1) admits a 6-dimensional isometry group G that is either $SO(1,3)$, $E(3)$ or $SO(4)$, acting on the orbits $G/SO(3)$, with the spatial 3-dimensional sections H^3, E^3 or S^3, respectively. Important notice: Weyl tensor $C^{\text{FLRW}}_{ijkl} = 0$.

- Present Universe acceleration and early Universe inflation are defined by

$$\ddot{a}(t) > 0 \ , \ \text{or equivalently} \ , \ \frac{d}{dt}\left(\frac{H^{-1}}{a}\right) < 0$$

where $H = \dot{a}/a$ is Hubble 'constant'. The amount of inflation (# e-foldings) is given by

$$N = \ln\frac{a(t_{\text{end}})}{a(t_{\text{start}})} = \int_{t_{\text{start}}}^{t_{\text{end}}} H\, dt$$

$\boxed{R+R^2 \text{ gravity toy-model (well studied)}}$

In 4 dimensions, there are only 3 independent quadratic curvature invariants: $R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}$, $R^{\mu\nu}R_{\mu\nu}$ and R^2. In addition,

$$\int d^4x\,\sqrt{-g}\left(R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right)$$

is topological for any metric, while

$$\int d^4x\,\sqrt{-g}\left(3R^{\mu\nu}R_{\mu\nu} - R^2\right)$$

is also topological for any FLRW metric. Hence, as regards the FLRW metrics, the most general quadratic curvature action is given by $(8\pi G_N = 1)$

$$S = -\frac{1}{2}\int d^4x\,\sqrt{-g}\left(R - 2\Lambda + \alpha R^2\right)$$

There is the exact inflationary dS solution (with $\Lambda = 0$) to this model: $a(t) \propto e^{Ht}$ and $H^2 = (24\alpha)^{-1}$. It is stable (attractor!) when $\alpha > 0$.

The simplest $(R+R^2)$-gravity toy-model is not phenomenologically viable for the present Universe (Navarro, van Acoyelen, 2006), so let's now discuss a generic $f(R)$ gravity, in the model-independent way.

$f(R)$ gravity theories

An $f(R)$ gravity is specified by the action $\boxed{S_f = -\frac{1}{2\kappa^2}\int d^4x\, f(R)}$ where R is the Ricci scalar curvature of a metric $g_{\mu\nu}(x)$, and κ is the gravitational coupling constant, $\kappa^2 = 8\pi G_N$. A matter action S_m, minimally coupled to the metric, is supposed to be added to S_f. We use the 'mostly minus' spacetime signature.

The gravitational equations of motion derived from the action $S_f + S_m$ read

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + g_{\mu\nu}\Box f'(R) - \nabla_\mu\nabla_\nu f'(R) = \kappa^2 T_{\mu\nu}$$

where the primes denote differentiation. Those equations of motion are the 4th-order differential equations with respect to the metric (ie. with the higher derivatives). Taking the trace of the equation above yields

$$\Box f'(R) + \frac{1}{3}f'(R)R - \frac{2}{3}f(R) = \kappa^2 T$$

Hence, in contrast to General Relativity having $f'(R) = const.$, in $f(R)$ gravity the field $\phi = f'(R)$ is dynamical and represents an independent propagating (scalar) degree of freedom. In terms of the fields $(g_{\mu\nu}, \phi)$ the equations of motion are of the 2nd order in the derivatives.

$f(R)$ gravity and present Universe acceleration

The equations of motion in the $f(R)$-FLRW cosmology (generalizing Friedmann and Raychaudhuri equations, respectively) are given by

$$H^2 = \frac{\kappa^2}{3f'}\rho_R \quad \text{and} \quad \frac{\ddot{a}}{a} = -\frac{\kappa^2}{2f'}(\rho_R + 3P_R)$$

with the energy density ρ_R and pressure P_R are due to the curvature modification,

$$\rho_R = \frac{Rf' - f}{2} - 3H\,\dot{R}\,f'' , \quad P_R = 2H\,\dot{R}\,f'' + \ddot{R}\,f'' + \frac{1}{2}\left(f - f'R\right) + f'''\,\dot{R}^{\,2}$$

The ρ_R and P_R identically vanish in Einstein gravity, where $f(R) = R$.

It is not diffiicult to choose the function $f(R)$ in order to get $\ddot{a} > 0$, with a desired equation of state for DE. Actually, it is possible a reconstruction of the function $f(R)$ from any desired scale factor (history) $a(t)$, with $R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]$

$f(R)$ gravity = quintessence

- The easiest way to make a connection between $f(R)$ gravity and scalar-tensor gravity is to apply a Legendre-Weyl transform. The action S_f is classically equivalent to

$$S_A = \frac{-1}{2\kappa^2} \int d^4x \sqrt{-g} \, \{AR - V(A)\}$$

where the real scalar $A(x)$ is related to the scalar curvature R by the Legendre transformation

$$R = V'(A) \qquad \text{and} \qquad f(R) = RA(R) - V(A(R))$$

A Weyl transformation of the metric $g_{\mu\nu}(x) \to \exp\left[\frac{2\kappa\phi(x)}{\sqrt{6}}\right] g_{\mu\nu}(x)$ with an arbitrary field parameter $\phi(x)$ yields

$$\sqrt{-g}\, R \to \sqrt{-g}\, \exp\left[\frac{2\kappa\phi(x)}{\sqrt{6}}\right] \left\{ R - \sqrt{\frac{6}{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi\right) \kappa - \kappa^2 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\}$$

Hence, when choosing $A(\kappa\phi) = \exp\left[\frac{-2\kappa\phi(x)}{\sqrt{6}}\right]$ and ignoring the total derivative, we can rewrite the above action to the form

$$S_\phi = \int d^4x\,\sqrt{-g}\left\{\frac{-R}{2\kappa^2} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2\kappa^2}\exp\left[\frac{4\kappa\phi(x)}{\sqrt{6}}\right]V(A(\kappa\phi))\right\}$$

in terms of the physical (and canonically normalized) scalar field $\phi(x)$.

- Applicability of the Legendre-Weyl transform implies the invertibility of the function $f'(R)$ at the reference point R_0 ie. both $f'(R_0)$ and $f''(R_0) \neq 0$. The $R_0 = const.$ is a solution to the pure $f(R)$-gravity provided that $f'(R_0)R_0 = 2f(R_0)$. When considering small perturbations, $R = R_0 + Z$, and linearizing the equations of motion with respect to Z, one gets

$$\left(\Box + m^2\right)Z = 0 \qquad \text{with} \qquad m^2 = \frac{1}{3}\left[R_0 - f'(R_0)/f''(R_0)\right]$$

Hence, the R_0-background is stable when $m^2 > 0$ (Starobinsky,1988).

- After the Weyl transform, the gravity-coupled matter fields in S_m become conformally coupled to ϕ. Hence, some stabilization mechanism is needed for ϕ.

Local tests of $f(R)$ gravity

- Any modification of Einstein gravity has to be consistent with local physics constraints, eg., those coming from our Solar System. There are many choices of $f(R)$ that fit all known experimental data. For instance, the function

$$f(R) = R + \lambda R_0 \left[\frac{1}{\left(1 + \frac{R^2}{R_0^2}\right)^n} - 1 \right]$$

with properly chosen parameters $R_0 \sim H_0^2$, $\lambda > 0$ and $n > 0$, may fit all known Solar System observations (Starobinsky et al, 2009)

- To recover Einstein gravity for stellar systems, the extra scalar should effectively decouple when one gets close to a star — it is known as the Chameleon effect (Khoury, Veltman, 2003). The relevant distance from the star of mass M (inside it the scalar gets hidded or screened) is given by the Vainstein radius $R_V = \left(G_N M / H_0^2\right)^{1/3}$. The mass of the scalar depends upon the background, it should be at least of $10^3 H_0$, in order to have the Chameleon effect, and it should also meet the experimental bound on the (absence of) 5th force (fine-tuning).

$F(R)$ Supergravity

- Supersymmetry is the symmetry between bosons and fermions, it is well motivated in particle physics beyond the SM, and it is also needed for the consistency of strings. Supergravity is the theory of local supersymmetry. Supergravity is also the only known consistent route to couple spin-3/2 particles (gravitinos).
- Most of studies of superstring- and brane- cosmology are based on their effective description in the 4-dimensional $N = 1$ supergravity.
- An $N = 1$ locally supersymmetric generalization of $f(R)$ gravity is possible (Gates Jr., SVK, 2009). It is non-trivial because, despite of the apparent presence of the higher derivatives, there should be no ghosts, and the auxiliary freedom (Gates Jr., 1996) is to be preserved. The modified supergravity action turns out to be classically equivalent to the standard $N = 1$ Poincaré supergravity coupled to a dynamical chiral superfield whose Kähler potential and superpotential are dictated by a single holomorphic function (= super-quintessence).

A possible connection to the Loop Quantum Gravity was investigated by Gates Jr., N. Yunes and SVK, in Phys. Rev. D80 (2009) 065003, arXiv:0906.4978 [hep-th].

Basic facts about 4-dim, $N = 1$ supergravity in superspace

A concise and manifestly supersymmetric description of supergravity is given by Superspace. We use here the natural units $c = \hbar = \kappa = 1$.

The chiral superspace density (in the supersymmetric gauge-fixed form) reads

$$\mathcal{E}(x,\theta) = e(x)\left[1 - 2i\theta\sigma_a\bar{\psi}^a(x) + \theta^2 B(x)\right] \ , \tag{1}$$

where $e = \sqrt{-\det g_{\mu\nu}}$, $g_{\mu\nu}$ is a spacetime metric, $\psi^a_\alpha = e^a_\mu \psi^\mu_\alpha$ is a chiral gravitino, $B = S - iP$ is the complex scalar auxiliary field. We use the lower case middle greek letters $\mu, \nu, \ldots = 0,1,2,3$ for curved spacetime vector indices, the lower case early latin letters $a, b, \ldots = 0,1,2,3$ for flat (target) space vector indices, and the lower case early greek letters $\alpha, \beta, \ldots = 1,2$ for chiral spinor indices. Supergravity $\neq$ curved Superspace (off-shell SUSY constraints needed)!

The solution of the superspace Bianchi identitiies and the constraints defining the $N = 1$ Poincaré-type *minimal* supergravity results in only three covariant tensor superfields $\mathcal{R}$, $\mathcal{G}_a$ and $\mathcal{W}_{\alpha\beta\gamma}$, subject to the off-shell relations:

$$\mathcal{G}_a = \bar{\mathcal{G}}_a\ , \qquad \mathcal{W}_{\alpha\beta\gamma} = \mathcal{W}_{(\alpha\beta\gamma)}\ , \qquad \bar{\nabla}_{\dot{\alpha}}\mathcal{R} = \bar{\nabla}_{\dot{\alpha}}\mathcal{W}_{\alpha\beta\gamma} = 0\ , \tag{2}$$

and

$$\bar{\nabla}^{\dot{\alpha}}\mathcal{G}_{\alpha\dot{\alpha}} = \nabla_{\alpha}\mathcal{R}\ , \qquad \nabla^{\gamma}\mathcal{W}_{\alpha\beta\gamma} = \tfrac{i}{2}\nabla_{\alpha}{}^{\dot{\alpha}}\mathcal{G}_{\beta\dot{\alpha}} + \tfrac{i}{2}\nabla_{\beta}{}^{\dot{\alpha}}\mathcal{G}_{\alpha\dot{\alpha}}\ , \tag{3}$$

where $(\nabla_{\alpha}, \bar{\nabla}_{\dot{\alpha}}.\nabla_{\alpha\dot{\alpha}})$ represent the curved superspace $N = 1$ supercovariant derivatives, and bars denote complex conjugation.

The covariantly chiral complex scalar superfield $\mathcal{R}$ has the scalar curvature R as the coefficient at its θ^2 term, the real vector superfield $\mathcal{G}_{\alpha\dot{\alpha}}$ has the traceless Ricci tensor, $R_{\mu\nu} + R_{\nu\mu} - \frac{1}{2}g_{\mu\nu}R$, as the coefficient at its $\theta\sigma^a\bar{\theta}$ term, whereas the covariantly chiral, complex, totally symmetric, fermionic superfield $\mathcal{W}_{\alpha\beta\gamma}$ has the self-dual part of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ as the coefficient at its linear θ^{δ}-dependent term. A generic higher-derivative supergravity Lagrangian (e.g., representing the supergravitational part of the superstring effective action) is given by

$$\mathcal{L} = \mathcal{L}(\mathcal{R}, \mathcal{G}, \mathcal{W}, \ldots) \tag{4}$$

where the dots stand for arbitrary supercovariant derivatives of the superfields.

New proposal: $F(\mathcal{R})$ supergravity

Let's concentrate on the particular sector of the generic higher-derivative supergravity (4), by ignoring the tensor superfields $\mathcal{W}_{\alpha\beta\gamma}$ and $\mathcal{G}_{\alpha\dot{\alpha}}$, as well as the derivatives of the scalar superfield $\mathcal{R}$:

$$S_F = \int d^4x d^2\theta\, \mathcal{E} F(\mathcal{R}) + \text{H.c.} \tag{5}$$

with a holomorphic function $F(\mathcal{R})$. Besides manifest local $N = 1$ supersymmetry, the action (5) also possess the auxiliary freedom, since the auxiliary field B does not propagate. It distinguishes the action (5) from other possible truncations of eq. (4). The action (5) gives rise to the spacetime torsion generated by gravitino, while its bosonic terms have the form

$$S_f = \int d^4x \sqrt{-g}\, f(R) \tag{6}$$

Hence, eq. (5) can also be considered as a locally $N = 1$ supersymmetric extension of the $f(R)$-type modified gravity, often used to 'explain' the present Universe acceleration.

Getting fields from superfields

Applying the superspace chiral density formula

$$\int d^4x d^2\theta\, \mathcal{E}\mathcal{L} = \int d^4x\, e\,\{\mathcal{L}_{\text{last}} + B\mathcal{L}_{\text{first}}\} \tag{7}$$

to our action (5) yields its bosonic part as

$$F'(\bar{X})\left[\tfrac{1}{3}R_* + 4\bar{X}X\right] + 3XF(\bar{X}) + \text{H.c.} \tag{8}$$

where primes denote differentiation. We have used the notation

$$X = \tfrac{1}{3}B\,, \qquad R_* = R + \frac{i}{2}\varepsilon^{abcd}R_{abcd} \tag{9}$$

Varying eq. (8) with respect to the auxiliary fields X and $\bar{X}$ gives rise to merely algebraic equation on the auxiliary fields,

$$3\bar{F} + X(4\bar{F}' + 7F') + 4\bar{X}XF'' + \tfrac{1}{3}F''R_* = 0 \tag{10}$$

- Let's first consider the simple special case when

$$F'' = 0 \qquad \text{or, equivalently,} \qquad F(\mathcal{R}) = f_0 + f_1\mathcal{R} \tag{11}$$

with some complex constants f_0 and f_1, where $\mathrm{Re} f_1 < 0$. Then eq. (10) is easily solved as

$$\bar{X} = \frac{-3(f_0 + f_1 R_*)}{4f_1 + 7\bar{f}_1} \tag{12}$$

Substituting the solution (12) back into the Lagrangian (8) yields

$$\tfrac{2}{3}(\mathrm{Re} f_1)R_* - \frac{9\,|f_0|^2}{14(\mathrm{Re} f_1)} \equiv -\frac{1}{2\kappa^2}R_* - \Lambda = -\frac{1}{2\kappa^2}R(\Gamma + T) - \Lambda \tag{13}$$

where we have reintroduced the standard gravitational constant $\kappa_0 = M^{-1}_{\mathrm{Planck}}$ in terms of the (reduced) Planck mass, the standard supergravity connection (i.e. Christoffel symbols Γ plus torsion T), and a cosmological constant Λ,

$$\kappa = \sqrt{\frac{3}{4\,|\mathrm{Re} f_1|}}\ , \qquad \Lambda = \frac{-9\,|f_0|^2}{14\,|\mathrm{Re} f_1|} \tag{14}$$

Hence, the cosmological constant in the standard supergravity is always negative or zero. (A large negative Λ forces a universe to collapse quickly.) It is yet another reason to go to the modified supergravity having $F'' \neq 0$.

Supersymmetric Legendre-Weyl-Kähler transformation

The superfield action (5) is classically equivalent to

$$S_V = \int d^4x d^2\theta \, \mathcal{E} \left[\mathcal{Z}\mathcal{R} - V(\mathcal{Z}) \right] + \text{H.c.} \tag{15}$$

with the covariantly chiral superfield $\mathcal{Z}$ as the Lagrange multiplier. Varying the action (15) with respect to $\mathcal{Z}$ gives back the original action (5) provided that

$$F(\mathcal{R}) = \mathcal{R}\mathcal{Z}(\mathcal{R}) - V(\mathcal{Z}(\mathcal{R})) \tag{16}$$

where the function $\mathcal{Z}(\mathcal{R})$ is defined by inverting the function

$$\mathcal{R} = V'(\mathcal{Z}) \tag{17}$$

Equations (16) and (17) define the superfield Legendre transform, and imply

$$F'(\mathcal{R}) = Z(\mathcal{R}) \qquad \text{and} \qquad F''(\mathcal{R}) = Z'(\mathcal{R}) = \frac{1}{V''(\mathcal{Z}(\mathcal{R}))} \tag{18}$$

where $V'' = d^2V/d\mathcal{Z}^2$. The second formula (18) is the duality relation between the supergravitational function F and the chiral superpotential V.

A super-Weyl transform of the acton (15) can be done entirely in superspace. In terms of components, the super-Weyl transform amounts to a Weyl transform, a chiral rotation and a (superconformal) S-supersymmetry transformation (Howe). The chiral density superfield $\mathcal{E}$ is a chiral compensator of the super-Weyl transformations,

$$\mathcal{E} \to e^{3\Phi}\mathcal{E} \;, \tag{19}$$

whose parameter Φ is an arbitrary covariantly chiral superfield, $\bar{\nabla}_{\dot{\alpha}}\Phi = 0$. Under the transformation (19) the covariantly chiral superfield $\mathcal{R}$ transforms as

$$\mathcal{R} \to e^{-2\Phi}\left(\mathcal{R} - \frac{1}{4}\bar{\nabla}^2\right)e^{\bar{\Phi}} \quad . \tag{20}$$

The super-Weyl chiral superfield parameter Φ can be traded for the chiral Lagrange multiplier $\mathcal{Z}$ by using a generic gauge condition

$$\mathcal{Z} = \mathcal{Z}(\Phi) \tag{21}$$

where $\mathcal{Z}(\Phi)$ is a holomorphic function of Φ. It results in the equivalent action

$$S_{\Phi} = \int d^4x d^4\theta\, E^{-1}e^{\Phi+\bar{\Phi}}\left[\mathcal{Z}(\Phi) + \text{H.c.}\right] - \int d^4x d^2\theta\, \mathcal{E}e^{3\Phi}V(\mathcal{Z}(\Phi)) + \text{H.c.} \tag{22}$$

Equation (22) has the standard form of the action of a chiral matter superfield coupled to supergravity,

$$S[\Phi,\bar{\Phi}] = \int d^4x d^4\theta\, E^{-1}\Omega(\Phi,\bar{\Phi}) + \left[\int d^4x d^2\theta\, \mathcal{E} P(\Phi) + \text{H.c.}\right] , \qquad (23)$$

in terms of a 'Kähler' potential $\Omega(\Phi,\bar{\Phi})$ and a chiral superpotential $P(\Phi)$. In our case (22) we find

$$\Omega(\Phi,\bar{\Phi}) = e^{\Phi+\bar{\Phi}}\left[\mathcal{Z}(\Phi)+\bar{\mathcal{Z}}(\bar{\Phi})\right] , \quad P(\Phi) = -e^{3\Phi}V(\mathcal{Z}(\Phi)) \ . \qquad (24)$$

The truly Kähler potential $K(\Phi,\bar{\Phi})$ is given by

$$K = -3\ln(-\frac{\Omega}{3}) \quad \text{or} \quad \Omega = -3e^{-K/3} , \qquad (25)$$

because of the invariance of the action (23) under the supersymmetric Kähler-Weyl transformations

$$K(\Phi,\bar{\Phi}) \to K(\Phi,\bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) , \quad \mathcal{E} \to e^{\Lambda(\Phi)}\mathcal{E} , \qquad (26)$$

$P(\Phi) \to -e^{-\Lambda(\Phi)}P(\Phi)$, with an arbitrary chiral superfield parameter $\Lambda(\Phi)$.

Scalar potential

(in components) is given by the standard formula (Cremmer *et al*, 1979)

$$\mathcal{V}(\phi,\bar{\phi}) = e^{\Omega}\left\{\left|\frac{\partial P}{\partial \Phi} + \frac{\partial \Omega}{\partial \Phi}P\right|^2 - 3\,|P|^2\right\}\Bigg| \tag{27}$$

where all superfields are restricted to their leading field components, $\Phi| = \phi(x)$. Equation (27) can be simplified by making use of the Kähler-Weyl invariance (26) that allows us to choose the gauge

$$P = 1 \tag{28}$$

It is equivalent to the well known fact that the scalar potential (27) is actually governed by the single (Kähler-Weyl-invariant) potential

$$G(\Phi,\bar{\Phi}) = \Omega + \ln P + \ln \bar{P} \tag{29}$$

In our case (24) we have

$$G = e^{\Phi+\bar{\Phi}}\left[\mathcal{Z}(\Phi) + \bar{\mathcal{Z}}(\bar{\Phi})\right] + 3(\Phi+\bar{\Phi}) + \ln(-V(\mathcal{Z}(\Phi)) + \ln(-\bar{V}(\bar{\mathcal{Z}}(\bar{\Phi})) \tag{30}$$

Let's now specify our gauge (21) by choosing the condition

$$3\Phi + \ln(-V(\mathcal{Z}(\Phi)) = 0 \quad \text{or} \quad V(\mathcal{Z}(\Phi)) = -e^{-3\Phi} \tag{31}$$

that is equivalent to eq. (28). Then the potential (30) gets simplified to

$$G = \Omega = e^{\Phi+\bar{\Phi}}\left[\mathcal{Z}(\Phi) + \bar{\mathcal{Z}}(\bar{\Phi})\right] \tag{32}$$

Equations (16), (17) and (32) are the one-to-one algebraic relations between a holomorphic function $F(\mathcal{R})$ in our modified supergravity action (5) and a holomorphic function $\mathcal{Z}(\Phi)$ defining the scalar potential (27)

$$\mathcal{V} = e^{G}\left[\left(\frac{\partial^2 G}{\partial\Phi\partial\bar{\Phi}}\right)^{-1}\frac{\partial G}{\partial\Phi}\frac{\partial G}{\partial\bar{\Phi}} - 3\right]\Bigg| \tag{33}$$

in the classically equivalent scalar-tensor supergravity. The latter may be used for embedding a slow-roll inflation into supergravity. In our setup the correspondence can be promoted even further, by embedding the slow-roll inflation into the 'purely geometrical' (apparently) higher-derivative supergravity theory (5), defined in terms of a single holomorphic function.

No-scale modified supergravity

The no-scale supergravity arises by demanding the scalar potential (33) to vanish. It results in the vanishing cosmological constant without fine-tuning. The no-scale supergravity potential G has to obey the non-linear 2nd-order partial differential equation

$$3\frac{\partial^2 G}{\partial\Phi\partial\bar{\Phi}} = \frac{\partial G}{\partial\Phi}\frac{\partial G}{\partial\bar{\Phi}} \tag{34}$$

A gravitino mass $m_{3/2}$ is given by the vacuum expectation value (Wess, Bagger)

$$m_{3/2} = \left\langle e^{G/2} \right\rangle \tag{35}$$

so that the spontaneous supersymmetry breaking scale can be arbitrarily chosen.

- We get extra bonus, when identifying the $N = 1$ modified supergravity gravitino with CDM particle (cf. Moroi, Murayama, 1993). The gravitino CDM is the darkest DM candidate since it cannot be produced at colliders. If gravitino is LSP, it would also be stable.

Imposing the no-scale condition (34) in our case (32) gives rise to the 1st order non-linear differential equation

$$3\left(e^{\bar{\Phi}}X' + e^{\Phi}\bar{X}'\right) = \left|e^{\bar{\Phi}}X' + e^{\Phi}\bar{X}\right|^2 \tag{36}$$

where we have introduced the notation

$$\mathcal{Z}(\Phi) = e^{-\Phi}X(\Phi)\ , \qquad X' = \frac{dX}{d\Phi} \tag{37}$$

Accordingly, the gravitino mass (35) is given by

$$m_{3/2} = \left\langle \exp\frac{1}{2}\left(e^{\bar{\Phi}}X + e^{\Phi}\bar{X}\right)\right\rangle \tag{38}$$

- We are not aware of any non-trivial holomorphic exact solution to eq. (36).

For instance, should it obey the holomorphic differential equation

$$X' = e^{\Phi}g(X,\Phi) \tag{39}$$

with a holomorphic function $g(X,\Phi)$, eq. (36) gives rise to the functional equation

$$3\,(g+\bar{g}) = \left| e^{\bar{\Phi}} g + \bar{X} \right|^2 \tag{40}$$

- Being restricted to the real variables $\Phi = \bar{\Phi} \equiv y$ and $X = \bar{X} \equiv x$, eq. (36) reads

$$6x' = e^y (x'+x)^2\,, \quad \text{where} \quad x' = \frac{dx}{dy} \tag{41}$$

This equation can be integrated after the change of variables *

$$x = e^{-y} u \tag{42}$$

that leads to a quadratic (!) equation with respect to $u' = du/dy$,

$$(u')^2 - 6u' + 6u = 0 \tag{43}$$

It follows

$$y = \int^u \frac{d\xi}{3 \pm \sqrt{3(3-2\xi)}} = \mp \sqrt{1 - \tfrac{2}{3}u} + \ln\left(\sqrt{3(3-2u)} \pm 3 \right) + C\,. \tag{44}$$

*I am grateful to A. Starobinsky who pointed it out to me.

Conclusion

- $f(R)$ gravity may pass local gravity tests, while modifying Einstein gravity at cosmological distances. The characteristic distance entering these modifications is of the order 10 pc for the Sun and of the order 100 Kpc for a galaxy;

- within the Vainstein distance, linearlization of modified gravity breaks down (van Acoyelen, 2006), ie. gravity is in the non-perturbative regime (Dvali, 2006);

- in $f(R)$ gravity the present Universe acceleration is the manifestation of a new geometry of the Universe (presumably generated by Quantum Gravity or Superstrings - however, no claim yet);

- $f(R)$ gravity is classically equivalent to the quintessense (or Dynamical Dark Energy). The late time acceleration of the Universe may be driven e.g., by the cosmological constant $\Lambda = \langle V \rangle_0$ in the quintessence scalar potential V;

- $f(R)$ gravity may provide a unified description of the early Universe inflation and the present Universe acceleration;

- the transition from the ordinary matter-dominated FLRW Universe to the modified evolution is controlled by the extra dimensional constant R_0, whose value is of the order of the present Hubble parameter H_0;

- though it may be impossible to distinguish between a modified gravity and DE in the FLRW Universe, it may well be possible in an inhomogeneous Universe;

- It is possible to avoid ghosts in a higher-derivative gravity and supergravity. In some special cases, a higher-derivatve gravity may be classically equivalent to a scalar-tensor gravity without ghosts or higher derivatives;

- A locally $N=1$ supersymmetric extension of the modified $f(R)$ gravity exist, and is parametrized by a single holomorphic function. It is classically equivalent to the standard theory of a chiral scalar superfield (with non-trivial Kähler potential and chiral superpotential) coupled to the (minimal) $N=1$ Poincaré supergravity in four space-time dimensions;
- $F(R)$ supergravity gives rise to the spacetime torsion and has the natural candidate for a CDM-particle: a massive gravitino. We conjectured the identification of the dynamical chiral superfield in the modified supergravity and the dilaton-axion chiral superfield in superstring theory (more opportunities for CDM and hybrid inflation!);
- we computed a scalar potential in the dual version of the modified supergravity via the Legendre–Kähler-Weyl transform in superspace. The Kähler potential, the superpotential and the scalar potential of the dual theory are all governed by a single holomorphic function;
- we also found the conditions for the vanishing cosmological constant and spontanenous supersymmetry breaking, without fine-tuning, defining the no-scale $F(\mathcal{R})$ modified supergravity.

Comments and Discussion

Warped compactification to 4 spacetime dimensions

- To match the constraints imposed by particle physics, M-theory should be compactified on a 7-dimensional space of special (G_2) holonomy (Atiyah, Witten)
- In the presence of fluxes, we consider a warped compactification

$$ds_{11}^2 = e^{2A(y)} ds_{\mathrm{FRW}}^2 + e^{-2A(y)} ds_7^2 \tag{45}$$

where $ds_{\mathrm{FRW}}^2(x)$ is the (uncompactified) 4-dim FLRW metric, $ds_7^2(y)$ is a (compactified) 7-dim metric, and $A(y)$ is a warp factor, $\nabla^2 A \propto$ (fluxes)^2 .

- Since we are interested in the purely gravitational terms in 4 dims, the explicit form of the 7-metric ds_7^2 is not needed. After dimensional reduction, the only gravitational terms in 4 dims are

$$S_4 = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \, (R + \beta J_R) \tag{46}$$

where we have introduced the Einstein coupling κ in four dimensions, and the four-dimensional counterpart J_R of J in eq. (1), $i, j = 0, 1, 2, 3$,

$$J_R = R^{mijn} R_{pijq} R_m{}^{rsp} R^q{}_{rsn} + \frac{1}{2} R^{mnij} R_{pqij} R_m{}^{rsp} R^q{}_{rsn} + O(R_{mn}) \tag{47}$$

Couplings and scales

The relation between the coupling constants κ_{11} and κ is $\kappa^2 = e^{5A} M^7_{\text{KK}} \kappa^2_{11}$ where the Kaluza-Klein (KK) compactification scale is $M^{-7}_{\text{KK}} = Vol_7 \equiv \int d^7y \sqrt{g_7}$, and the average warp factor A (of integer weight p) is defined by

$$e^{pA} = \frac{1}{Vol_7} \int d^7y \sqrt{g_7}\, e^{pA(y)} \tag{48}$$

It follows

$$\beta = \frac{1}{3} \left(\frac{\kappa^2}{2^{23/2} \pi^5 e^{14A} M^7_{\text{KK}}} \right)^{2/3} \tag{49}$$

of mass dimension -6. When using the Planck scale $\kappa \approx 10^{-33}$cm and $M^{-1}_{\text{KK}} \approx 10^{-15}$cm, and ignoring the warp factor, $A = 0$, we get (in fact, unacceptable) value $\beta \approx 10^{-118}\,\text{cm}^6$. Altogether it leads to the modified gravitational equations

$$R_{ij} - \frac{1}{2} g_{ij} R + \beta \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ij}} \left(\sqrt{-g} J_R \right) = \kappa^2 T_{ij} \tag{50}$$

where T_{ij} stands for the energy-momentum tensor of all matter fields (including dilaton and axion)

On-shell structure of the quartic gravity terms

Detailed structure and physical meaning of the quartic curvature terms are revealed via their connection to the four-dimensional Bel-Robinson (BR) tensor

$$T_R^{iklm} = R^{ipql} R^k{}_{pq}{}^m + {}^*R^{ipql*} R^k{}_{pq}{}^m \qquad (51)$$

whose structure is quite similar to that of the Maxwell stress-energy tensor, *cf.*

$$T_{ij}^{\text{Maxwell}} = F_{ik} F_j{}^k + {}^*F_{ik}{}^*F_j{}^k \;\; , \quad F_{ij} = \partial_i A_j - \partial_j A_i \qquad (52)$$

Weyl cousin T_C^{ijlm} of the BR tensor is obtained by replacing all curvatures by Weyl tensors. It is factorized in the 2-component (spinor) formalism,

$$(T_C)_{\alpha\beta\gamma\delta\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = C_{\alpha\beta\gamma\delta} \bar{C}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \qquad (53)$$

Important! : here we consider all the quartic terms on-shell, i.e. modulo Ricci-tensor-dependent terms. Then we find

$$T^2_{ijkl} = 8 J_R = \tfrac{1}{4} (R_{ijkl} R^{ijkl})^2 + \tfrac{1}{4} ({}^*R_{ijkl} R^{ijkl})^2 \qquad (54)$$

and

$$T^2_{ijkl} = 8J_R = -\tfrac{1}{4}({}^*R^2_{ijkl})^2 + \tfrac{1}{4}({}^*R_{ijkl}R^{ijkl})^2$$
$$= \tfrac{1}{4}(P_4^2 - E_4^2) = \tfrac{1}{4}(P_4 + E_4)(P_4 - E_4) \tag{55}$$

in terms of the Euler and Pontryagin topological densities in four dimensions.

The on-shell BR tensor is fully symmetric and traceless,

$$T_{ijkl} = T_{(ijkl)} \ , \qquad T^i_{ikl} = 0 \tag{56}$$

is covariantly conserved

$$\nabla^i T_{ijkl} = 0 \tag{57}$$

and has a positive ‘energy’ density,

$$T_{0000} > 0 \tag{58}$$

When using *Riemann Normal Coordinates* (RNC) at a given spacetime point, one has

$$g_{ij} = \eta_{ij} \ , \quad g_{ij,k} = 0 \ , \quad g_{ij,mn} = -\tfrac{1}{3}(R_{imjn} + R_{injm}) \tag{59}$$

and

$$\Gamma^i_{jk,l} = -\frac{1}{3}(R^i{}_{jkl} + R^i{}_{kjl}) \tag{60}$$

Raising and lowering of vector indices are done with Minkowski metric η_{ij} and its inverse η^{ij}, so that all traces in the last two eqs. (59) and (60) vanish.

BR tensor is simply related to the gravitational energy-momentum pseudo-tensors in RNC (Deser):

$$T_{ijkl} = \partial_k \partial_l \left(t^{LL}_{ij} + \frac{1}{2} t^E_{ij} \right) \tag{61}$$

where the symmetric Landau-Lifshitz gravitational pseudo-tensor is

$$\begin{aligned}(t_{LL})^{ij} = &- \eta^{ip}\eta^{jq}\Gamma^k_{pm}\Gamma^m_{qk} + \Gamma^i_{mn}\Gamma^j_{pq}\eta^{mp}\eta^{nq} \\ &- \left(\Gamma^m_{np}\Gamma^j_{mq}\eta^{in}\eta^{pq} + \Gamma^m_{np}\Gamma^i_{mq}\eta^{jn}\eta^{pq}\right) + h^{ij}\Gamma^m_{np}\Gamma^n_{mq}\eta^{pq}\end{aligned} \tag{62}$$

and the non-symmetric Einstein gravitational pseudo-tensor is

$$(t^E)^i_j = \left(-2\Gamma^i_{mp}\Gamma^m_{jq} + \delta^i_j \Gamma^n_{pm}\Gamma^m_{qn}\right)\eta^{pq} \tag{63}$$

Modified cosmological (Friedmann) equations

Substituting the FRW metric into a generic (about 100 different terms!) quartic gravity equations of motion yields the generalized Friedmann equation having the form ($k = 0$)

$$3H^2 \equiv 3\left(\frac{\dot{a}}{a}\right)^2 = \beta P_8\left(\frac{\dot{a}}{a}, \frac{\ddot{a}}{a}, \frac{\dddot{a}}{a}, \frac{\ddddot{a}}{a}\right) \quad , \tag{64}$$

where P_8 is a polynomial with respect to its arguments,

$$P_8 = \sum_{\substack{n_1+2n_2+3n_3+4n_4=8,\\ n_1,n_2,n_3,n_4\geq 0}} c_{n_1n_2n_3n_4}\left(\frac{\dot{a}}{a}\right)^{n_1}\left(\frac{\ddot{a}}{a}\right)^{n_2}\left(\frac{\dddot{a}}{a}\right)^{n_3}\left(\frac{\ddddot{a}}{a}\right)^{n_4} \tag{65}$$

The sum goes over the integer partitions $(n_1, 2n_2, 3n_3, 4n_4)$ of 8, the dots stand for the derivatives with respect to time t, and $c_{n_1n_2n_3n_4}$ are some real coefficients. The highest derivative enters linearly at most, $n_4 = 0, 1$.

For instance, the FLRW Ansatz with $k = 0$ results in the non-vanishing curvatures

$$R^0{}_{\mu 0\nu} = \delta_{\mu\nu} \ddot{a}\, a, \;\; R^{\mu}{}_{\nu\lambda\rho} = \left(\delta^{\mu}_{\lambda}\delta_{\nu\rho} - \delta^{\mu}_{\rho}\delta_{\nu\lambda}\right)(\dot{a})^2, \;\; R^{\mu}_{\nu} = -\delta^{\mu}_{\nu}\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2\right] \quad (66)$$

where $\mu, \nu, \lambda, \rho = 1, 2, 3$. As regards the $(BR)^2$ gravity, we find

$$3H^2 + \beta\left[9\left(\frac{\ddot{a}}{a}\right)^4 - 36H^2\left(\frac{\ddot{a}}{a}\right)^3 + 84H^4\left(\frac{\ddot{a}}{a}\right)^2 - 36H\left(\frac{\ddot{a}}{a}\right)^2\left(\frac{\dddot{a}}{a}\right) + 63H^8 - 72H^3\left(\frac{\ddot{a}}{a}\right)\left(\frac{\dddot{a}}{a}\right) + 48H^6\left(\frac{\ddot{a}}{a}\right) - 24H^5\left(\frac{\dddot{a}}{a}\right)\right] = 0 \quad (67)$$

It is remarkable that the 4th order time derivatives cancel, whereas the square of the 3rd order time derivative of the scale factor, $\dddot{a}^{\,2}$, does not appear at all.

Exact non-perturbative inflationary solutions

The structure of eqs. (64) and (65) admits the existence of rather generic exact inflationary solutions without a spacetime singularity, when using the most naive (de Sitter) Ansatz for the scale factor,

$$a(t) = a_0 e^{Bt} \tag{68}$$

with some real positive constants a_0 and B. Substituting eq. (68) into eq. (64), we get $3B^2 = (\#)\beta B^8$, whose coefficient $(\#)$ is just a sum of all c-coefficients in eq. (65). Assuming the $(\#)$ to be positive, we find (68) as the exact solution with

$$B = \left(\frac{3}{\#\beta}\right)^{1/6} \tag{69}$$

This solution in non-perturbative in β, i.e. it is impossible to get it when considering the quartic curvature terms as a perturbation. The assumption that we are dealing with the leading correction, implies $Bt \ll 1$. Because of eqs. (49) and (69), it leads to the natural hierarchy

$$\kappa M_{\rm KK} \ll 1 \quad \text{or} \quad l_{\rm Pl} \ll l_{\rm KK} \tag{70}$$

where we have introduced the four-dimensional Planck scale $l_{\rm Pl} = \kappa$ and the compactification scale $l_{\rm KK} = M_{\rm KK}^{-1}$.

The effective Hubble scale B of eq. (69) should be lower than the effective (with warping) KK scale $M_{\rm KK}^{\rm eff.} = e^A M_{\rm KK}$, in order to validate our four-dimensional description of gravity, i.e. our ignorance of all KK modes,

$$B < M_{\rm KK}^{\rm eff.} \tag{71}$$

It rules out the naive KK reduction (with $A = 0$) but still allows the warped compactification (45), when the average warp factor is tuned as

$$e^A < \frac{(\kappa M_{\rm KK})^{2/5}}{(9/\#)^{3/10} 2^{23/10} \pi} \sim \mathcal{O}\left(10^{-3}\right) \tag{72}$$

where we have used eq. (49) and estimated $(\#)$ as of order 10.

Our exact solution (68) is non-singular, while it describes an inflationary isotropic and homogeneous universe; cf. Starobinsky (1980), Maeda, Ohta (2004)

Stability analysis

To be truly inflationary solutions, eqs. (68) and (69) should correspond to the stable fixed points (or attractors). When using the parametrization

$$a(t) = e^{\lambda(t)} \tag{73}$$

we find

$$\begin{aligned}
\frac{\dot{a}}{a} &= \dot{\lambda} \quad , \\
\frac{\ddot{a}}{a} &= \ddot{\lambda} + (\dot{\lambda})^2 \quad , \\
\frac{\dddot{a}}{a} &= \dddot{\lambda} + 3\ddot{\lambda}\dot{\lambda} + (\dot{\lambda})^3 \quad , \\
\frac{\ddddot{a}}{a} &= \ddddot{\lambda} + 4\dddot{\lambda}\dot{\lambda} + 6\ddot{\lambda}(\dot{\lambda})^2 + 3(\ddot{\lambda})^2 + (\dot{\lambda})^4
\end{aligned} \tag{74}$$

The polynomial (65) now takes the form

$$P_8 = \sum_{\substack{n_1+2n_2+3n_3+4n_4=8, \\ n_1,n_2,n_3,n_4 \geq 0}} d_{n_1 n_2 n_3 n_4} \left(\dot{\lambda}\right)^{n_1} \left(\ddot{\lambda}\right)^{n_2} \left(\dddot{\lambda}\right)^{n_3} \left(\ddddot{\lambda}\right)^{n_4} \tag{75}$$

where the d-coefficients are some simple linear combinations of the c-coefficients.

Equations (68) and (69) are also simplified as

$$\lambda(t) = Bt + \lambda_0 \ , \qquad \text{where} \qquad a_0 = e^{\lambda_0} \quad \text{and} \quad d_{8000} = \# \tag{76}$$

The equations of motion (64) can be rewritten to the form

$$3y_1^2 = \beta P_8(y_1, y_2, y_3, \dot{y}_3) \equiv \beta P_{8,0}(y_1, y_2, y_3) + \beta P_4(y_1, y_2, y_3)\, \dot{y}_3 \ , \tag{77}$$

where we have introduced the notation

$$y_1 = \dot{\lambda} \ , \quad y_2 = \ddot{\lambda} \ , \quad y_3 = \dddot{\lambda} \ . \tag{78}$$

Equation (77) can now be put into the autonomous form

$$\begin{aligned} \dot{y}_1 &= y_2 \ , \\ \dot{y}_2 &= y_3 \ , \\ \dot{y}_3 &= \frac{3y_1^2 - \beta P_{8,0}(y_1, y_2, y_3)}{\beta P_4(y_1, y_2, y_3)} \equiv f(y_1, y_2, y_3) \end{aligned} \tag{79}$$

that is quite suitable for the stability analysis against small perturbations about the fixed point, $y_a = y_a^{\text{fixed}} + \delta y_a$, where $a = 1, 2, 3$.
We find

$$\begin{aligned}
\delta \overset{\bullet}{y}_1 &= \delta y_2 \ , \\
\delta \overset{\bullet}{y}_2 &= \delta y_3 \ , \\
\delta \overset{\bullet}{y}_3 &= \left.\frac{\partial f}{\partial y_1}\right| \delta y_1 + \left.\frac{\partial f}{\partial y_2}\right| \delta y_2 + \left.\frac{\partial f}{\partial y_3}\right| \delta y_3 \ ,
\end{aligned} \tag{80}$$

where all the partial derivatives are taken at the fixed point (denoted by $|$). The fixed point is stable when all the eigenvalues of the matrix

$$\widehat{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \left.\frac{\partial f}{\partial y_1}\right| & \left.\frac{\partial f}{\partial y_2}\right| & \left.\frac{\partial f}{\partial y_3}\right| \end{pmatrix} \tag{81}$$

in eq. (80) are negative or have negative real parts. Then our fixed point is a stable attractor.

Scale factor duality

More restrictions on the cosmological polynomial P_8 of eq. (65) arise in specific superstring-inspired cosmological scenarios, e.g. in the pre-Big Bang scenario of Gasperini-Veneziano. They imposed the scale factor duality, by requiring the cosmological Friedmann equation to be invariant under the duality transformation

$$a(t) \leftrightarrow \frac{1}{a(t)} \equiv b(t) \tag{82}$$

This duality is a cosmological version of the genuine stringy T-duality (which is a symmetry of the non-perturbative string spectrum), in the case of time-dependent backgrounds. The scale factor duality is merely the symmetry of the perturbative equations of motion of the background fields. It is used to exclude the cosmological singularity (Big Bang) in the pre-Big Bang scenario.

In the λ-parametrization (73) the scale-factor duality transformation (82) takes the very simple form

$$\lambda(t) \leftrightarrow -\lambda(t) \tag{83}$$

The equations of motion in the form (77) are manifestly invariant under the transform $\lambda(t) \to \lambda(t) + \lambda_0$, where λ_0 is an arbitrary constant. We also find

$$
\begin{aligned}
\frac{\dot{a}}{a} &= -\frac{\dot{b}}{b}\,, \quad \frac{\ddot{a}}{a} = -\frac{\ddot{b}}{b} + 2\left(\frac{\dot{b}}{b}\right)^2 \,, \\
\frac{\dddot{a}}{a} &= -\frac{\dddot{b}}{b} + 6\left(\frac{\dot{b}}{b}\right)\left(\frac{\ddot{b}}{b}\right) - 6\left(\frac{\dot{b}}{b}\right)^3 \,, \\
\frac{\ddddot{a}}{a} &= -\frac{\ddddot{b}}{b} + 6\left(\frac{\ddot{b}}{b}\right)^2 + 8\left(\frac{\dot{b}}{b}\right)\left(\frac{\dddot{b}}{b}\right) - 36\left(\frac{\dot{b}}{b}\right)^2\left(\frac{\ddot{b}}{b}\right) + 24\left(\frac{\dot{b}}{b}\right)^4
\end{aligned}
\tag{84}
$$

To illustrate, how the scale factor duality affects the polynomial P_8, let's consider the case with the 3rd order time derivatives, motivated by eq. (67), in the notation

$$
\frac{\dot{a}}{a} = x\,, \quad \frac{\ddot{a}}{a} = y\,, \quad \frac{\dddot{a}}{a} = z \tag{85}
$$

The duality invariance condition reads

$$P_8(-x, 2x^2 - y, 6xy - 6x^3 - z) = P_8(x, y, z) \tag{86}$$

The structure of the polynomial P_8 in eq. (65), as the sum over partitions of 8, restricts a solution to eq. (86) to be most quadratic in z,

$$P_8(x, y, z) = a_2(x, y)z^2 + b_5(x, y)z + c_8(x, y) \ , \tag{87}$$

whose coefficients are polynomials in (x, y), of order given by subscript,

$$\begin{aligned} a_2(x, y) =&\ a_0x^2 + a_1y \ , \\ b_5(x, y) =&\ b_0x^5 + b_1x^3y + b_2xy^2 \ , \\ c_8(x, y) =&\ c_4y^4 + c_3y^3x^2 + c_2y^2x^4 + c_1yx^6 + c_0x^8 \end{aligned} \tag{88}$$

Their substitution into eq. (86) yields an overdetermined system of linear equations on the coefficients. Nevertheless, we find a consistent general solution,

$$\begin{aligned} P_8(x, y, z) = a_0x^2z^2 &+ (b_0x^5 - 3a_0xy^2)z \\ &+ c_4y^4 + (9a_0 - 4c_4)y^3x^2 + c_2y^23x^4 \\ &+ (8c_4 - 18a_0 - 3b_0 - 2c_2)yx^6 + c_0x^8 \end{aligned} \tag{89}$$

parameterized by five real coefficients $(a_0, b_0, c_4, c_2, c_0)$.

Demanding the existence of the exact solution (68), i.e. positivity of $(\#)$ in eq. (69), yields

$$5c_4 + c_0 > 11a_0 + 2b_0 + c_2 \tag{90}$$

- As regards the $(BR)^2$ gravity representing the 'minimal' candidate for the off-shell superstring quartic effective action, we checked that neither the duality invariance nor the inequality (90) are satisfied by the coefficients present in eq. (67). We interpret it as the clear indications that some additional Ricci-dependent terms have to be added to the $(BR)^2$ terms or, equivalently, the $(BR)^2$ gravity is ruled out as an off-shell quartic effective action for superstrings.

- Our results are generalizable to any higher order in the spacetime curvatures, because it amounts to increasing the order of the polynomial P. It leads to the ALL-orders or non-perturbative speculative cosmological equation:

$$H^2 = \frac{\dot{a}^2}{a^2} = \beta P[a(t)] \tag{91}$$

whose function P is subject to the scale-factor duality condition

$$P[a(t)] = P[1/a(t)] \; . \tag{92}$$

Problems with String-generated Quartic Gravity

- The Quartic Gravity terms are merely the leading terms in the gravitational superstring effective action. When they become large, there is no reason why the higher-order curvature terms are to be ignored;

- the higher-order (full) curvature terms, when considered non-perturbatively, may easily lead to unphysical solutions, and violate unitarity and causality.

- We treated the Quartic Gravity as the toy model only.

Remarks

- Finding a Graceful Exit from the geometrical inflation to a matter-driven inflation, as well as getting the right number of e-foldings, need further investigation

- A possibility to get inflation by modifying Einstein equations with the 2nd-order curvature terms (representing the gravitational anomalies of matter fields) was first discovered a long time ago by Starobinsky (1980). A similar mechanism is known in the four-dimensional supergravity, with inflation being generated by the $\mathcal{R}^2$-term originating from the one-loop Kähler anomaly (Cardoso, Ovrut)

- Instabilities in the cosmological scenarios based on the 2nd-order curvature terms, against adding higher order curvature terms, were analyzed by Maeda

- In General Relativity, only the spin-2 part of a metric is dynamical. A dynamical generation of a massive scalar field is known to occur already in the presence of the quadratic curvature terms (Buchbinder *et al*) out of the spin-0 part of the metric. In supergravity, as we found, the whole chiral scalar superfield becomes dynamical, while it can be identified with the super-Weyl compensator.

- In superstring theory, the superspin-0 part of the supervielbein is given by a chiral scalar superfield, whose leading complex component represents a dilaton-axion field, $\phi| = \varphi(x) + iB(x)$. It is tempting to identify $\varphi(x)$ with a superstring dilaton, and B with a superstring B-field (or axion). In string theory, the dilaton field controls the superstring loops and (D-brane) instantons, which may be the source of the function $F(\mathcal{R})$. The B-field is the source of the non-minimal space-time torsion in string theory.

- Unfortunately, the string theory technology at present does not allow us to compute the function $F(\mathcal{R})$ in eq. (5). It is mainly because of the on-shell nature of the known string theory. However, its $\mathcal{R}$-dependence may be fixed by some additional (off-shell) physical requirements such as no-ghosts, stability, and the scale-factor self-duality, etc.

- Sometimes, a positive cosmological constant and a slow-roll inflation are achieved by demanding a shift symmetry of the Kähler potential (i.e. its flatness in one direction), and then perturbing the scalar potential around that flat direction.

- In the context of superstrings, the effective supergravity potential (32) may capture some 'stringy' features, such as T-duality and maximal curvature.

- Since we used the (old) *minimal* formulation of an off-shell supergravity multiplet, with a fixed superconnection, adding the minimal coupling to a (scalar- or vector-type) supermatter is straightforward in superspace, just by using the supercovariant derivatives, while it does not change our results. However, adding a non-minimal coupling of matter to the scalar supercurvature would drastically change everything, so it deserves a separate investigation (work in progress).