SUPERSYMMETRIC DEFORMATIONS GAUGED SUPERGRAVITIES AND THE EMBEDDING TENSOR

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Extended gauged supergravities and fluxes

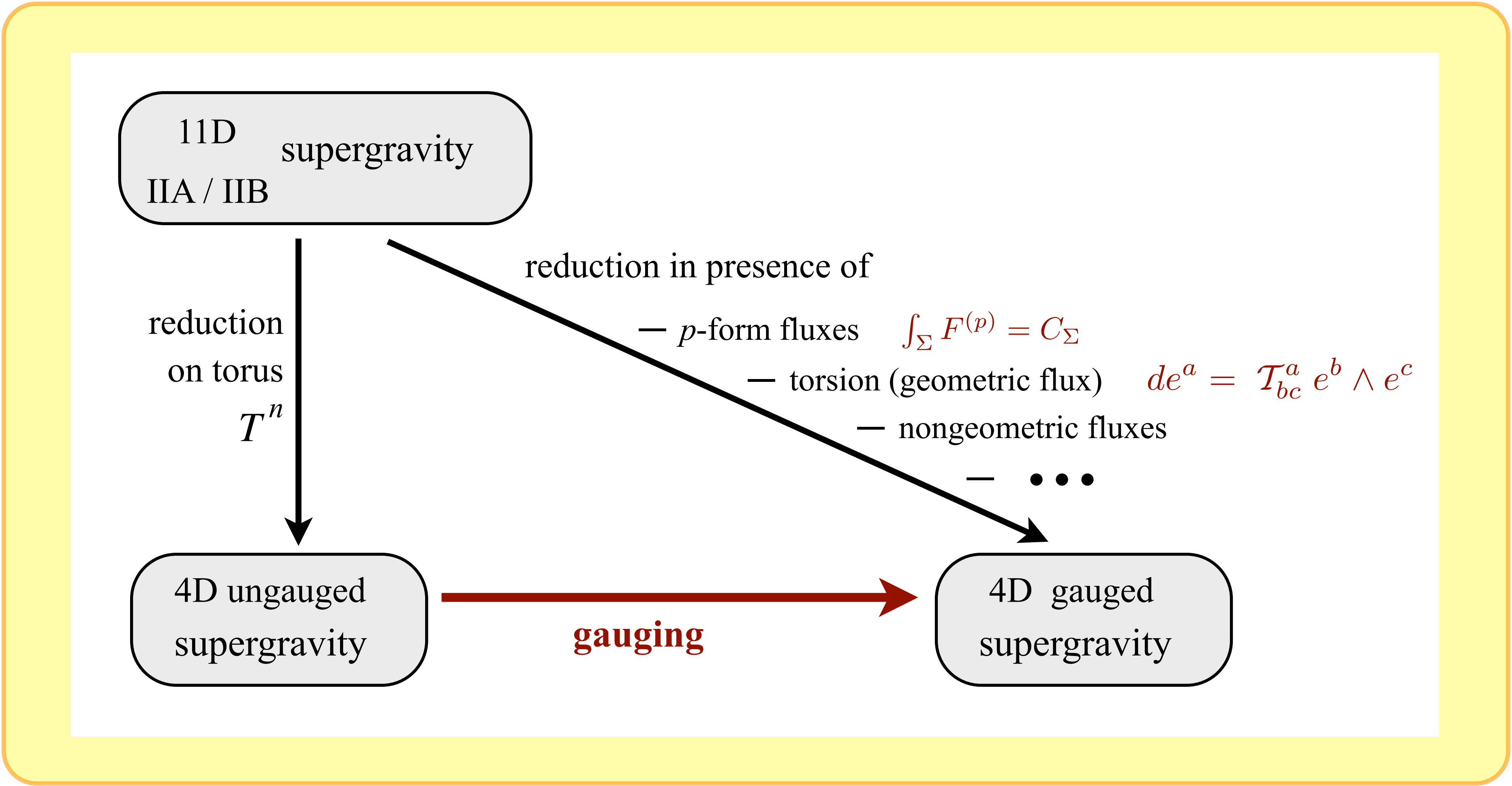
or

Supersymmetric deformations of extended supergravities

deformation parameters:
charges ~ fluxes ~ embedding tensor

They can often be discussed in the context of M-Theory compactifications.

The embedding tensor method is applicable to any field theory with vector fields!



Samtleben, 0808.4076

Truncation of the infinite tower of KK states. The embedding of the gauged theory in the original theory differs from the embedding of the ungauged theory.

The possible gaugings may teach us something about BPS states of M-Theory that are not contained in the supergravity approximation

TOPICS

HIDDEN SYMMETRIES

GAUGING AND GAUGE GROUP EMBEDDINGS

HIERARCHY OF p-FORM FIELDS

THE p-FORM HIERARCHY IN 4 SPACE-TIME DIMENSIONS

MAXIMAL SUPERGRAVITIES

LIFE AT THE END OF THE p-FORM HIERARCHY

HIDDEN SYMMETRIES

The toroidal compactification of pure gravity (Kaluza-Klein)

$$\boxed{\mathcal{M}^D \to \mathcal{M}^d \times T^n} \qquad (D = d + n)$$

$$g_{MN} \quad \to \quad g_{\mu\nu} + A_\mu{}^n + g_{mn}$$

→ massless states: graviton, n gauge fields (KK photons), $\frac{1}{2}n(n+1)$ scalar fields

infinite tower of massive graviton states

resulting theory is invariant under the group $\mathrm{GL}(n)$

non-linearly realized on the scalars: $\boxed{\dfrac{\mathrm{GL}(n)}{\mathrm{SO}(n)}}$

the massive states carry KK photon charges

charge lattice of KK tower: symmetry restricted to $\boxed{\mathrm{GL}(n, \mathbb{Z})}$

Lower space-time dimensions do not follow the generic pattern:

three space-time dimensions: the vector fields can be dualized to scalars (Hodge duality)

massless: graviton (no states), $\frac{1}{2}n(n+3)$ scalars

symmetry non-linearly realized on the scalars $\boxed{\dfrac{\mathrm{SL}(n+1)}{\mathrm{SO}(n+1)}}$

Systematic features of toroidal compactifications:

★ the rank of the invariance group increases with *n*

★ when starting with scalars that parametrize a homogeneous target space, the target space remains homogeneous

★ the presence of the massive states breaks the symmetry group to an arithmetic subgroup

Another example: graviton-tensor theory

the symmetry of the resulting compactified theory depends sensitively on the original theory

$$\mathcal{L}_D = -\tfrac{1}{2}\sqrt{g}\,R - \tfrac{3}{4}\sqrt{g}\big(\partial_{[M}B_{NP]}\big)^2$$

$$\begin{aligned} g_{MN} &\rightarrow g_{\mu\nu} + A_\mu{}^m + g_{mn} \\ B_{MN} &\rightarrow B_{\mu\nu} + B_{m\mu} + B_{mn} \end{aligned}$$

$\Rightarrow$ $\mathrm{G} \subset \mathrm{SO}(n,n;\mathbb{Z})$

$\Rightarrow$ massless states: graviton, tensor, $2n$ spin-1 states, and n^2 spinless states

$\Rightarrow$ tower of massive graviton and tensor states not the generic pattern in five, four and three space-time dimensions !

e.g. upon including a dilaton in the original theory, one finds :

$$\begin{array}{llll} d>5 & : & \mathrm{G}=\mathbb{R}^{+}\times \mathrm{SO}(n,n;\mathbb{Z}) & (n,n)\ \text{vectors} \\ d=5 & : & \mathrm{G}=\mathbb{R}^{+}\times \mathrm{SO}(n,n;\mathbb{Z}) & (n,n)+1\ \text{vectors} \\ d=4 & : & \mathrm{G}=\mathrm{SL}(2;\mathbb{Z})\times \mathrm{SO}(n,n;\mathbb{Z}) & (n,n)+1\ \text{vectors} \\ d=3 & : & \mathrm{G}=\mathrm{SO}(n+1,n+1;\mathbb{Z}) & 0\ \text{vectors} \end{array}$$

→ GOAL: study all possible deformations induced by gauging subgroups of G

The Hodge dilemma:

★ to increase the symmetry ⇒ dualize to lower-rank form fields

★ the presence of certain form fields may be an obstacle to certain gauge groups

★ what to do when the theory contains no (vector) gauge fields

Example: maximal supergravity in 3 space-time dimensions

Nicolai, Samtleben, 2000

gauging versus scalar-vector-tensor duality

128 scalars and *128* spinors, but no vectors !

obtained by dualizing vectors in order to realize the symmetry $E_{8(8)}(\mathbb{R})$

solution:

introduce *248* vector gauge fields with Chern-Simons terms

$$\mathcal{L}_{\rm CS} \propto g\,\varepsilon^{\mu\nu\rho}\,A_\mu{}^M\,\Theta_{MN}\left[\partial_\nu A_\rho{}^N - \frac{1}{3}g\,f_{PQ}{}^N A_\nu{}^P A_\rho{}^Q\right]$$

EMBEDDING TENSOR

vectors ‘invisible’ at the level of the toroidal truncation

First: general analysis of gauge group embeddings.

GAUGING AND GAUGE GROUP EMBEDDINGS

There are restrictions on the possible gaugings

The gauge group must be a subgroup of the full rigid symmetry group of the Lagrangian and/or the equations of motion.

Restrictions follow from the consistency of the combined p-form gauge transformations.

They can also follow from supersymmetry.

The restrictions are subtle!
A gauge group must be a proper subgroup. In spite of that it may nevertheless not be realizeable for a certain (ungauged) Lagrangian.

Hence the field content is important

But also the space-time dimension is relevant.
In particular even and odd dimensions are different

Gauge group embeddings

gauge a subgroup of G, the symmetry group of the ungauged theory

with gauge fields $A_\mu{}^M$ transforming in some representation of G

gauge group encoded into the EMBEDDING TENSOR $\Theta_M{}^\alpha$

$$X_M = \Theta_M{}^\alpha \, t_\alpha$$

gauge group generators ↵ ↳ G generators

$\Theta_M{}^\alpha$ treated as a spurionic quantity,
transforming under the action of G
according to a product representation

dW, Nicolai, Samtleben,
Trigiante, 2000-2008

This representation branches into irreducible representations.
Not all these representations are allowed !!
(for instance, because of supersymmetry)

➟ **Representation (linear) constraint**

EMBEDDING TENSORS FOR MAXIMAL SUPERGRAVITY IN D = 3,4,5,6,7

7	$\mathrm{SL}(5)$	$\mathbf{10} \times \mathbf{24} = \mathbf{10} + \boxed{\mathbf{15}} + \boxed{\overline{\mathbf{40}}} + \mathbf{175}$
6	$\mathrm{SO}(5,5)$	$\mathbf{16} \times \mathbf{45} = \mathbf{16} + \boxed{\mathbf{144}} + \mathbf{560}$
5	$\mathrm{E}_{6(6)}$	$\mathbf{27} \times \mathbf{78} = \mathbf{27} + \boxed{\mathbf{351}} + \overline{\mathbf{1728}}$
4	$\mathrm{E}_{7(7)}$	$\mathbf{56} \times \mathbf{133} = \mathbf{56} + \boxed{\mathbf{912}} + \mathbf{6480}$
3	$\mathrm{E}_{8(8)}$	$\mathbf{248} \times \mathbf{248} = \boxed{\mathbf{1}} + \mathbf{248} + \boxed{\mathbf{3875}} + \mathbf{27000} + \mathbf{30380}$

dW, Samtleben, Trigiante, 2002

$\uparrow$ D $\quad \uparrow$ G $\quad \uparrow$ M $\quad \uparrow$ α

- characterize all possible gaugings
- group-theoretical classification
- universal Lagrangians

➡ Closure (quadratic) constraint

closure: $[X_M, X_N] = f_{MN}{}^P X_P$

$$\underbrace{\Theta_M{}^\beta \Theta_N{}^\gamma f_{\beta\gamma}{}^\alpha}_{-(X_M)_\gamma{}^\alpha} = f_{MN}{}^P \Theta_P{}^\alpha = -\underbrace{\Theta_M{}^\beta t_{\beta N}{}^P}_{X_{MN}{}^P \in \mathrm{g}} \Theta_P{}^\alpha$$

$\Leftrightarrow$ $\Theta_M{}^\alpha$ is invariant under the gauge group

$\Leftrightarrow$ $[X_M, X_N] = X_{MN}{}^P X_P$

$\boxed{X_{MN}{}^P}$ contains the gauge group structure constants, but is in general not symmetric in lower indices, unless contracted with the embedding tensor !!!!

$Z^M{}_{NP} \equiv X_{(NP)}{}^M$ $\qquad$ $Z^M{}_{NP}\,\Theta_M{}^\alpha = 0$

Constraint :

$$\boxed{\mathcal{Q}_{MN}{}^{\alpha} \equiv \delta_M \Theta_N{}^{\alpha} = \Theta_M{}^{\beta}\, \delta_\beta \Theta_N{}^{\alpha}}$$

must vanish!

$$\Longrightarrow \quad \mathcal{Q}_{(MN)}{}^{\alpha} = Z^P{}_{MN}\, \Theta_P{}^{\alpha} = 0$$

Jacobi identity **affected** :

$$X_{[NP}{}^{R}\, X_{Q]R}{}^{M} = \tfrac{2}{3} Z^M{}_{R[N}\, X_{PQ]}{}^{R}$$

in special basis:

$$X_{MN}{}^{P} = \begin{pmatrix} -f_{M*}{}^{*} & \\ & \end{pmatrix}$$

problematic !!

The gauge fields $A_\mu{}^M$ not involved in the gauging can still carry charges.
This is known to be inconsistent ! To see this:

covariant derivative $D_\mu = \partial_\mu - g\, A_\mu{}^M\, X_M$

Ricci identity $[D_\mu, D_\nu] = -g\, \mathcal{F}_{\mu\nu}{}^M\, X_M$

field strength

$$\mathcal{F}_{\mu\nu}{}^M = \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + g\, X_{NP}{}^M\, A_{[\mu}{}^N A_{\nu]}{}^P$$

anti-symmetric part

Palatini identity

$$\delta\mathcal{F}_{\mu\nu}{}^{M} = 2\,D_{[\mu}\delta A_{\nu]}{}^{M} - 2\,g\,Z^{M}{}_{PQ}\,\delta A_{[\mu}{}^{P}A_{\nu]}{}^{Q}$$

NOT covariant indeed !

options:

★ try to enlarge/change the gauge group

or .....

★ introduce an extra gauge transformation $\delta_{\Xi}A_{\mu}{}^{M} = -g\,Z^{M}{}_{NP}\Xi_{\mu}{}^{NP}$

and

introduce 2-form gauge fields $B_{\mu\nu}{}^{MN}$ whose variation cancels the undesirable terms:

$$\boxed{\mathcal{F}_{\mu\nu}{}^{M} \to \mathcal{H}_{\mu\nu}{}^{M} = \mathcal{F}_{\mu\nu}{}^{M} + g\,Z^{M}{}_{NP}B_{\mu\nu}{}^{NP}}$$

$\boxed{Z^{M}{}_{NP}}$ acts as an intertwining tensor between the gauge field representation and the 2-form field representation

subtle: regard (NP) as a single index, which does not map into the full symmetric tensor product !

This leads to, e.g.

$$\begin{aligned}
\delta B_{\mu\nu}{}^{MN} &= 2\,D_{[\mu}\Xi_{\nu]}{}^{MN} - 2\,\Lambda^{\lceil M}\mathcal{H}_{\mu\nu}{}^{N\rfloor} \\
&\quad + 2\,A_{[\mu}{}^{\lceil M}\delta A_{\nu]}{}^{N\rfloor} \\
&\quad - g\,Y^{MN}{}_{P\lceil RS\rfloor}\,\Phi_{\mu\nu}{}^{P\lceil RS\rfloor}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_{\mu\nu\rho}{}^{MN} &= 3\,D_{[\mu}B_{\nu\rho]}{}^{MN} \\
&\quad + 6\,A_{[\mu}{}^{\lceil M}\left(\partial_\nu A_{\rho]}{}^{N\rfloor} + \tfrac{1}{3}gX_{[PQ]}{}^{N\rfloor}A_\nu{}^P A_{\rho]}{}^Q\right) \\
&\quad + g\,Y^{MN}{}_{P\lceil RS\rfloor}\,C_{\mu\nu\rho}{}^{P\lceil RS\rfloor}
\end{aligned}$$

etcetera

where

$\Phi_{\mu\nu}{}^{P\lceil RS\rfloor}$ new gauge parameter

$C_{\mu\nu\rho}{}^{P\lceil RS\rfloor}$ new tensor field

$Y^{MN}{}_{P\lceil RS\rfloor}$ new covariant tensor proportional to the embedding tensor, orthogonal to $Z^M{}_{NP}$

Potentially there are complete *p*-form representations

HIERARCHY OF p-FORM FIELDS

this structure continues indefinitely

$$A_\mu{}^{M} \longrightarrow B_{\mu\nu}{}^{MN} \longrightarrow C_{\mu\nu\rho}{}^{MNP} \longrightarrow \cdots$$

(p-form gauge fields)

$$\Lambda^{M} \longrightarrow \Xi_\mu{}^{MN} \longrightarrow \Phi_{\mu\nu}{}^{MNP} \longrightarrow \cdots$$

(transformation parameters)

$$Z^{M}{}_{NP} \longrightarrow Y^{MN}{}_{PQR} \longrightarrow Y^{MNP}{}_{QRST} \longrightarrow \cdots$$

(intertwining tensors)

The covariant intertwining tensors are all proportional to the embedding tensor and mutually orthogonal.
The intertwining tensors have been determined by induction.

dW, Samtleben, 2005
dW, Nicolai, Samtleben, 2008

Alternative deformations (digression)

An obvious question is whether the gaugings discussed so far are the only viable deformations. While it is true that other deformations are known in supergravity, there are indications that these deformations are already incorporated in the present approach.

$$
\begin{aligned}
\mathcal{H}_{\mu\nu}{}^{M} &= \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + g\, X_{NP}{}^M\, A_{[\mu}{}^N A_{\nu]}{}^P \\
&\quad + g\, Z^M{}_{NP} B_{\mu\nu}{}^{NP} \\
\mathcal{H}_{\mu\nu\rho}{}^{MN} &= 3\, D_{[\mu} B_{\nu\rho]}{}^{MN} \\
&\quad + 6\, A_{[\mu}{}^{\lceil M} \left(\partial_\nu A_{\rho]}{}^{N\rfloor} + \tfrac{1}{3} g X_{[PQ]}{}^{N\rfloor} A_\nu{}^P A_{\rho]}{}^Q \right) \\
&\quad + g\, Y^{MN}{}_{P\lceil RS\rfloor}\, C_{\mu\nu\rho}{}^{P\lceil RS\rfloor}
\end{aligned}
$$

✓ $\mathcal{O}(g^0)$ *: survives* $g = 0$ *limit* *(known from Einstein-Maxwell SG)*

✓ $Z^M{}_{NP}\, \Theta_M{}^\alpha = 0 \Longrightarrow \Theta = 0\,,\ Z \neq 0$

(Romans massive deformation)

At this point there is no Lagrangian yet. (There exist universal Lagrangians!) In the context of a Lagrangian the transformations of the gauge hierarchy are subject to change.

Often the hierarchy breaks off at some point and higher rank forms do not appear in the Lagrangian (projection)

The physical degrees of freedom are shared between the various tensor fields in a way which depends on the embedding tensor.

studied/applied in $D = 2,3,4,5,6,7$ space-time dimensions
in $D=4$, for $N = 0,1,2,4,8$ supergravities
in $D=3$, for $N = 1,\ldots,6,8,9,10,12,16$ supergravities

by e.g.: Bergshoeff, Derendinger, de Vroome, dW, Herger, Hohm, Nicolai, Petropoulos, Ortin, Prezas, Riccione, Samtleben, Schön, Sezgin, Trigiante, Van Proeyen, van Zalk, Weidner, West, Zagermann, etc.
Related work by, e.g.:D’Auria, Ferrara, Hull, Louis, Micu, Reid-Edwards, Sommovigo, Vaula, etc.

Another example: 5 space-time dimensions

42 scalars and 27 vectors, and no tensors !
in order to realize the symmetry $E_{6(6)}^{\rm rigid} \times {\rm USp}(8)^{\rm local}$.

introduce a local subgroup such as $E_{6(6)} \rightarrow {\rm SO}(6)^{\rm local} \times {\rm SL}(2)$

Günaydin, Romans, Warner, 1986

inconsistent!
vectors decompose according to: $\overline{\mathbf{27}} \quad \rightarrow \quad (\mathbf{15},\mathbf{1}) + (\overline{\mathbf{6}},\mathbf{2})$

charged vector fields
must be (re)converted to tensor fields !

- linear constraint follows from supersymmetry:

$$\Theta_M{}^\alpha \in \mathbf{351} \longrightarrow \mathbf{27}\times\mathbf{78} = \cancel{\mathbf{27}} + \mathbf{351} + \cancel{\overline{\mathbf{1728}}}$$

- quadratic constraint follows from closure:

$$(\mathbf{351}\times\mathbf{351})_{\rm s} = \cancel{\overline{\mathbf{27}}} + \cancel{\mathbf{1728}} + \mathbf{351}' + \mathbf{7722} + \mathbf{17550} + \mathbf{34398}$$

dW, Samtleben, Trigiante, 2005

digression:

consider the representations appearing in $(\mathbf{27}\times\mathbf{27})_{\rm s} = (\overline{\mathbf{27}} + \mathbf{351}')$

$$X_{(MN)}{}^{P} = d_{I,MN}\, Z^{P,I} \qquad\qquad d_{MNI} \;:\; E_{6(6)}$$ invariant tensor(s)

two possible representations can be associated with the new index $\begin{cases} \overline{\mathbf{27}} \\ \cancel{\mathbf{351}}' \end{cases}$

$$\overline{\mathbf{27}}\times(\mathbf{27}\times\mathbf{27})_{\rm s} = \boxed{\mathbf{351}} + \mathbf{27} + \mathbf{27} + \overline{\mathbf{351}}' + \overline{\mathbf{1728}} + \overline{\mathbf{7722}}$$

indeed: $(\overline{\mathbf{27}}\times\overline{\mathbf{27}})_{\rm a} = \mathbf{351} \longrightarrow X_{(MN)}{}^{P} = d_{MNQ}\, Z^{PQ}$

anti-symmetric !

from the closure constraint:

$$Z^{MN}\,\Theta_N{}^{\alpha} = 0 \quad \to \quad Z^{MN}\,X_N = 0$$ orthogonality

$$X_{MN}{}^{[P}\,Z^{Q]N} = 0$$ gauge invariant tensor

this structure is generic !

Rather than converting and tensors into vectors and reconverting some of them them when a gauging is switched on, we introduce both vectors and tensors from the start, transforming into the representations $\overline{\mathbf{27}}$ and $\mathbf{27}$, respectively.

extra gauge invariance

$$\delta A_\mu^M = \partial_\mu \Lambda^M - g\, X_{[PQ]}{}^M\, \Lambda^P\, A_\mu^Q - g\, Z^{MN}\, \Xi_{\mu\, N}$$

$$\mathcal{F}_{\mu\nu}{}^M = \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + g\, X_{[NP]}{}^M\, A_\mu{}^N A_\nu{}^P$$

not fully covariant

introduce fully covariant field strength $\boxed{\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + g\, Z^{MN}\, B_{\mu\nu\, N}}$

to compensate for lack of closure:

$$\begin{aligned}\delta B_{\mu\nu\, M} \;=\;& 2\,\partial_{[\mu}\Xi_{\nu]N} - g\, X_{PN}{}^Q\, A_{[\mu}{}^P\, \Xi_{\nu]Q} + g\, Z^{MN}\, \Lambda^P\, X_{PN}{}^Q\, B_{\mu\nu\, Q} \\ & - g\Big(2\, d_{MPQ}\, \partial_{[\mu} A_{\nu]}{}^P - g\, X_{RM}{}^P\, d_{PQS} A_{[\mu}{}^R\, A_{\nu]}{}^S\Big)\Lambda^Q\end{aligned}$$

because of the extra gauge invariance, the degrees of freedom remain unchanged (subtle)

upon switching on the gauging there will be a balanced decomposition of vector and tensor fields

Universal invariant Lagrangian containing
kinetic terms for the tensor fields combined with a
Chern-Simons term for the vector fields

projects higher-*p* gauge transformations

$$
\begin{aligned}
\mathcal{L}_{\rm VT} \;=\; & \tfrac{1}{2} i\varepsilon^{\mu\nu\rho\sigma\tau}\Big\{ g Z^{MN} B_{\mu\nu\,M}\Big[D_\rho B_{\sigma\tau\,N} + 4\, d_{NPQ}\, A_\rho{}^P \Big(\partial_\sigma A_\tau{}^Q + \tfrac{1}{3} g\, X_{[RS]}{}^Q A_\sigma{}^R A_\tau{}^S\Big)\Big] \\
& - \tfrac{8}{3} d_{MNP}\Big[A_\mu{}^M\, \partial_\nu A_\rho{}^N\, \partial_\sigma A_\tau{}^P \\
& + \tfrac{3}{4} g\, X_{[QR]}{}^M\, A_\mu{}^N A_\nu{}^Q A_\rho{}^R \Big(\partial_\sigma A_\tau{}^P + \tfrac{1}{5} g\, X_{[ST]}{}^P A_\sigma{}^S A_\tau{}^T\Big)\Big]\Big\}
\end{aligned}
$$

zeroth order in the coupling constant !

this term is present for ALL gaugings
there is no other restriction than the constraints on
the embedding tensor

dW, Samtleben, Trigiante, 2005

The embedding tensor approach yields universal results for any theory of interest.

Crucial: one works with complete duality representations of all the p-forms. Therefore there is a considerable redundancy of degrees of freedom which are controlled by the extra gauge invariances. There are also (unexpected) additional symmetries in the context of specific actions.

The previous examples concerned odd space-time dimensions. Now we turn to even dimensions and consider D=4.

THE *p*-FORM HIERARCHY IN *4* SPACE-TIME DIMENSIONS

Here the ungauged Lagrangian is not unique because of electric/magnetic duality

Consider with n abelian gauge fields $A_\mu{}^\Lambda$

Field equations & Bianchi identities: $\boxed{\partial_{[\mu}F_{\nu\rho]}{}^\Lambda = 0 = \partial_{[\mu}G_{\nu\rho]\,\Lambda}}$

where $G_{\mu\nu\,\Lambda} = \varepsilon_{\mu\nu\rho\sigma}\,\dfrac{\partial\mathcal{L}}{\partial F_{\rho\sigma}{}^\Lambda}$

2n-component vector of electric and magnetic fields and inductions:

$$G_{\mu\nu}{}^M = \begin{pmatrix} F_{\mu\nu}{}^\Lambda \\ G_{\mu\nu\Lambda} \end{pmatrix}$$

Its rotations leave the field equations and Bianchi identities invariant!

$$\begin{pmatrix} F^{\Lambda} \\ G_{\Lambda} \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{F}^{\Lambda} \\ \tilde{G}_{\Lambda} \end{pmatrix} = \begin{pmatrix} U^{\Lambda}{}_{\Sigma} & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_{\Lambda}{}^{\Sigma} \end{pmatrix} \begin{pmatrix} F^{\Sigma} \\ G_{\Sigma} \end{pmatrix}$$

The equations can be described on the basis of a new Lagrangian provided the rotation matrix is symplectic,

i.e. when it leaves the matrix $\Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$ *invariant.*

The new Lagrangian, which describes equivalent field equations and Bianchi identities, does not follow from straightforward substitution. Instead:

$$\boxed{\tilde{\mathcal{L}}(\tilde{F}) + \tfrac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\,\tilde{F}_{\mu\nu}{}^{\Lambda}\,\tilde{G}_{\rho\sigma\Lambda} = \mathcal{L}(F) + \tfrac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\,F_{\mu\nu}{}^{\Lambda}\,G_{\rho\sigma\Lambda}}$$

"Hamiltonian"

The Lagrangian does not transform as a function: $\tilde{\mathcal{L}}(\tilde{F}) \neq \mathcal{L}(F)$

but $\mathcal{L}(F) + \tfrac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\,F_{\mu\nu}{}^{\Lambda}\,G_{\rho\sigma\Lambda}$ does.

Invariance when $\tilde{\mathcal{L}}(\tilde{F}) = \mathcal{L}(\tilde{F})$

Electric groups $(Z=0)$: $\tilde{F}^{\Lambda}_{\mu\nu} = U^{\Lambda}{}_{\Sigma}\, F_{\mu\nu}{}^{\Sigma}$

then $\mathcal{L}(U^{\Lambda}{}_{\Sigma}\, F^{\Sigma}) = \mathcal{L}(F^{\Lambda}) - \tfrac{1}{8}\varepsilon^{\mu\nu\rho\sigma}(U^{\mathrm{T}}W)_{\Lambda\Sigma}\, F_{\mu\nu}{}^{\Lambda} F_{\rho\sigma}{}^{\Sigma}$

"Peccei-Quinn"

Electric gaugings

$$\delta_{\rm local}\mathcal{L} = \tfrac{1}{8}\,\varepsilon^{\mu\nu\rho\sigma}\,\Lambda^{\Lambda}\,X_{\Lambda\Sigma\Gamma}\,\mathcal{F}_{\mu\nu}{}^{\Sigma}\mathcal{F}_{\rho\sigma}{}^{\Gamma}$$

function of coordinates   non-abelian field strengths

this requires an extra term

$$\mathcal{L}_{\rm top} = \tfrac{1}{3}g\,\varepsilon^{\mu\nu\rho\sigma}\,X_{\Lambda\Sigma\Gamma}\,A_{\mu}{}^{\Lambda}A_{\nu}{}^{\Sigma}(\partial_{\rho}A_{\sigma}{}^{\Gamma} + \tfrac{3}{8}g\,X_{\Xi\Delta}{}^{\Gamma}\,A_{\rho}{}^{\Xi}A_{\sigma}{}^{\Delta})$$

dW, Lauwers, Van Proeyen, 1985

The gauge generators should be consistent with the symplectic property of the electro/magnetic duality transformations:

$$X_{M[N}{}^{Q}\,\Omega_{P]Q} = 0$$

and are subject to a representation (linear) constraint:

$$X_{(MN}{}^{Q}\,\Omega_{P)Q} = 0 \Longrightarrow \begin{cases} X^{(\Lambda\Sigma\Gamma)} = 0 \\ 2X^{(\Gamma\Lambda)}{}_{\Sigma} = X_{\Sigma}{}^{\Lambda\Gamma} \\ X_{(\Lambda\Sigma\Gamma)} = 0 \\ X_{(\Gamma\Lambda)}{}^{\Sigma} = X^{\Sigma}{}_{\Lambda\Gamma} \end{cases}$$

hence, not in general anti-symmetric !

Consider also:

$$X_{(MN)}{}^{P} = Z^{P}{}_{MN} = \tfrac{1}{2}\Omega^{PR}\Theta_{R}{}^{\alpha}\, t_{\alpha M}{}^{Q}\,\Omega_{NQ} = Z^{P,\alpha}\, d_{\alpha MN}$$

This leads to the definitions:

$$d_{\alpha\, MN} \equiv (t_{\alpha})_{M}{}^{P}\,\Omega_{NP}$$

$$Z^{M,\alpha} \equiv \tfrac{1}{2}\Omega^{MN}\Theta_{N}{}^{\alpha} \quad\Longrightarrow\quad \begin{cases} Z^{\Lambda\alpha} = \tfrac{1}{2}\Theta^{\Lambda\alpha} \\ Z_{\Lambda}{}^{\alpha} = -\tfrac{1}{2}\Theta_{\Lambda}{}^{\alpha} \end{cases}$$

magnetic electric

➞ *2-forms transform in adjoint representation*

Quadratic constraint:

$$Z^{M\,\alpha}\,\Theta_{M}{}^{\beta}\, d_{\alpha PQ} = \tfrac{1}{2}\Omega^{MN}\,\Theta_{M}{}^{\beta}\Theta_{N}{}^{\alpha}\, d_{\alpha PQ} = 0$$

Possibly stronger version: $\Omega^{MN}\,\Theta_{M}{}^{\beta}\Theta_{N}{}^{\alpha} = 0$

➞ *there exists a purely electric duality frame!*

The Lagrangian:

1 - Define new electric and magnetic covariant field strengths:

$$\mathcal{H}_{\mu\nu}{}^{M} = \mathcal{F}_{\mu\nu}{}^{M} + gZ^{M,\alpha}B_{\mu\nu\,\alpha}$$

where $B_{\mu\nu\alpha} = d_{\alpha MN}B_{\mu\nu}{}^{MN}$

2 - Include electric and magnetic gauge fields in the covariant derivatives and replace the (electric) field strengths by the modified ones given above.

3 - Add the following term to the Lagrangian:

$$\begin{aligned}\mathcal{L}_{\rm top} \;=\;& \tfrac{1}{8}g\varepsilon^{\mu\nu\rho\sigma}\Theta^{\Lambda\alpha}B_{\mu\nu\alpha}\left(2\partial_\rho A_{\sigma\Lambda} + gX_{MN\Lambda}A_\rho{}^M A_\sigma{}^N - \tfrac{1}{4}g\Theta_\Lambda{}^\beta B_{\rho\sigma\beta}\right)\\ &+ \tfrac{1}{3}g\varepsilon^{\mu\nu\rho\sigma}X_{MN\Lambda}A_\mu{}^M A_\nu{}^N\left(\partial_\rho A_\sigma{}^\Lambda + \tfrac{1}{4}gX_{PQ}{}^\Lambda A_\rho{}^P A_\sigma{}^Q\right)\\ &+ \tfrac{1}{6}g\varepsilon^{\mu\nu\rho\sigma}X_{MN}{}^\Lambda A_\mu{}^M A_\nu{}^N\left(\partial_\rho A_{\sigma\Lambda} + \tfrac{1}{4}gX_{PQ\Lambda}A_\rho{}^P A_\sigma{}^Q\right)\end{aligned}$$

This represents the universal Lagrangian for any gauging. It depends on the embedding tensor whose constraints ensure its full gauge invariance !

4 - In principle the tensor fields can be integrated out. One then finds a conventional Lagrangian with electric gaugings written in an another electric/magnetic duality frame.

MAXIMAL SUPERGRAVITIES

Apply the embedding tensor formalism to the maximal supergravities, with the duality group, the representations of the vector gauge fields and the embedding tensor as input.

At this point, the number of space-time dimensions is not used!

This purely group-theoretic analysis yields all the representations for the hierarchy of *p*-form fields.

Leads to :

rank ⇨		1	2	3	4	5	6
7	$\mathrm{SL}(5)$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\boxed{\mathbf{24}}$	$\overline{\mathbf{15}}+\mathbf{40}$
6	$\mathrm{SO}(5,5)$	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\boxed{\mathbf{45}}$	$\mathbf{144}_s$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
5	$\mathrm{E}_{6(+6)}$	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\boxed{\mathbf{78}}$	$\mathbf{351}$	$\mathbf{27}+\mathbf{1728}$	
4	$\mathrm{E}_{7(+7)}$	$\mathbf{56}$	$\boxed{\mathbf{133}}$	$\mathbf{912}$	$\mathbf{133}+\mathbf{8165}$		
3	$\mathrm{E}_{8(+8)}$	$\boxed{\mathbf{248}}$	$\mathbf{3875}$	$\mathbf{3875}+\mathbf{147250}$			

Striking feature:

rank *D-2* : adjoint representation of the duality group

dW, Samtleben, Nicolai, 2008

note: restricted representation, not the full symmetric tensor product

	rank ⇨	1	2	3	4	5	6
7	$\mathrm{SL}(5)$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\boxed{\overline{\mathbf{15}}+\mathbf{40}}$
6	$\mathrm{SO}(5,5)$	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\mathbf{45}$	$\boxed{\mathbf{144}_s}$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
5	$\mathrm{E}_{6(+6)}$	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$	$\boxed{\mathbf{351}}$	$\mathbf{27}+\mathbf{1728}$	
4	$\mathrm{E}_{7(+7)}$	$\mathbf{56}$	$\mathbf{133}$	$\boxed{\mathbf{912}}$	$\mathbf{133}+\mathbf{8165}$		
3	$\mathrm{E}_{8(+8)}$	$\mathbf{248}$	$\boxed{\mathbf{3875}}$	$\mathbf{3875}+\mathbf{147250}$			

Striking feature:

rank *D-1* : embedding tensor !

rank ⇨		1	2	3	4	5	6
7	$\mathrm{SL}(5)$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}} + \mathbf{40}$
6	$\mathrm{SO}(5,5)$	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\mathbf{45}$	$\mathbf{144}_s$	$\boxed{\mathbf{10}+\mathbf{126}_s+\mathbf{320}}$
5	$\mathrm{E}_{6(+6)}$	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$	$\mathbf{351}$	$\boxed{\mathbf{27}+\mathbf{1728}}$	
4	$\mathrm{E}_{7(+7)}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\boxed{\mathbf{133}+\mathbf{8165}}$		
3	$\mathrm{E}_{8(+8)}$	$\mathbf{248}$	$\mathbf{3875}$	$\boxed{\mathbf{3875}+\mathbf{147250}}$			

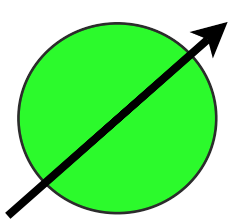
Striking feature:

rank *D* : closure constraint on the embedding tensor !

	rank ⇨	1	2	3	4	5	6
7	SL(5)	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}} + \mathbf{40}$
6	SO(5, 5)	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\mathbf{45}$	$\mathbf{144}_s$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
5	$E_{6(+6)}$	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$	$\mathbf{351}$	$\mathbf{27}+\mathbf{1728}$	
4	$E_{7(+7)}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{133}+\mathbf{8165}$		
3	$E_{8(+8)}$	$\mathbf{248}$	$\mathbf{3875}$	$\mathbf{3875}+\mathbf{147250}$			

Perhaps most striking:

implicit connection between space-time electric/magnetic (Hodge) duality and the U-duality group



Θ dial

Probes new states in M-Theory!

M-theory implications:

		1	2	3	4	5	6
7	$\mathrm{SL}(5)$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}}+\mathbf{40}$
6	$\mathrm{SO}(5,5)$	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\mathbf{45}$	$\mathbf{144}_s$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
5	$\mathrm{E}_{6(+6)}$	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$	$\mathbf{351}$	$\mathbf{27}+\mathbf{1728}$	
4	$\mathrm{E}_{7(+7)}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{133}+\mathbf{8165}$		
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The table coincides substantially with results based on several rather different conceptual starting points:

- M(atrix)-Theory compactified on a torus: duality representations of states
- Correspondence between toroidal compactifications of M-Theory and del Pezzo surfaces
- ***E11*** decompositions

• Algebraic Aspects of Matrix Theory on T^n

Elitzur, Giveon, Kutasov, Rabinovici, 1997

Based on the correspondence between super-Yang-Mills on T^n and M-Theory on $\tilde{T}^n$, a rectangular torus with radii $R_1, R_2, \ldots R_n$ in the infinite-momentum frame.

Invariance group consist of permutations of the R_i combined with the *T*-duality relations ($i \neq j \neq k$) :

$$R_i \to \frac{l_{\mathrm{p}}^3}{R_j R_k} \qquad R_j \to \frac{l_{\mathrm{p}}^3}{R_k R_i} \qquad R_k \to \frac{l_{\mathrm{p}}^3}{R_i R_j} \qquad l_{\mathrm{p}}^3 \to \frac{l_{\mathrm{p}}^6}{R_i R_j R_k}$$

generate a group isomorphic with the Weyl group of $\mathrm{E}_{n(n)}$

The explicit duality multiplets arise as representations of this group.

Example $n=4$ ⟶ $D=7$

4 KK states on T^n $\qquad M \sim \frac{1}{R_i}$

6 2-brane states wrapped on T^n $\qquad M \sim \frac{R_j R_k}{l_{\rm p}^3} \qquad j \neq k$

4 2-brane states wrapped on $T^n \times x^{11}$ $\qquad M \sim \frac{R_{11} R_i}{l_{\rm p}^3}$

1 5-brane state wrapped on $T^n \times x^{11}$ $\qquad M \sim \frac{R_{11} R_1 R_2 R_3 R_4}{l_{\rm p}^6}$

the dimensions of these two multiplets coincide with those of the multiplets presented previously for vectors and tensors

for higher *n* the multiplets are sometimes incomplete, because they are not generated as a single orbit by the Weyl group.

- **A Mysterious Duality** Iqbal, Neitzke, Vafa, 2001

This cannot be a coincidence!

It is important to uncover the physical interpretation of these duality relations. One possibility is that the del Pezzo surface is the moduli space of some probe in M-Theory. It must be a U-duality invariant probe ....

Such probe is the gauging encoded in the embedding tensor!

- ***E11*** **decomposition**

Based on the conjecture that *E11* is the underlying symmetry of M-Theory. Decomposing the relevant *E11* representation to dimensions *D*<*11* yields representations that substantially overlap with those generated for the gaugings.

West et. al., 2001-2007
Bergshoeff et. al.,2005-2007

LIFE AT THE END OF THE *p*-FORM HIERARCHY

		1	2	3	4	5	6
7	$\mathrm{SL}(5)$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}}+\mathbf{40}$
6	$\mathrm{SO}(5,5)$	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\mathbf{45}$	$\mathbf{144}_s$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
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4	$\mathrm{E}_{7(+7)}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{133}+\mathbf{8165}$		
3	$\mathrm{E}_{8(+8)}$	$\mathbf{248}$	$\mathbf{3875}$	$\mathbf{3875}+\mathbf{147250}$			

It is possible to construct the hierarchy starting from the intermediate *(D-3)*-forms, assuming that they transform according to the conjugate of the representation associated with the vector fields. In this way one generates the *(D-2)*-, the *(D-1)*-, and the *D*-form fields, in accordance we the results found in the table. Note that the latter two forms are not related to any other forms by Hodge duality!

p-forms transforming in the conjugate of the representations of the *1*-forms, the adjoint representation, the embedding tensor and the constraints:

$$\Delta \overset{[D-3]}{C}{}_{M} = D \overset{[D-4]}{\Phi}{}_{M} + \cdots$$ DUAL to *1*-forms

$$\Delta \overset{[D-2]}{C}{}_{\alpha} = D \overset{[D-3]}{\Phi}{}_{\alpha} + \cdots$$ DUAL to *0*-forms (adjoint)

$$\Delta \overset{[D-1]}{C}{}^{M}{}_{\alpha} = D \overset{[D-2]}{\Phi}{}^{M}{}_{\alpha} + \cdots$$ DUAL to embedding tensor

$$\Delta \overset{[D]}{C}{}^{MN}{}_{\alpha} = D \overset{[D-1]}{\Phi}{}^{MN}{}_{\alpha} + \cdots$$ DUAL to quadratic constraint

$$\Delta \overset{[D+1]}{C}{}^{PQR}{}_{\alpha} = D \overset{[D]}{\Phi}{}^{PQR}{}_{\alpha} + \cdots$$ not relevant

p-forms transforming in the conjugate of the representations of the *1*-forms, the adjoint representation, the embedding tensor and the constraints:

$$\Delta \overset{[D-3]}{C}{}_{M} = D \overset{[D-4]}{\Phi}{}_{M} + \cdots - \boxed{Y_M{}^{\alpha}} \overset{[D-3]}{\Phi}{}_{\alpha}$$

$$\Delta \overset{[D-2]}{C}{}_{\alpha} = D \overset{[D-3]}{\Phi}{}_{\alpha} + \cdots - \boxed{Y_{\alpha,M}{}^{\beta}} \overset{[D-2]}{\Phi}{}^{M}{}_{\beta}$$

$$\Delta \overset{[D-1]}{C}{}^{M}{}_{\alpha} = D \overset{[D-2]}{\Phi}{}^{M}{}_{\alpha} + \cdots - \boxed{Y^{M}{}_{\alpha,PQ}{}^{\beta}} \overset{[D-1]}{\Phi}{}^{PQ}{}_{\beta}$$

$$\Delta \overset{[D]}{C}{}^{MN}{}_{\alpha} = D \overset{[D-1]}{\Phi}{}^{MN}{}_{\alpha} + \cdots - \boxed{Y^{MN}{}_{\alpha,PQR}{}^{\beta}} \boxed{\overset{[D]}{\Phi}{}^{PQR}{}_{\beta}}$$

$$\Delta \cancel{\overset{[D+1]}{C}{}^{PQR}{}_{\alpha}} = \cancel{D \overset{[D]}{\Phi}{}^{PQR}{}_{\alpha}} + \cdots - \cdots$$

intertwiners

closure constraint

$$\boxed{\mathcal{Q}_{MN}{}^{\alpha} \equiv \delta_M \Theta_N{}^{\alpha} = \Theta_M{}^{\beta}\, \delta_\beta \Theta_N{}^{\alpha}}$$

intertwiners

$$\begin{aligned}
Y_M{}^{\alpha} &= \Theta_M{}^{\alpha} \\
Y_{\alpha,M}{}^{\beta} &= \delta_\alpha \Theta_M{}^{\beta} \\
Y^M{}_{\alpha,PQ}{}^{\beta} &= \frac{\delta}{\delta\, \Theta_M{}^{\alpha}}\, \mathcal{Q}_{PQ}{}^{\beta} \\
Y^{MN}{}_{\alpha,PQR}{}^{\beta} &= -\delta^M_P\, Y^N{}_{\alpha,QR}{}^{\beta} + X_{PQ}{}^{M}\, \delta^N_R \delta^\beta_\alpha + X_{PR}{}^{N}\, \delta^M_Q \delta^\beta_\alpha - X_{P\alpha}{}^{\beta}\, \delta^N_R \delta^M_Q
\end{aligned}$$

Alternative form for the intertwiners

(closer to the generic formulae that follow by induction)

$$
\begin{aligned}
Y_{\alpha,M}{}^{\beta} &= t_{\alpha M}{}^{N}\, Y_N{}^{\beta} - X_M{}^{\beta}{}_{\alpha}\ , \\
Y^{M}{}_{\alpha,PQ}{}^{\beta} &= -\,\delta_P{}^{M}\, Y_{\alpha,Q}{}^{\beta} - (X_P)_Q{}^{\beta,M}{}_{\alpha}\ , \\
Y^{MN}{}_{\alpha,PQR}{}^{\beta} &= -\,\delta_P{}^{M}\, Y^{N}{}_{\alpha,QR}{}^{\beta} - (X_P)_{QR}{}^{\beta,MN}{}_{\alpha}
\end{aligned}
$$

orthogonality:

$$
\boxed{
\begin{aligned}
Y \times Y' &\propto Q_{MN}{}^{\alpha} \\
Y^{MN}{}_{\alpha,PQR}{}^{\beta}\, Q_{MN}{}^{\alpha} &= 0
\end{aligned}
}
$$

What is the role of the higher form fields ?

This construction supports the following idea which has been worked out completely for three and four space-time dimensions:

Regard the embedding tensor as a space-time field transforming in the appropriate representation, but not satisfying the quadratic closure constraint. Add the gauge invariant Lagrangian with *(D-1)*- and *D*-form fields:

$$\begin{aligned}\mathcal{L} \quad = \quad & g\,\varepsilon^{\mu_1\mu_2\cdots\mu_D}\,C_{\mu_1\cdots\mu_{D-1}}{}^{M}{}_{\alpha}\,D_{\mu_D}\Theta_M{}^{\alpha} \\ & + g^2\,\varepsilon^{\mu_1\mu_2\cdots\mu_D}\,C_{\mu_1\cdots\mu_D}{}^{MN}{}_{\alpha}\,Q_{MN}{}^{\alpha}\end{aligned}$$

dW, Samtleben, Nicolai, 2008
dW, van Zalk, 2009

Conclusions

✦ General gaugings of a large variety of theories can be constructed and studied in the framework of the embedding tensor technique, which, in principle, entails a hierarchy of *p*-forms.

✦ Maximal supergravity theories contain subtle information about M-Theory. This may be interpreted as an indication that supergravity needs to be extended towards string/M-theory.