Statistics/Link Invariants of Quantum Matter & Old/New Topological Boundary Conditions

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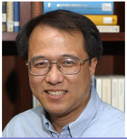



どうもありがとうございました。
IPMUで私の研究成果を発表
することは大変光栄です。
私のセミナーに来
ていただきありがとうございます。

Special thanks to

Masahito, Masaki, Yuji, Kazuya, Rie, and others...

Part I (Wed 07/12: about $1 \sim 1\frac{1}{4}$ hour):
Based on arXiv 1602.05951 and 1612.09298.
(w/ Putrov, Wen, Yau)

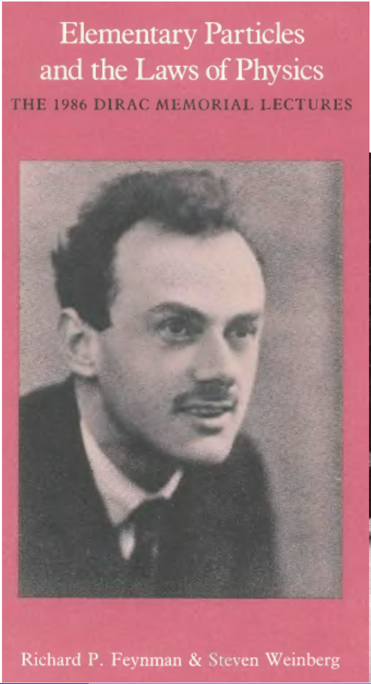
Part II (Wed 07/19: about $1 \sim 1\frac{1}{4}$ hour):
Based on arXiv 1705.06728.
(w/ Wen, Witten)

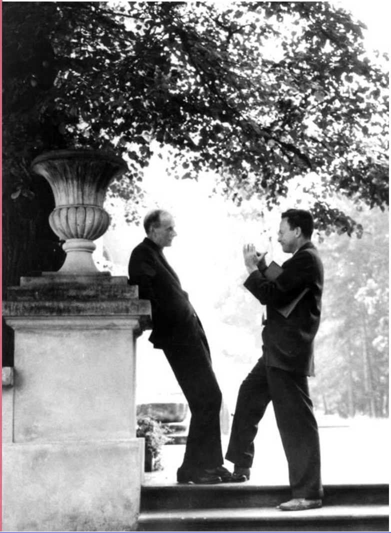
p.s. My Office is at 4F B04 IPMU.

Part I

Braiding Statistics/Link Invariants of Quantum Matter in 2+1 and 3+1D

Based on arXiv 1602.05951 and 1612.09298.
(w/ Putrov, Wen, Yau)


Elementary Particles
and the Laws of Physics
THE 1986 DIRAC MEMORIAL LECTURES
Richard P. Feynman & Steven Weinberg



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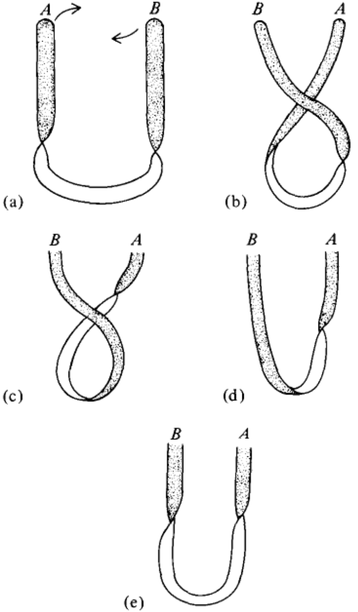

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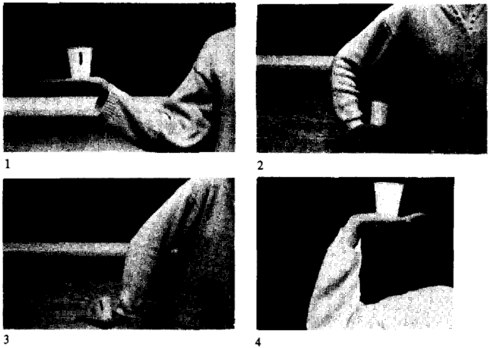
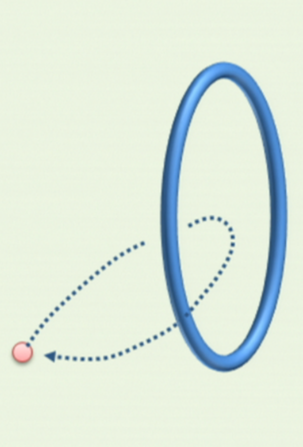
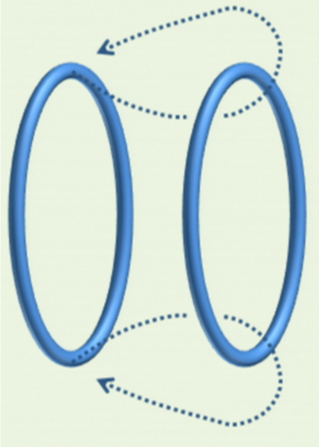


Key Messages :

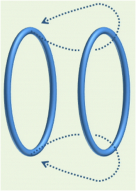
1. In 2+1D, particle(worldline)-particle braiding. In 3+1D, more than particle-string and two string(worldsheet) braidings,





Why do we need a multi-string braiding process?

- i. Two string braiding cannot capture much (no more than particle-loop braiding). No link in 3+1D.



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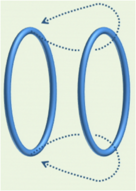
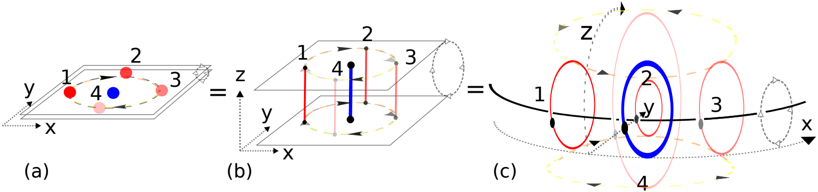


- ii. Dimensional reduction 3+1D $\rightarrow$ 2+1D picture:



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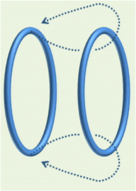

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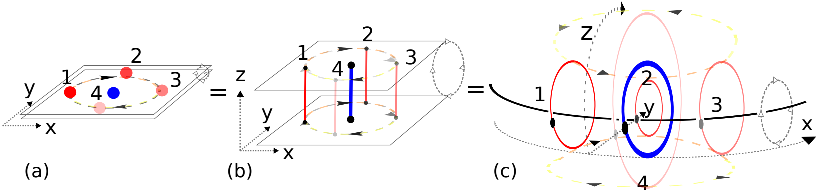
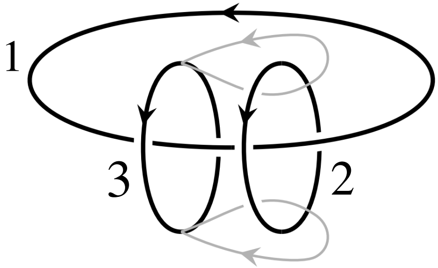


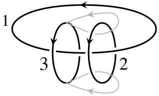
e.g. (de-)Categorification.
Reduce SL(3,$\mathbb{Z}$) Rep to SL(2,$\mathbb{Z}$) Rep.

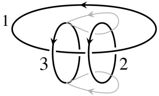
Part 1 talk, 1st Key Message

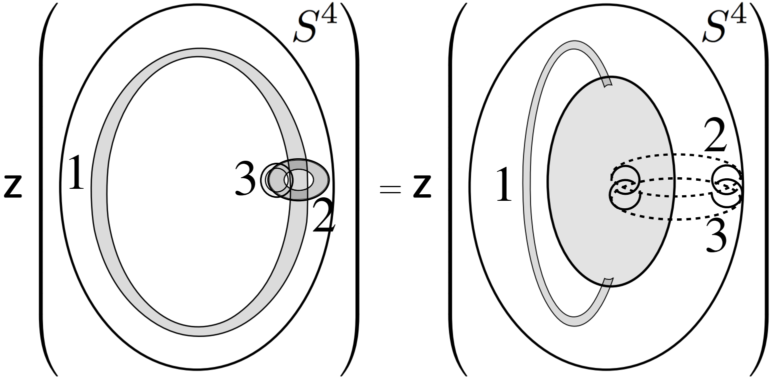
1. Multi-string braiding statistics characterize 3+1D topological orders/TQFTs.

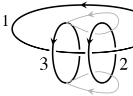


Levin's (1403.7437), Ran's (1404.1062), JW-Wen (1404.7854).

Recall in 3D space:  .

Recall in 3D space:  . In spacetime:



Recall in 3D space: 

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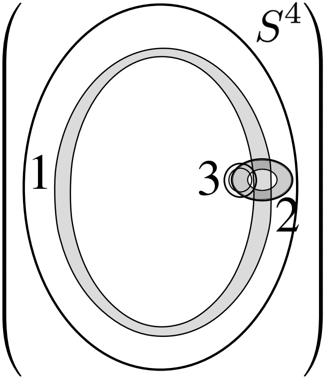
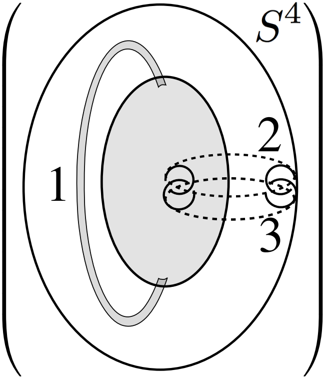
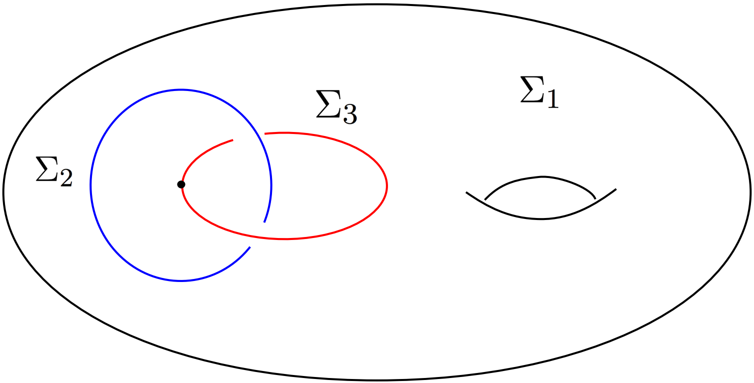
Z 

= Z 

$= \mathbf{Z}(S^4; \mathrm{Tri}[\mu_1, \mu_2, \mu_3])$

$= \mathbf{Z}(S^4; \mathrm{Link}[\mathrm{Spun}[\mathrm{Hopf}[\mu_3, \mu_2]], \mu_1])$


Σ_3
Σ_1
Σ_2

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- Let us recall a Z_2 gauge theory in 2+1D
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- Then we consider a 3+1D TQFTs:

$$\int_{M^4} \sum_{I=1}^{3} \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_1 N_2\ p}{(2\pi)^2 N_{12}} A^1 \wedge A^2 \wedge dA^3$$

- We discuss the field theory first, then the lattice model.

First field theory - compute the Berry phase, path integral, and derive link invariants.

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- Z_N gauge in any dD, an action $S[A, B] = \int_{M^d} \frac{N}{2\pi} B \wedge dA$

$$\langle \Phi \rangle = \frac{1}{Z} \int DA\, DB\ \exp[iS[A, B]]\ \exp[ie \int_{\gamma^1} A] \exp[iq \int_{S^{d-2}} B]$$

$\diamond$ Integrate out B, the EOM imposes: $dA = -\frac{2\pi}{N} q\ \delta^{\perp}(S^{d-2})$.

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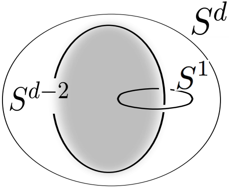
$\diamond$ But $\int_{\gamma^1} A = \int_{M_d} A \wedge \delta^{\perp}(\gamma^1)$. Together we derive

$\langle \Phi \rangle = \exp[-\frac{2\pi i}{N} qe \int_{M_d} \delta^{\perp}(\gamma^1) \wedge \delta^{\perp}(\mathcal{V}^{d-1})]$

$= \exp[-\frac{2\pi i}{N} qe\ \#(\gamma^1 \cap \mathcal{V}^{d-1})]$

$= \exp[-\frac{2\pi i}{N} qe\ \mathrm{Lk}(\gamma^1, S^{d-2})]$.



- Next, a 3+1D TQFTs (Gauge group $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$):

$$S[A,B] = \int_{M^4} \sum_{I=1}^{3} \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_1 N_2\, p}{(2\pi)^2 N_{12}} A^1 \wedge A^2 \wedge dA^3$$

$\diamond$ Gauge transform (Only $\epsilon^{12} = -\epsilon^{21} = 1$):

$$A^I \to A^I + dg^I, \qquad B^I \to B^I + d\eta^I + \frac{N_1 N_2 p}{2\pi N_{12} N_I} \epsilon^{IJ} dg^J \wedge A^3.$$

$\diamond$ Surface operators:

$$W_{123} = \exp\Big(i \sum_{I,J}^{3} q_I \left\{ \int_{\Sigma_I} B^I + \frac{N_1 N_2 p\, \epsilon^{IJ}}{2\pi\, N_{12} N_I} \int_{\mathcal{V}_I} A^J \wedge dA^3 \right\}\Big)$$

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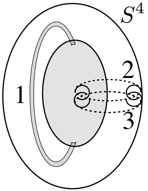
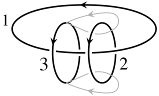
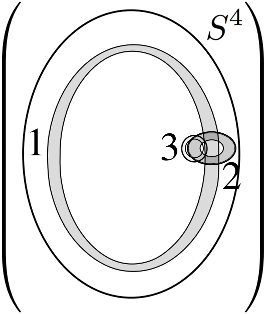
◇ Integrate out B, the EOM imposes: $A^I = -\frac{2\pi q_I}{N_I} \delta^\perp(\mathcal{V}_I)$.

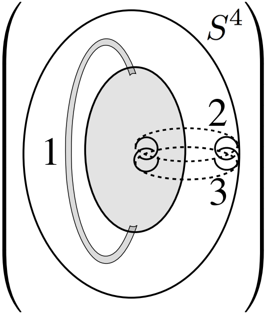
$$\langle W_{123} \rangle = \exp[\frac{2\pi i \, p \, q_1 q_2 q_3}{N_{123}} \#(\mathcal{V}_1 \cap \mathcal{V}_2 \cap \Sigma_3)]$$
$$= \exp[\frac{2\pi i \, p \, q_1 q_2 q_3}{N_{123}} \mathrm{Tlk}(\Sigma_1, \Sigma_3, \Sigma_2)].$$



Recall in 3D space: 

. In spacetime:

Z 

$/\mathbf{Z}[S^4] =$ **Z** 

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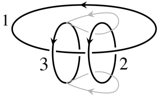
Recall in 3D space: 

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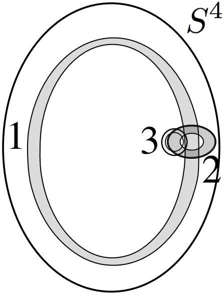
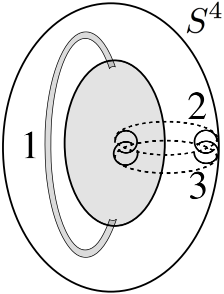
$\mathbf{Z}\Bigg($ 

$\Bigg)/\mathbf{Z}[S^4] = \mathbf{Z}\Bigg($ 

$\Bigg)/\mathbf{Z}[S^4]$

$$= \exp[\frac{2\pi i\, p\, q_1 q_2 q_3}{N_{123}} \#(\mathcal{V}_1 \cap \mathcal{V}_2 \cap \Sigma_3)]$$

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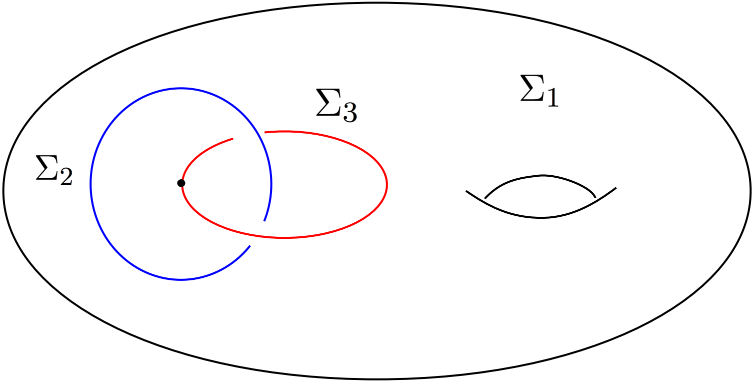
So far we discussed Abelian statistics.
There are also non-Abelian statistics, in 2+1D/3+1D.

HOMEWORK exercise: braiding statistics and link invariants in 2+1D/3+1D.

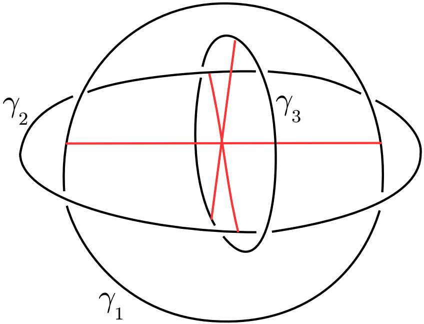
Abelian braiding statistics:

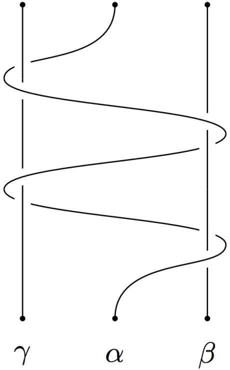


2+1D $\frac{N_I}{2\pi} B^I \wedge dA^I + \frac{p}{(4\pi)} A^1 \wedge dA^2$

$\to$ 3+1D $\frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_{I'} N_{J'}\ p}{(2\pi)^2 N_{I'J'}} A^{I'} \wedge A^{J'} \wedge dA^{K'}$

non-Abelian braiding statistics:

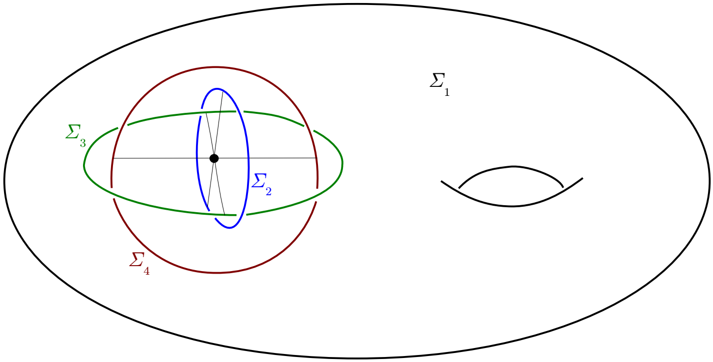




2+1D $\frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_1 N_2 N_3\, p}{(2\pi)^2 N_{123}} A^1 \wedge A^2 \wedge A^3$

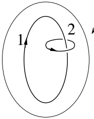
Related work: C.Wang-Levin 1412.1781 and He et al 1608.05393

non-Abelian braiding statistics:

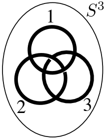


$$2+1\text{D}\ \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_1 N_2 N_3\, p}{(2\pi)^2 N_{123}} A^1 \wedge A^2 \wedge A^3$$
$$\to 3+1\text{D}\ \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_1 N_2 N_3 N_4\, p}{(2\pi)^3 N_{1234}} A^1 \wedge A^2 \wedge A^3 \wedge A^4$$

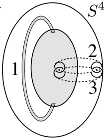
Path integral link $\leftrightarrow$ Field theory action

Z ()

$\leftrightarrow \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{p}{(4\pi)} A^1 \wedge dA^2$

Z ()

$\leftrightarrow \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_1 N_2 N_3 \; p}{(2\pi)^2 N_{123}} A^1 \wedge A^2 \wedge A^3$

Z ()

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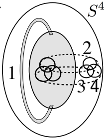
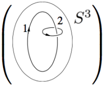
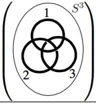
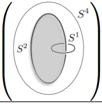
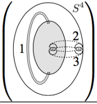
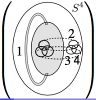
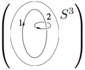
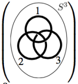
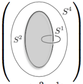
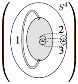
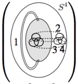
Z ()

$\leftrightarrow \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_1 N_2 N_3 N_4 \; p}{(2\pi)^3 N_{1234}} A^1 \wedge A^2 \wedge A^3 \wedge A^4$

e.g. Continuum field formulation of Dijkgraaf-Witten group cohomology.

(i). Path-integral linking topological invariances; Quantum statistic braiding data	(ii). Group-cohomology class	(iii). TQFT actions characterized by the spacetime-braiding in (i)
	2+1D	
Z 	$\mathcal{H}^3(\prod_{I=1}^{2} \mathbb{Z}_{N_I}, U(1))$	$\int \frac{N_I}{2\pi} B^I \wedge \mathrm{d}A^I + c_{IJ} A^I \wedge \mathrm{d}A^J$
Z 	$\mathcal{H}^3(\prod_{I=1}^{3} \mathbb{Z}_{N_I}, U(1))$	$\int \frac{N_I}{2\pi} B^I \wedge dA^I + c_{123} A^1 \wedge A^2 \wedge A^3$
	3+1D	
Z 	1	$\int \frac{N_I}{2\pi} B^I \wedge \mathrm{d}A^I$
Z 	$\mathcal{H}^4(\prod_{I=1}^{3} \mathbb{Z}_{N_I}, U(1))$	$\int \frac{N_I}{2\pi} B^I \wedge dA^I + c_{IJK} A^I \wedge A^J \wedge \mathrm{d}A^K$
Z 	$\mathcal{H}^4(\prod_{I=1}^{4} \mathbb{Z}_{N_I}, U(1))$	$\int \frac{N_I}{2\pi} B^I \wedge dA^I + c_{1234} A^1 \wedge A^2 \wedge A^3 \wedge A^4$

(i). Path-integral linking topological invariances; Quantum statistic braiding data	(ii). Group-cohomology cocycles distinguished by the braiding in (i)	(iii). TQFT actions characterized by the spacetime-braiding in (i)
2+1D		
Z  $= Z[S^3;\mathrm{Hopf}[\sigma_1,\sigma_2]] = \mathcal{S}_{\bar{\sigma}_1\sigma_2}$	$\exp\left(\frac{2\pi \mathrm{i} p_{IJ}}{N_I N_J}\, a_I(b_J + c_J - [b_J+c_J])\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \mathrm{i} c_{IJ} A^I \wedge dA^J$
Z  $= Z[S^3;\mathrm{BR}[\sigma_1,\sigma_2,\sigma_3]$; Also $Z[T^3_{xyt};\sigma'_{1x},\sigma'_{2y},\sigma'_{3t}]$	$\exp\left(\frac{2\pi \mathrm{i} p_{123}}{N_{123}}\, a_1 b_2 c_3\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \mathrm{i} c_{123} A^1 \wedge A^2 \wedge A^3$
3+1D		
Z  $= \mathrm{L}^{(S^2,S^1)}_{\mu\sigma}$	1	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I$
Z  $= Z[S^4;\mathrm{Link}[\mathrm{Spun}[\mathrm{Hopf}[\mu_3,\mu_2]],\mu_1]] = \mathrm{L}^{\mathrm{Tri}}_{\mu_3,\mu_2,\mu_1}$	$\exp\left(\frac{2\pi \mathrm{i} p_{IJK}}{(N_{IJ}\cdot N_K)}(a_I b_J)(c_K + d_K - [c_K+d_K])\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \mathrm{i} c_{IJK} A^I \wedge A^J \wedge dA^K$
Z  $= Z[S^4;\mathrm{Link}[\mathrm{Spun}[\mathrm{BR}[\mu_4,\mu_3,\mu_2]],\mu_1]]$; Also $Z[T^4 \# S^2 \times S^2;\mu'_4,\mu'_3,\mu'_2,\mu'_1]$	$\exp\left(\frac{2\pi \mathrm{i} p_{1234}}{N_{1234}} a_1 b_2 c_3 d_4\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \mathrm{i} c_{1234} A^1 \wedge A^2 \wedge A^3 \wedge A^4$

What have we learned?

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Some TQFT gauge theories (topological orders).

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We derive their Quantum Statistics (Braiding).

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Some TQFT gauge theories (topological orders).

We derive their Quantum Statistics (Braiding).

Next, we derive some constraints between
Quantum Statistics (Braiding and Fusion)
and
Spacetime Topology

Part 1 talk, 2nd Key Questions :

2. What are the constraints of quantum statistics (the emergent particles for many-body systems)?

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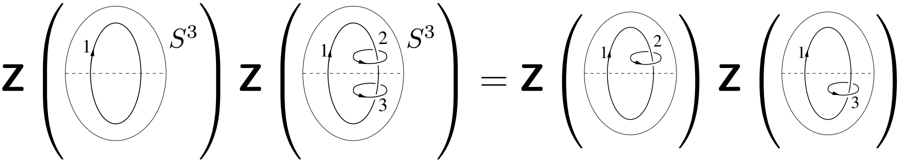
Verlinde's formula (S^1 and S^1 linking in S^3)



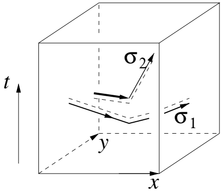
$$\boxed{\mathcal{S}^{\text{line}}_{\bar{\sigma}_1 0}\sum_{\sigma_4}\mathcal{S}^{\text{line}}_{\bar{\sigma}_1\sigma_4}\mathcal{N}^{\sigma_4}_{\sigma_2\sigma_3}=\mathcal{S}^{\text{line}}_{\bar{\sigma}_1\sigma_2}\mathcal{S}^{\text{line}}_{\bar{\sigma}_1\sigma_3}}$$

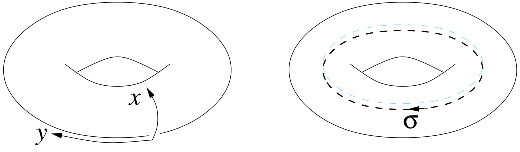
Verlinde, Witten, Moore-Seiberg (1989).

Quantum statistics and spacetime surgery

Motivation: Beyond dimensional reduction and gauge theory.

Key ideas: Spacetime process, quantum amplitudes by partition function $\mathbf{Z} = \mathbf{Z}(\mathcal{M}; \text{worldline } W, \text{worldsheet } V, \text{links})$, and spacetime surgery.





(a) (b)

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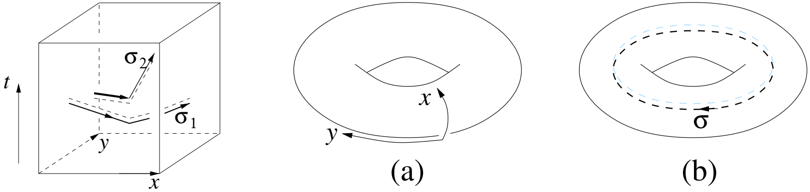

(a) (b)

(a) Solid torus: $D^2_{tx} \times S^1_y$, represents a quantum state: $|0_{D^2_{tx} \times S^1_y}\rangle$.

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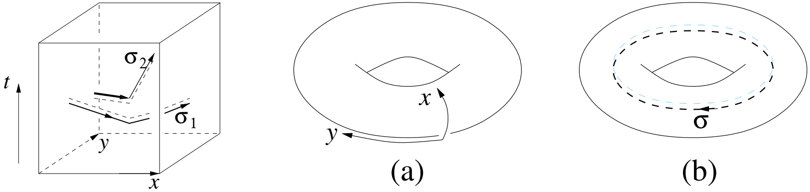


(a) Solid torus: $D^2_{tx} \times S^1_y$, represents a quantum state: $|0_{D^2_{tx} \times S^1_y}\rangle$.

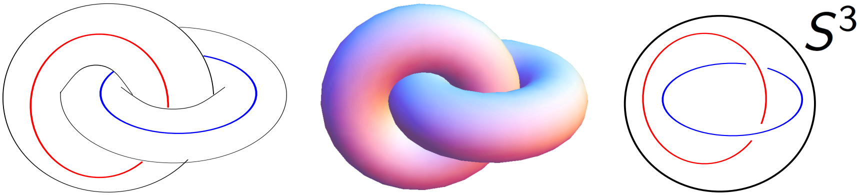
(b) Other ground-states by a worldline operator: $W^y_\sigma |0_{D^2_{tx} \times S^1_y}\rangle$.

$$G^\alpha_\sigma = \langle \alpha | W^y_\sigma | 0_{D^2_{xt} \times S^1_y} \rangle, \quad F^\alpha_{\sigma_1 \sigma_2} = \langle \alpha | W^y_{\sigma_1} W^y_{\sigma_2} | 0_{D^2_{xt} \times S^1_y} \rangle.$$

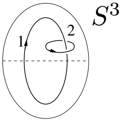
2+1D statistics and surgery: $\mathcal{S}^{xy}$

Glue two solid 2-tori along T^2 by $\hat{\mathcal{S}}^{xy}(x, y) \to (-y, x)$ to 3-sphere:

$$S^3 = D^2_{xt} \times S^1_y \cup_{T^2, \mathcal{S}} D^2_{xt} \times S^1_y.$$



here $\mathbf{Z}(S^3) = \sum_{\alpha\beta} (G_0^\alpha)^* \mathcal{S}_{\alpha\beta} G_0^\beta, \quad T^2 = S^1_x \times S^1_y$

$\mathbf{Z}\Bigg($  $\Bigg)$

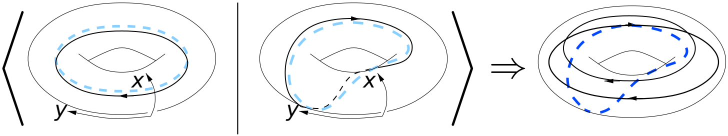
$$= \sum_{\alpha\beta} (G^\alpha_{\sigma_1})^* \mathcal{S}_{\alpha\beta} G^\beta_{\sigma_2} \equiv \mathcal{S}^{\text{line}}_{\bar{\sigma}_1 \sigma_2}.$$

2+1D statistics and surgery: $\mathcal{T}^{xy}$

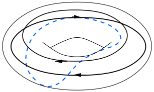
Glue two solid 2-tori by $\hat{\mathcal{T}}^{xy}(x, y) \to (x + y, y)$.

$$S^2 \times S^1 = D^2_{xt} \times S^1_y \cup_{T^2, \mathcal{T}} D^2_{xt} \times S^1_y$$

here $T^2 = S^1_x \times S^1_y$.



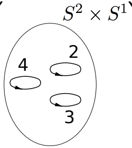
$S^2 \times S^1$

Z () $= \sum_{\alpha\beta} (G^{\alpha}_{\sigma_1})^* \mathcal{T}_{\alpha\beta} G^{\beta}_{\sigma_2} = \delta_{\sigma_1\sigma_2} \mathrm{e}^{\mathrm{i}\theta_{\sigma_2}} = \mathcal{T}_{\sigma_1\sigma_2}$

Two wordlines are not linked. One of them is **twisted by** 2π **spin**.

One more data: Fusion tensor $\mathcal{N}^{a}_{bc}$

$\mathbf{Z}\Bigg($



$$\Bigg) = \langle 0_{D^2_{xt} \times S^1_y} | (W^y_{\sigma_4})^\dagger W^y_{\sigma_2} W^y_{\sigma_3} | 0_{D^2_{xt} \times S^1_y} \rangle$$

$$= \sum_{\beta'} (G^{-1})^{\sigma_4}_{\beta'} F^{\beta'}_{\sigma_2 \sigma_3} \equiv \mathcal{N}^{\sigma_4}_{\sigma_2 \sigma_3}.$$

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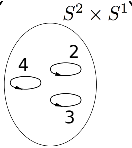
$\mathbf{Z}\Bigg($ 

$\Bigg) = \langle 0_{D^2_{xt}\times S^1_y} | (W^y_{\sigma_4})^\dagger W^y_{\sigma_2} W^y_{\sigma_3} | 0_{D^2_{xt}\times S^1_y} \rangle$

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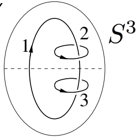
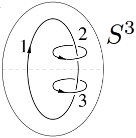
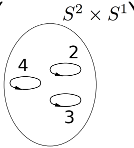
$\mathbf{Z}\Bigg($ 

$\Bigg) = \langle 0_{D^2_{xt}\times S^1_y} | (W^y_{\sigma_1})^\dagger |\alpha\rangle\langle\alpha|\hat{\mathcal{S}}|\beta\rangle\langle\beta| W^y_{\sigma_2} W^y_{\sigma_3} | 0_{D^2_{xt}\times S^1_y} \rangle$

$$= \mathcal{S}^{\text{line}}_{\bar{\sigma}_1\sigma_4} \tilde{\mathcal{N}}^{\sigma_4}_{\sigma_2\sigma_3}$$

Compare to Witten's approach (1989):

$\mathbf{Z}\Big($ 

$\Big) = \mathcal{S}_{\bar{\sigma}_1\sigma_4}\mathbf{Z}\Big($ 

$\Big) = \mathcal{S}_{\bar{\sigma}_1\sigma_4}\mathcal{N}^{\sigma_4}_{\sigma_2\sigma_3}.$

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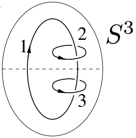
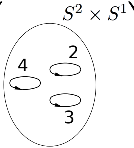
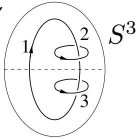
$$\mathbf{Z}\left(\right.$$


$$\left.\right) = \mathcal{S}_{\bar{\sigma}_1\sigma_4}\mathbf{Z}\left(\right.$$


$$\left.\right) = \mathcal{S}_{\bar{\sigma}_1\sigma_4}\mathcal{N}^{\sigma_4}_{\sigma_2\sigma_3}.$$

Our approach:

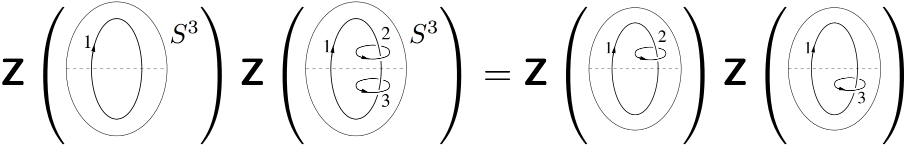
$$\mathbf{Z}\left(\right.$$


$$\left.\right) = \langle 0_{D^2_{xt}\times S^1_y}|(W^y_{\sigma_1})^\dagger|\alpha\rangle\langle\alpha|\hat{\mathcal{S}}|\beta\rangle\langle\beta|W^y_{\sigma_2}W^y_{\sigma_3}|0_{D}$$

$$= \sum_{\alpha\beta}(G^\alpha_{\sigma_1})^*\mathcal{S}_{\alpha\beta}G^\beta_{\sigma_4}\sum_{\beta'}(G^{-1})^{\sigma_4}_{\beta'}F^{\beta'}_{\sigma_2\sigma_3} = \mathcal{S}^{\text{line}}_{\bar{\sigma}_1\sigma_4}\tilde{\mathcal{N}}^{\sigma_4}_{\sigma_2\sigma_3}$$

2+1D statistics and surgery: constraints on fusion $\mathcal{N}_{ij}^{k}$ and braiding statistics $\mathcal{S}_{ab}$

S^1 and S^1 linking in S^3. Def: $\mathcal{N}_{\sigma_2\sigma_3}^{\sigma_4} \equiv \sum_\beta (G^{-1})_\beta^{\sigma_4} F_{\sigma_2\sigma_3}^{\beta}$.



$$\sum_{\alpha\beta} (G_{\sigma_1}^{\alpha})^* \mathcal{S}_{\alpha\beta} G_0^{\beta} \sum_{\alpha\beta} (G_{\sigma_1}^{\alpha})^* \mathcal{S}_{\alpha\beta} F_{\sigma_2\sigma_3}^{\beta} = \sum_{\alpha\beta} (G_{\sigma_1}^{\alpha})^* \mathcal{S}_{\alpha\beta} G_{\sigma_2}^{\beta} \sum_{\alpha\beta} (G_{\sigma_1}^{\alpha})^* \mathcal{S}_{\alpha\beta} G_{\sigma_3}^{\beta}$$

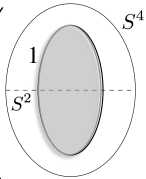
$$\boxed{\mathcal{S}_{\bar\sigma_1 0}^{\text{line}} \sum_{\sigma_4} \mathcal{S}_{\bar\sigma_1\sigma_4}^{\text{line}} \mathcal{N}_{\sigma_2\sigma_3}^{\sigma_4} = \mathcal{S}_{\bar\sigma_1\sigma_2}^{\text{line}} \mathcal{S}_{\bar\sigma_1\sigma_3}^{\text{line}}}.$$

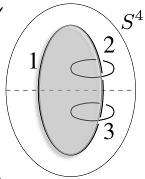
3+1D statistics and spacetime surgery I

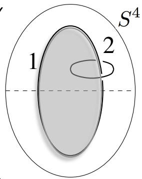
Glue two 4-manifolds along $S^2 \times S^1$ to 4-sphere S^4:

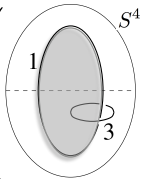
$$S^4 = (D^3 \times S^1) \cup_{S^2 \times S^1} (S^2 \times D^2).$$

One S^2 and two S^1 Linkings in S^4.

Z (

) **Z** (

) = **Z** (

) **Z** (

) .

$$\boxed{\mathrm{L}^{\mathsf{Lk}(S^2,S^1)}_{\mu_1 0} \sum_{\sigma_4} \mathrm{L}^{\mathsf{Lk}(S^2,S^1)}_{\mu_1 \sigma_4} (F^{S^1})^{\sigma_4}_{\sigma_2 \sigma_1} = \mathrm{L}^{\mathsf{Lk}(S^2,S^1)}_{\mu_1 \sigma_2} \mathrm{L}^{\mathsf{Lk}(S^2,S^1)}_{\mu_1 \sigma_3}}.$$

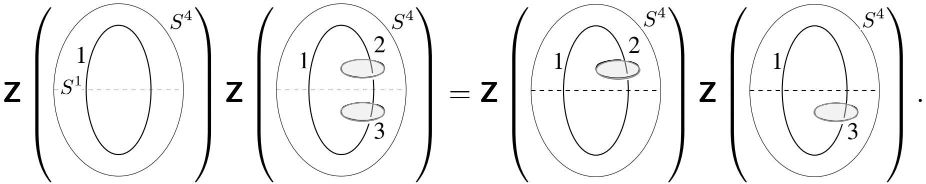
We derive generalized Verlinde's formulas (2015).

3+1D statistics and spacetime surgery II

Glue two 4-manifolds along $S^2 \times S^1$ to 4-sphere S^4:

$$S^4 = (D^3 \times S^1) \cup_{S^2 \times S^1} (S^2 \times D^2).$$

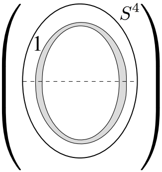
One S^1 and two S^2 Linkings in S^4.

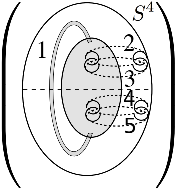


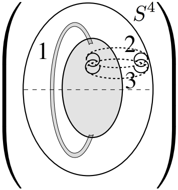
$$\boxed{\mathrm{L}^{\mathsf{Lk}(S^2,S^1)}_{0\sigma_1} \sum_{\sigma_4} \mathrm{L}^{\mathsf{Lk}(S^2,S^1)}_{\mu_4\sigma_1} (F^{S^2})^{\mu_4}_{\mu_2\mu_3} = \mathrm{L}^{\mathsf{Lk}(S^2,S^1)}_{\mu_2\sigma_1} \mathrm{L}^{\mathsf{Lk}(S^2,S^1)}_{\mu_3\sigma_1}}.$$

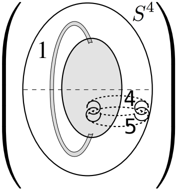
We derive generalized Verlinde's formulas (2015).

3+1D statistics and surgery:

Z 

Z 

$= Z$ 

Z 

$$\Rightarrow \boxed{\mathrm{L}^{\text{Tri}}_{0,0,\mu_1} \cdot \sum_{\Gamma,\Gamma',\Gamma_1,\Gamma'_1} (F^{T^2})^{\Gamma}_{\Gamma'_2,\Gamma'_4} (\mathcal{S}^{xyz})^{-1}_{\Gamma',\Gamma} (F^{T^2})^{\Gamma_1}_{\mu_1\Gamma'} \mathcal{S}^{xyz}_{\Gamma'_1,\Gamma_1} \; \mathrm{L}^{\text{Tri}}_{0,0,\Gamma'_1}}$$

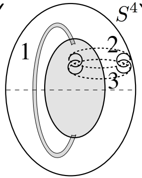
$$= \boxed{\sum_{\Gamma''_2,\eta_2,\eta'_2} (\mathcal{S}^{xyz})^{-1}_{\Gamma''_2,\Gamma'_2} (F^{T^2})^{\eta_2}_{\mu_1\Gamma''_2} \mathcal{S}^{xyz}_{\eta'_2,\eta_2} \; \mathrm{L}^{\text{Tri}}_{0,0,\eta'_2}}$$

$$\boxed{\cdot \sum_{\Gamma''_4,\eta_4,\eta'_4} (\mathcal{S}^{xyz})^{-1}_{\Gamma''_4,\Gamma'_4} (F^{T^2})^{\eta_4}_{\mu_1\Gamma''_4} \mathcal{S}^{xyz}_{\eta'_4,\eta_4} \; \mathrm{L}^{\text{Tri}}_{0,0,\eta'_4}}$$

We derive some generalized Verlinde's formulas (2015).

Require back-and-forth twice surgery between

$$S^4 \overset{\text{surgery}}{\longleftrightarrow} S^3 \times S^1 \# S^2 \times S^2.$$

$\mathbf{Z}$ (

) $\equiv \mathrm{L}^{\mathsf{Tri}}_{\mu_3,\mu_2,\mu_1}$

$$= \mathbf{Z}[S^4; \mathrm{Link}[\mathrm{Spun}[\mathrm{Hopf}[\mu_3,\mu_2]],\mu_1]]$$

$$= \sum_{\mu_3',\Gamma_2,\Gamma_2',\Gamma_2'',\eta_2,\eta_2'} \mathcal{S}^{xyz}_{\mu_3',\mu_3} (\mathcal{F}^{T^2})^{\Gamma_2}_{\mu_2\mu_3'} (\mathcal{S}^{xyz})^{-1}_{\Gamma_2',\Gamma_2}$$

$$(\mathcal{S}^{xyz})^{-1}_{\Gamma_2'',\Gamma_2'} (\mathcal{F}^{T^2})^{\eta_2}_{\mu_1\Gamma_2''} \mathcal{S}^{xyz}_{\eta_2',\eta_2}\ \mathrm{L}^{\mathsf{Tri}}_{0,0,\eta_2'}$$

What have we learned?

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Some TQFT gauge theories (topological orders).

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Some TQFT gauge theories (topological orders).

There are some connections between our previous

TQFT gauge theories (topological orders: TOs)

$\uparrow$

(gauging)

$\downarrow$

Symmetric protected topological states (SPTs)
(e.g. Topological Insulator/Superconductors)
(global symmetry)

So far we discussed low energy field theory from fundamental bosons. Dijkgraaf-Witten theory outputs bosonic TQFTs.

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Next we apply the idea that

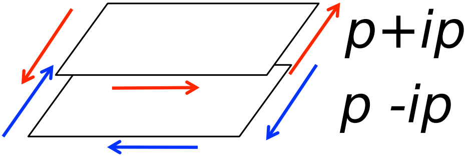
$$\boxed{\text{fermionic SPTs (global sym.)} \xrightarrow{\text{gauging}} \text{spin TQFTs}}$$

(fermionic gauge theory).

We consider fermionic Topological Superconductor (fTSC), that has onsite Z_2-Ising symmetry and Z_2^f fermionic parity symmetry ($Z_2^f \times Z_2$-fTSC as fSPTs).

Fermionic Topological Superconductor ($\nu \in \mathbb{Z}_8$-fTSC)

$Z_2^f \times Z_2$-fTSC as fSPTs with Z_2-Ising symmetry and Z_2^f fermionic parity symmetry.

$\nu = 1$: 

with Majorana-Weyl helical edge modes, combined non-chiral.

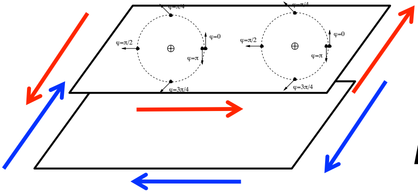
Left-right central charges $(c_L, c_R) = (\frac{1}{2}, -\frac{1}{2})$.

$\diamond$ DIII class ($T^2 = -1$, Pin^+) with onsite Z_2 symmetry, or DIII class with Reflection symmetry.

Fermionic Topological Superconductor ($\nu \in \mathbb{Z}_8$-fTSC)

Dynamical Z_2 fermionic spin TQFT, obtained by gauging $Z_2^f \times Z_2$-fTSC as fSPTs of Z_2-Ising symmetry.

$\nu = 1$:



◇ Z_2-gauge field is the difference of spin structures (form affine space over $H^1(M^3, Z_2)$).

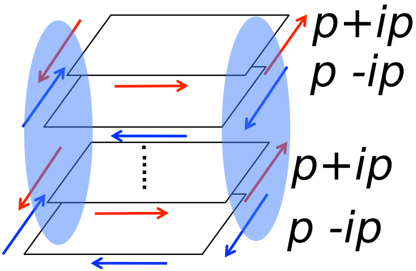
◇ Path integral Z: sum over Z_2-gauge fields (diff of spin structures).

Related physics:
(i) Dynamical (condensing) 1/2-quantum vortex of chiral p wave SC.

(ii) Moore-Read (Pfaffian) quantum Hall state

Fermionic Topological Superconductor ($\nu \in \mathbb{Z}_8$-fTSC)

$Z_2^f \times Z_2$-fTSC as fSPTs with Z_2-Ising symmetry and Z_2^f fermionic parity symmetry.

$\nu \in \mathbb{Z}_8$: 

with Majorana-Weyl edge modes, combined non-chiral.

Left-right central charges $(c_L, c_R) = (\frac{\nu}{2}, -\frac{\nu}{2})$ mod 4.

$\diamond$ fTSC partition function written as ABK (Arf-Brown-Kervaire) and Rokhlin invariants.

Fermionic Topological Superconductor ($\nu \in \mathbb{Z}_8$-fTSC)

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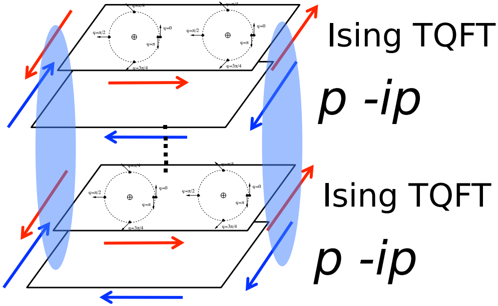

◇ Path integral Z: sum over Z_2-gauge fields (diff of spin structures).

◇ What are the spin TQFTs?

ν	TQFT description (Local action)	$\mathrm{GSD}_{T^2_o \mid T^2_e}$	$\mathcal{S}^{xy}$	$\mathcal{T}^{xy}$
0	level 2 BF theory $\cong$ level $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ $U(1)^2$ CS $\cong$ $\mathbb{Z}_2$-toric code	4b | 4b	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
1	Ising $\times$ $\overline{p - ip}$ $\cong$ Ising $\times$ $\overline{\text{spin-Ising}}$ $\cong$ $U(2)_{2,-4}$ $\times$ $(SO(3)_{-1} \times U(1)_1)$ CS	3f | 3b	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{\frac{\pi i}{8}} \end{pmatrix}$
2	level $K = \begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}$ $U(1)^2$ CS	4b | 4b	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{i}{2} & -\frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{i}{2} & \frac{i}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\frac{i\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-\frac{3}{4}i\pi} \end{pmatrix}$
3	$SU(2)_2 \times SO(3)_{-1}$ CS	3f | 3b	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{\frac{3\pi i}{8}} \end{pmatrix}$
4	level $K = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$ $U(1)^2$ CS $\cong$ $\mathbb{Z}_2$-double semions	4b | 4b	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$
5	$SU(2)_{-2} \times SO(3)_1$ CS	3f | 3b	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{-\frac{3\pi i}{8}} \end{pmatrix}$
6	level $K = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$ $U(1)^2$ CS	4b | 4b	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{i}{2} & \frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{i}{2} & -\frac{i}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-\frac{i\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{\frac{3}{4}i\pi} \end{pmatrix}$
7	$\overline{\text{Ising}}$ $\times$ $p + ip$ $\cong$ $\overline{\text{Ising}}$ $\times$ spin-Ising $\cong$ $U(2)_{-2,4}$ $\times$ $(SO(3)_1 \times U(1)_{-1})$ CS	3f | 3b	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{-\frac{\pi i}{8}} \end{pmatrix}$

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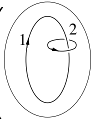
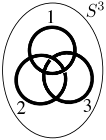
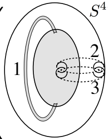
Path integral link $\leftrightarrow$ Field theory action

Z 

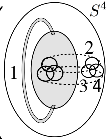
$\leftrightarrow \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{p}{(4\pi)} A^1 \wedge dA^2$

Z 

$\leftrightarrow \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_1 N_2 N_3\ p}{(2\pi)^2 N_{123}} A^1 \wedge A^2 \wedge A^3$

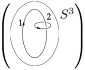
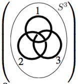
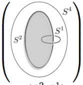
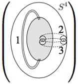
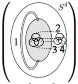
Z 

$\leftrightarrow \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_{I'} N_{J'}\ p}{(2\pi)^2 N_{I'J'}} A^{I'} \wedge A^{J'} \wedge dA^{K'}$

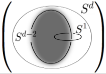
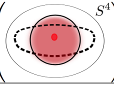
Z 

$\leftrightarrow \frac{N_I}{2\pi} B^I \wedge dA^I + \frac{N_1 N_2 N_3 N_4\ p}{(2\pi)^3 N_{1234}} A^1 \wedge A^2 \wedge A^3 \wedge A^4$

e.g. Dijkgraaf-Witten path integral **Z** &
Twisted generalization of quantum double models $\hat{H}$ (Kitaev, ...)

(i). Path-integral linking topological invariances; Quantum statistic braiding data	(ii). Group-cohomology cocycles distinguished by the braiding in (i)	(iii). TQFT actions characterized by the spacetime-braiding in (i)
	2+1D	
Z  $= Z[S^3; \text{Hopf}[\sigma_1, \sigma_2]]$ $= \mathcal{S}_{\bar{\sigma}_1 \sigma_2}$	$\exp\left(\frac{2\pi \mathrm{i} p_{IJ}}{N_I N_J}\, a_I(b_J + c_J - [b_J + c_J])\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \mathrm{i} c_{IJ} A^I \wedge dA^J$
Z  $= Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3]$; Also $Z[T^3_{xyt}; \sigma'_{1x}, \sigma'_{2y}, \sigma'_{3t}]$	$\exp\left(\frac{2\pi \mathrm{i} p_{123}}{N_{123}}\, a_1 b_2 c_3\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \mathrm{i} c_{123} A^1 \wedge A^2 \wedge A^3$
	3+1D	
Z  $= \mathrm{L}^{(S^2, S^1)}_{\mu\sigma}$	1	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I$
Z  $= Z[S^4; \text{Link}[\text{Spun}[\text{Hopf}[\mu_3, \mu_2]], \mu_1]]$ $= \mathrm{L}^{\text{Tri}}_{\mu_3, \mu_2, \mu_1}$	$\exp\left(\frac{2\pi \mathrm{i} p_{IJK}}{(N_{IJ} \cdot N_K)} (a_I b_J)(c_K + d_K - [c_K + d_K])\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \mathrm{i} c_{IJK} A^I \wedge A^J \wedge dA^K$
Z  $= Z[S^4; \text{Link}[\text{Spun}[\text{BR}[\mu_4, \mu_3, \mu_2]], \mu_1]]$; Also $Z[T^4 \# S^2 \times S^2; \mu'_4, \mu'_3, \mu'_2, \mu'_1]$	$\exp\left(\frac{2\pi \mathrm{i} p_{1234}}{N_{1234}} a_1 b_2 c_3 d_4\right)$	$\int \frac{\mathrm{i} N_I}{2\pi} B^I \wedge dA^I + \mathrm{i} c_{1234} A^1 \wedge A^2 \wedge A^3 \wedge A^4$

2. We obtain **lattice realizations** (UV-complete fully regularized, anomaly-free) with low energy IR **field theories**? Application to characterize and classify phases of cond-mat.? Yes.

	Any dimensions	
$\int \frac{N_I}{2\pi} B^I dA^I$ (Aharonov-Bohm) linking number $\mathrm{Lk}(S^{d-2}_m, C^1_n)$	Z () $/Z[S^d]$ $\exp[-\frac{2\pi i}{N} q_m e_n \mathrm{Lk}(S^{d-2}_m, C^1_n)]$	$\mathbb{Z}_N$ topological order $\mathbb{Z}_N$ spin liquid $\mathbb{Z}_N$ toric code $\mathbb{Z}_N$ gauge theory
	2+1D	
$\frac{\pi}{4} \int a \cup \mathrm{ABK}$ Arf invariant	Z () $/Z[S^3]$ ± 1	Gauged $\mathbb{Z}_2^f \times \mathbb{Z}_2$-fSPT model
$\pi \int a_1 \cup a_2 \cup \eta$ Sato-Levine invariant	Z () $/Z[S^3]$ ± 1	Gauged $\mathbb{Z}_2^f \times (\mathbb{Z}_2)^2$-fSPT model
	3+1D	
$\int \frac{N_I}{2\pi} B^I dA^I + \frac{p_{IJ} N_I N_J}{4\pi N_{IJ}} B^I B^J$ Intersection number of surfaces: $\#(\Sigma_I \cap \Sigma_J)$	Z () $/Z[S^4]$ $\exp(-\frac{\pi i p_{IJ} e_I e_J}{N_{IJ}} \#(\Sigma_I \cap \Sigma_J))$	Gauged SPT lattice model, Walker-Wang like model. (Spin TQFT for $p_{II} \in$ odd.)

Part I - Conclusion

1. String and particle quantum statistics of fusion and braiding in 2+1D and 3+1D. Modular $\mathrm{SL}(3, \mathbb{Z})$ representation of $\mathcal{S}^{xyz}$ and $\mathcal{T}^{xy}$ is derived. New link invariants and more for 3+1D topological order.

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5. Study 4d (3+1D) topological phases and 3d boundary physics. Condensed matter application.

どうもありがとうございました。