Doran-Harder-Thompson Conjecture via SYZ mirror symmetry

Atsushi Kanazawa

Kyoto University, Department of Mathematics

2016/11/28
Kavli IPMU String-Math Seminar

Goal of today's talk:

To prove the Doran-Harder-Thompson conjecture in the simplest case by using ideas from SYZ mirror symmetry.
The DHT conjecture claims that, given a degeneration of a CY manifold X to a union $X_1 \cup_Z X_2$ of quasi-Fano manifolds intersecting along $Z \in |-K_{X_i}|$, the mirror CY manifold of X can be constructed by gluing the mirror LG models of X_1 and X_2.

Plan of the first half (background material):

1. Mirror symmetry for CY manifolds
2. SYZ mirror symmetry
3. Mirror symmetry for (qausi-)Fano manifolds
4. Doran-Harder-Thompson conjecture

Definition A Calabi-Yau (CY) manifold X is a compact Kähler manifold X such that

1. the canonical bundle is trivial, $K_X = 0$,
   ($\exists\, \Omega$ nowhere vanishing holomorphic volume form)
2. $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.

$\dim X = 1$: elliptic curve,
$\dim X = 2$: K3 surface,
$\dim X = 3$: CY 3-fold (e.g. quintic 3-fold $X_5 \subset \mathbb{P}^4$),

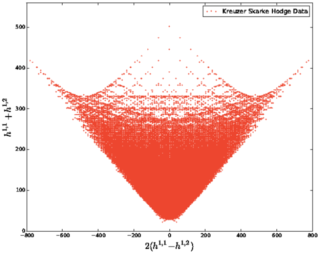
A CY manifold is a complex and symplectic manifold. Thanks to the triality (Riemann, complex and symplectic) of a Kähler manifold, we often use ”Kähler” instead of ”symplectic”.

Mirror symmetry conjecture

CY manifolds show up in pairs, say X and Y (called a mirror pair), in such a way that symplectic geometry of X is equivalent to complex geometry of Y, and vice versa.

Topological mirror symmetry: For a a mirror pair X and Y of Calabi-Yau n-fold, there should be exchange in Hodge #

$$h^{n-1,1}(X) = h^{1,1}(Y), \quad h^{1,1}(X) = h^{n-1,1}(Y).$$



Homological mirror symmetry (Kontsevich)

For a mirror pair X and Y of CY manifolds, there exits an equivalence of triangulated categories:

$$\begin{array}{ccc} \mathrm{D^bCoh}(X) & \xrightarrow[\text{equiv.}]{\cong} & \mathrm{D^bFuk}(Y) \\ \downarrow & & \downarrow \\ K(X) & \xrightarrow[\text{isom.}]{\cong} & H^n(Y, \mathbb{Z}) \end{array}$$

$\mathrm{D^bCoh}(X)$ derived category of coherent sheaves
Object: $E = \{\cdots \to E^i \to E^{i+1} \to \dots\}$ complex of sheaves
Morphism: $\mathrm{Ext}^\bullet(E_1, E_2)$ extension groups

$\mathrm{D^bFuk}(Y)$ (derived) Fukaya category
Object: (L, ∇) Lagrangian submanifold with flat $\mathrm{U}(1)$-connection
Morphism: $\mathrm{FH}^\bullet((L_1, \nabla_1), (L_2, \nabla_2))$ Lagrangian Floer homology

Strominger-Yau-Zaslow (SYZ) conjecture

Mirror CY n-folds X and Y carry dual sLag T^n-fibrations

$$X^{2n} \xrightarrow{\pi} B \xleftarrow{\pi^\vee} Y^{2n}$$

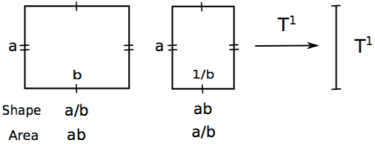
where $B \simeq S^n$. Here a submanifold L^n of a CY manifold X^{2n} is called special Lagrangian (sLag) if $\omega|_L = 0$ and $\mathrm{Im}\Omega|_L = 0$ for some Ω.

Roughly, mirror symmetry = Fourier transform (T-duality). This not only provides a way to construct a mirror Y out of X, but also explains why mirror symmetry should hold. Moreover, a mirror manifold depends on a choice of a sLag fibration on X (a choice of a large complex structure limit via Gromov-Hausdorff limit).

Example 1 Let $X := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be an elliptic curve equipped with $\omega = dx \wedge dy$ and $\Omega = dz$. Then $\exists$ smooth sLag T^1-fibration

$$\pi : X \to S^1 = \mathbb{R}/\mathrm{Im}(\tau), \quad z \mapsto \mathrm{Im}(z).$$

The mirror manifold Y is again an elliptic curve.



Example 2 Let $\pi : X \to \mathbb{P}^1$ be an elliptic fibration on a K3 surface. After the hyperKähler rotation given by $\Omega' = \omega + i\mathrm{Re}(\Omega)$ and $\omega' = \mathrm{Im}(\Omega)$, the same map $\pi : X' \to S^2$ is a sLag fibration (Harvey-Lawson).

The base B of the fibration $\pi : X^{2n} \to B^n$ is the moduli space of sLag T^n in X. The complement $B^o \subset B$ of discriminant loci carries 2 natural $\mathbb{Z}$-affine structures, which we call symplectic and complex. They are given by ω and $\mathrm{Im}\Omega$ respectively (I will discuss more about these later). For example, the symplectic $\mathbb{Z}$-affine structures are classically known as the action-angle variables. In recent studies, they appear to be more fundamental than complex and symplectic geometries.

Note that a $\mathbb{Z}$-affine structure means that changes of coordinates are in $\mathrm{GL_n}(\mathbb{Z}) \ltimes \mathbb{R}^n$. The symplectic $\mathbb{Z}$-affine structure is classically known as action-angle variables.

Semi-flat mirror symmetry

On the other hand, given a $\mathbb{Z}$-affine manifold B^n, $\exists$ smooth dual T^n-fibrations:

$$\begin{array}{ccccc} TB/\Lambda & & & & T^*B/\Lambda^* \\ & \searrow^{\pi} & & \swarrow^{\pi^\vee} & \\ & & B & & \end{array}$$

Here $\Lambda = \mathbb{Z}\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle$ is the fiberwise lattice generated by the $\mathbb{Z}$-affine coordinates. Then TB/Λ and T^*B/Λ^* are symplectic and complex manifolds respectively. To make them CY manifolds, we need a dual $\mathbb{Z}$-affine structure on B, given by Legendre transform w.r.t. a potential $f : B \to \mathbb{R}$ satisfying the $\mathbb{R}$-Monge-Ampére eq. This is a local model for SYZ mirror symmetry.

Mirror symmetry extends

Mirror symmetry can be generalized to open CY and non-CY manifolds in appropriate manners.

1. toric CY manifolds (Cheng-Klemm-Yau-Zaslow, ...)
2. (quasi-)Fano manifolds (Hori, Auroux, ...)
3. general types (Seidel, Effimov, Gross-Katzarkov-Ruddat, ...)
4. ...

In this talk, I will next focus on mirror symmetry for quasi-Fano manifolds. The SYZ program also extends to these.

Definition A quasi-Fano manifold X is a smooth variety X such that

1. $|-K_X|$ contains a smooth Calabi–Yau member,
2. $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.

A Fano manifold X is a quasi-Fano manifold if for example $\dim X < 3$. Keep in mind the projective line $\mathbb{P}^1$, Del Pezzo surfaces.

We consider a pair of quasi-Fano manifold X and an anti-canonical divisor $Z \in |-K_X|$. The complement $X \setminus Z$ can be thought of as a log CY manifold. $\exists\, \Omega$ nowhere vanishing volume form on $X \setminus Z$ with poles along Z.

Example Let $X = \mathbb{P}^1$ and $Z = \{0, \infty\}$ equipped with $\Omega = \frac{dz}{z}$. More generally, a toric Fano maniofld X and toric boundary Z equipped with $\Omega = \wedge_i \frac{dz_i}{z_i}$ (then $X \setminus Z = (\mathbb{C}^\times)^{\dim X}$).

Definition A Landau-Ginzburg (LG) model is a pair of a Kähler manifold Y and a proper holomorphic function $W : Y \to \mathbb{C}$. W is called a superpotential.

Conjecture

For a pair (X, Z) of a quasi-Fano manifold X and an anti-canonical divisor $Z \in |-K_X|$, $\exists$ a LG model (Y, W) such that

1. $\sum_j h^{d-i+j,j}(X) = h^i(Y, Z^\vee)$,
2. Z and $Z^\vee$ is a mirror CY pair,

where $Z^\vee$ is a generic fiber of W. The pair (X, Z) and (Y, W) is called a mirror pair. (We may have to deform Z to a smooth element.)

Remark A choice of an anti-canonical divisor Z is roughly a choice of a sLag torus fibration, which conjecturally corresponds to a large complex structure limit in the CY case.

Remark An anti-canonical divisor Z is an obstruction for X to be CY and W is an obstruction for Floer homology of a Lagrangian fiber of a sLag fibration L of X to be defined (FOOO, Cho-Oh). Namely, W is obtained as the weighted count of holomorphic discs of Maslov index $\mu(\beta) = 2$ with boundary in L

$$W = \sum_{\beta \in \pi_2(X,L)} n_\beta z_\beta.$$

where z_β is the semi-flat complex coordinate to be defined.

Example Let $X = \mathbb{P}^1$ and $Z = \{0, \infty\}$ equipped with $\Omega = \frac{dz}{z}$. Then the mirror LG model is given by $Y = \mathbb{C}^\times$,

$$W : \mathbb{C}^\times \to \mathbb{C}^1, \;\; z \mapsto z + \frac{q}{z},$$

where $q = \exp(-\int_{\mathbb{P}^1} \omega)$. This is called the Hori-Vafa mirror. One justification of duality, Fano $(\mathbb{P}^1, \{0, \infty\}) \leftrightarrow$ LG $(\mathbb{C}^\times, z + \frac{q}{z})$, is given by the ring isomorphism

$$\mathrm{QCoh}(\mathbb{P}^1) = \mathbb{C}[H^{\pm 1}]/(H^2 - q) \cong \mathbb{C}[z^{\pm 1}]/(z^2 - q) = \mathrm{Jac}(W).$$

Here $\mathrm{QCoh}(\mathbb{P}^1)$ is the quantum cohomology ring of $\mathbb{P}^1$ and $\mathrm{Jac}(W)$ is the Jacobian ring of the superpotential W.

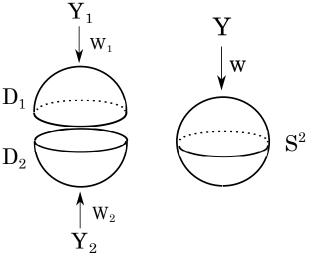
Definition A Tyurin degeneration is degeneration of a CY manifold X to a union $X_1 \cup_Z X_2$ of quasi-Fano manifolds X_1 and X_2 intersecting transversally along smooth $Z \in |-K_{X_i}|$. (This is an analogue of a Heegaard splitting $M^3 = N_+ \cup_\Sigma N_-$).

Conversely, consider a union $X_1 \cup_Z X_2$ of quasi-Fano manifolds X_1 and X_2 intersecting transversally along smooth $Z \in |-K_{X_i}|$. Assume $\exists\, L_i \in \mathrm{Pic}(X_i)$ ample such that $L_1|_Z = L_2|_Z \in \mathrm{Pic}(Z)$ ample.

Theorem (Kawamata-Namikawa) The union $X_1 \cup_Z X_2$ is smoothable to a CY manifold X iff $N_{Z/X_1} \cong N^{-1}_{Z/X_2}$. Such a CY manifold X is unique up to deformation.

Doran-Harder-Thompson Conjecture (2015)

Given a Tyurin degeneration of a Calabi–Yau manifold X to the union $X_1 \cup_Z X_2$ of quasi-Fano manifolds, then the mirror Landau–Ginzburg models $W_i : Y_i \to \mathbb{C}$ of (X_i, Z) can be glued together to form a Calabi–Yau manifold Y equipped with a Calabi–Yau fibration $W : Y \to \mathbb{P}^1$.



This conjecture relates mirror symmetry of CY manifolds and that of quasi-Fano manifolds, which have previously been studied independently.