# Dilaton Cosmology and their Gauge Theory Duals

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# Singularities and Holography

- Near singularities, the equations of General Relativity or their cousin – supergravity – break down.
- What replaces them ?
- Holographic correspondences which arise from String Theory provide a possible clue.
- Here, gravity in the bulk is an approximate description of possibly a more fundamental non-gravitational theory in lower number of dimensions.
- Question : can we use this description to ask what happens near singularities ?
- This talk : use the AdS/CFT correspondence.

- In this talk I will discuss one approach to understand this problem.
- There are several other approaches –

Craps, Hertog, Turok

Horowitz, Lawrence, Silverstein

The basic approach used in our work has been developed in some earlier papers,

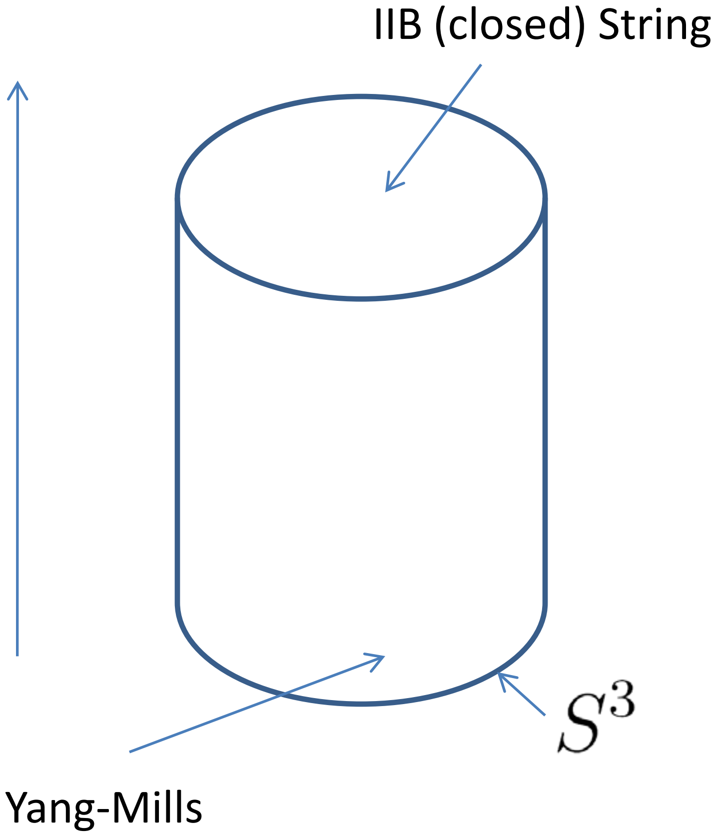
S.R.D., J. Michelson, K. Narayan and S. Trivedi, PRD 74:026002,2006.

A. Awad, S.R.D., K. Narayan and S. Trivedi, PRD 77:046008,2008.

# The Basic Setup

- Simplest setting – IIB string theory in $AdS_5 \times S^5$ dual to a $\mathcal{N}=4 \;\; SU(N)$ Yang-Mills theory living on the boundary.

$$\left(\frac{R}{l_s}\right)^4 = 4\pi\lambda$$



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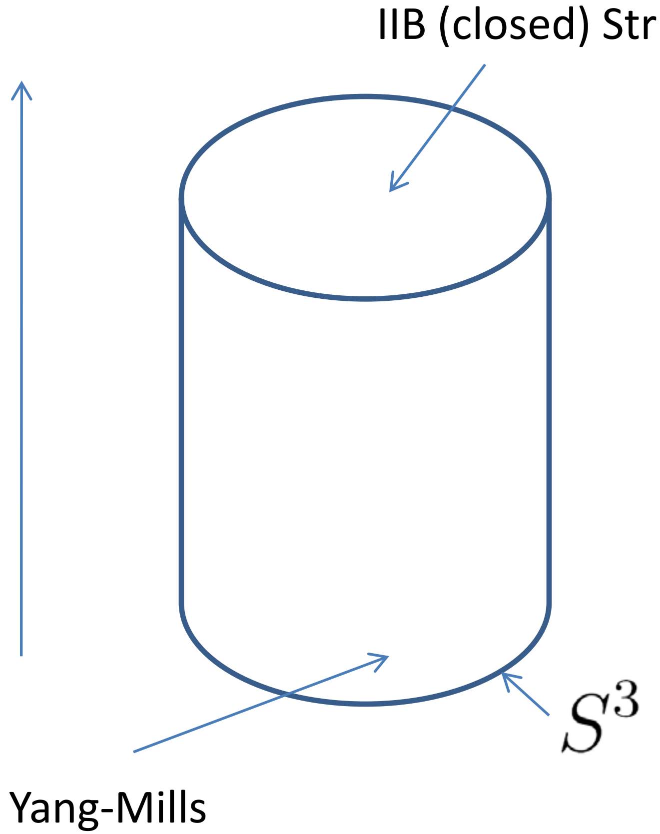
$$\left(\frac{R}{l_s}\right)^4 = 4\pi\lambda$$

- When $N \gg 1$ and the ‘t Hooft coupling is large, $\lambda = g_{YM}^2 N \gg 1$ the bulk theory may be approximated by (*classical*) supergravity and usual notions of space and time apply.



- When $\lambda \leq O(1)$ the gauge theory is still well-formulated.
- However, now the curvatures are large compared to the string scale. Supergravity breaks down and there is no meaningful interpretation in terms of a 10 dimensional local gravitational theory.

$$\lambda l_s^2 \sim \frac{1}{\mathcal{R}}$$

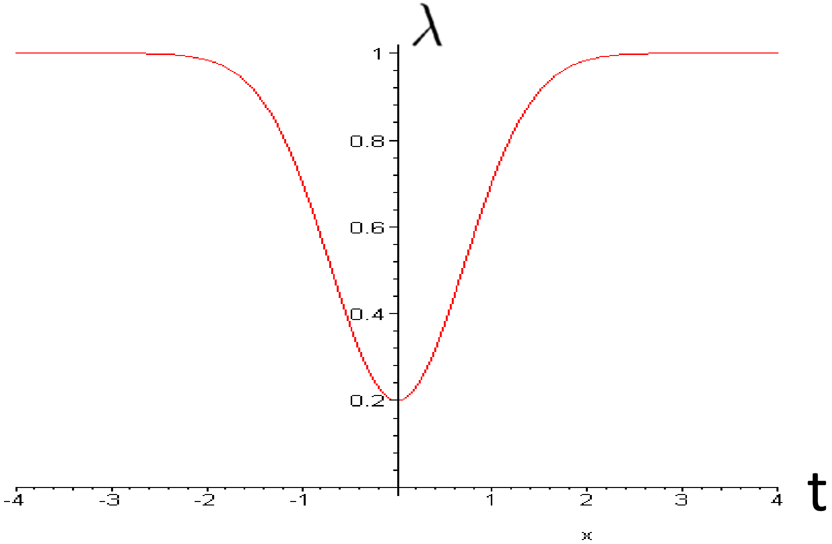
  From the supergravity point of view this may appear as a spacelike singularity for all physical purposes.
- If $N \gg 1$ we can still ignore quantum effects in this string theory, but we need to understand stringy effects
- Very little is known about even *classical* string theory in such backgrounds.
- The hope is that we can use the Yang-Mills description in this regime.

- In the supergravity regime, normalizable deformations of the conformally invariant $AdS_5 \times S^5$ geometry correspond to excited states of the gauge theory.
- Non-normalizable modes of the supergravity fields change the boundary values – these correspond to deformations of the Yang-Mills theory by addition of source terms to the action :

$$\mathcal{L}_{YM} \to \mathcal{L}_{YM} + \varphi \hat{O}$$

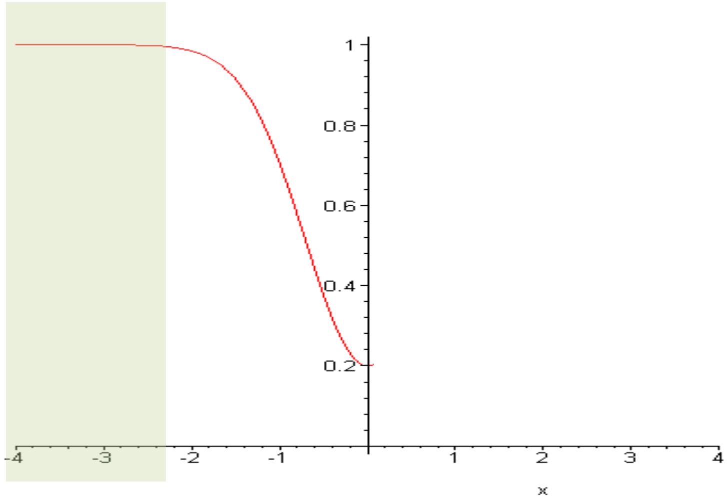
- Where $\hat{O}$ is the gauge theory operator dual to the mode $\varphi$
- The supergravity mode which is dual to the gauge coupling is the dilaton $\Phi$ .

- We will investigate toy models of cosmological singularities by considering $\mathcal{N}=4$ Yang-Mills theory with a time-dependent 't Hooft coupling .



In the bulk this corresponds to a time dependent dilaton $\Phi$ , and $N\ e^{\Phi}$ becomes small at some intermediate time , making the curvatures large.

- We will start the system in the vacuum of the gauge theory, with a large value of the ‘t Hooft coupling. The dual space-time is now pure $AdS_5 \times S^5$



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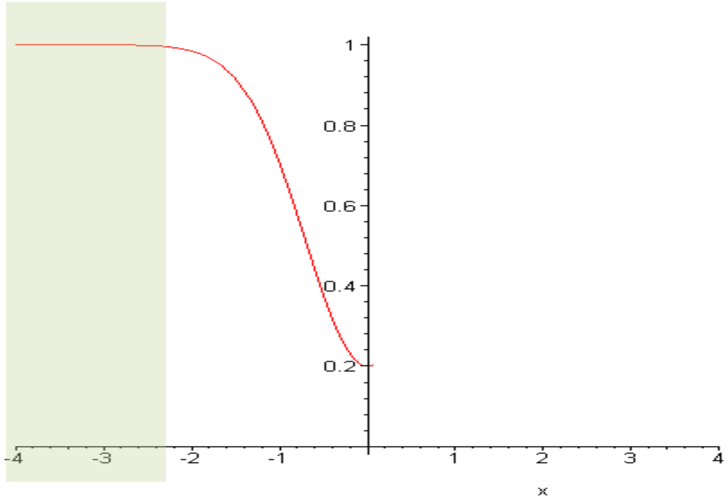
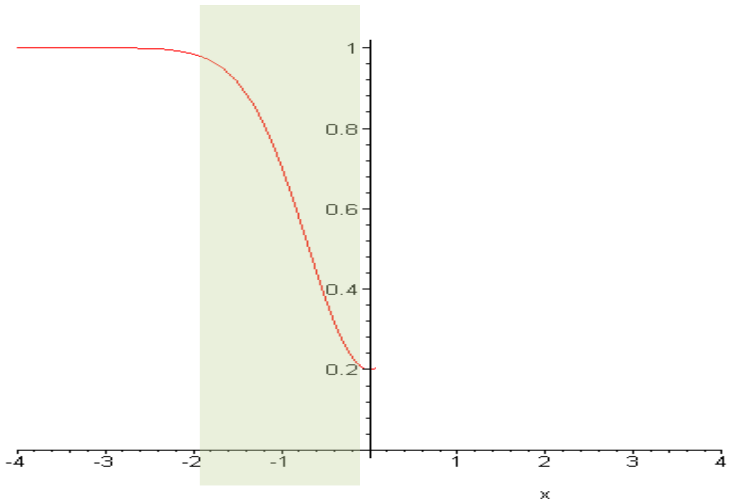


Once we turn on the time dependent source, the gauge theory evolves according to the deformed hamiltonian.

In this regime, the bulk is described by a **non-normalizable** dilaton mode . This evolves via the supergravity equations of motion – and produces a non-trivial metric by back-reaction.

- Once the gauge theory becomes weakly coupled, the supergravity description is not valid any more.



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We will ask if the gauge theory can meaningfully describe time evolution beyond this time.

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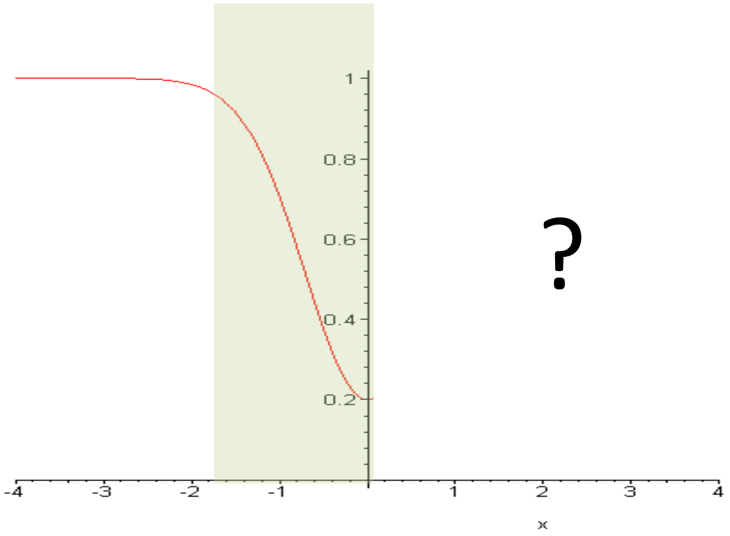

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We will ask if the gauge theory can meaningfully describe time evolution beyond this time.

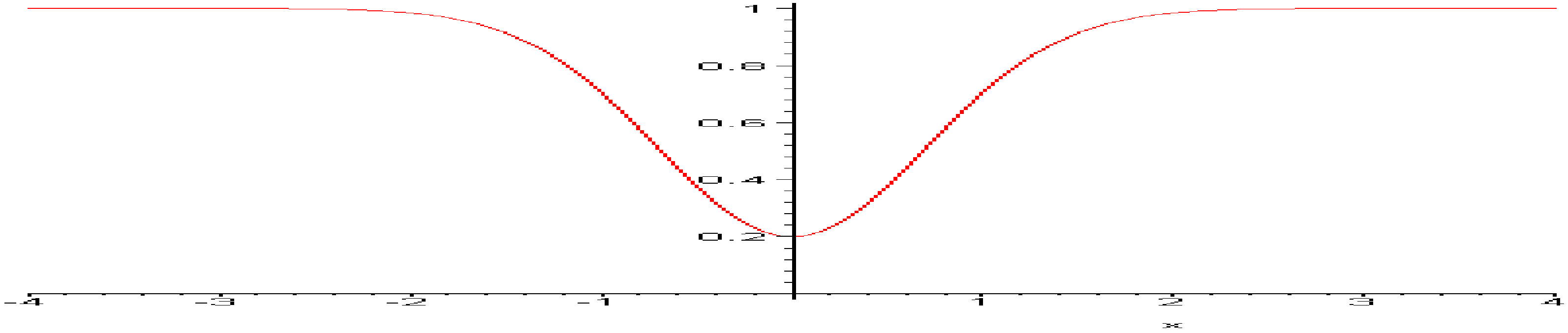
If not, we would like to learn – what precisely is the problem ?

# Slowly varying dilatons

- A breakdown of super-gravity can be achieved even by a coupling which is slowly varying, starting with a large $\lambda$ in the past.
- Since the gauge theory is defined on a $S^3$ whose radius can be taken to be $R$, slow variation means

$$R\partial_t = \epsilon \ll 1$$

- Therefore, if such a variation takes place over a timescale $t \sim \frac{R}{\epsilon}$ one can reach $\lambda \leq O(1)$



# Supergravity Solutions

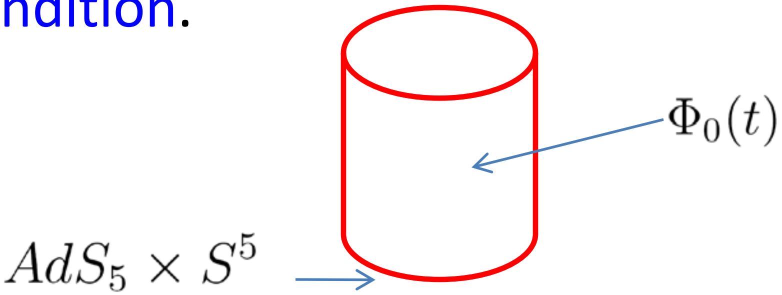
- In the infinite past (in terms of global time), the space-time is pure $AdS_5 \times S^5$ with a constant dilaton $\Phi_{-\infty}$ such that the string frame curvature is small in string units. The Einstein frame metric is

$$ds_0^2 = -(1 + \frac{r^2}{R^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2 d\Omega_3^2$$

This provides the initial condition.

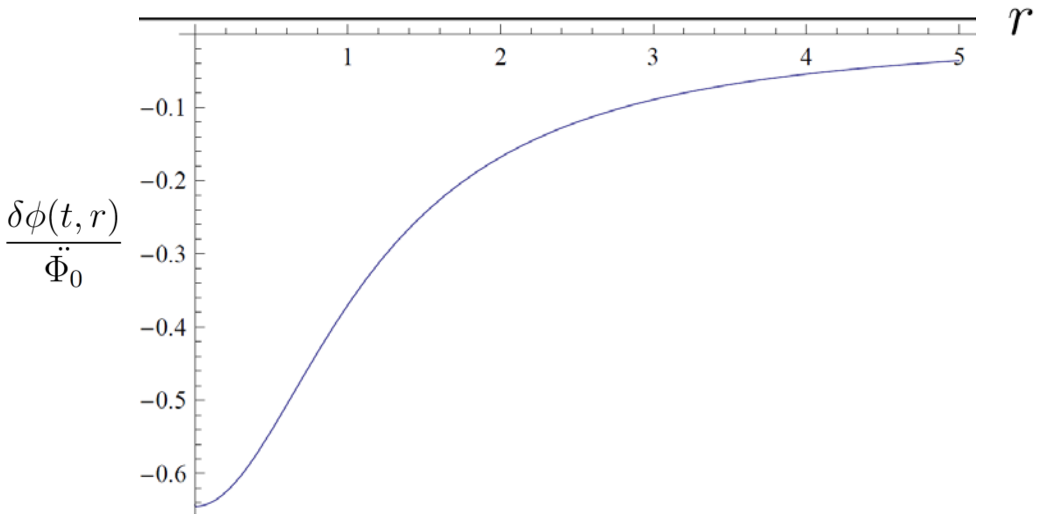
- The source on the boundary is a boundary value of the dilaton, which has been chosen as we described above - $\Phi_0(t)$

This provides the boundary condition.



- The solution to lowest order in $\epsilon$ is smooth everywhere – there are no horizons - no black holes are formed.

$$\Phi(t,r) = \Phi_0(t) + \frac{1}{4}\ddot{\Phi}_0(t)\left[\frac{1}{r^2}\log(1+r^2) - \frac{1}{2}(\log(1+r^2))^2 - \mathrm{dilog}(1+r^2) - \frac{\pi^2}{6}\right]$$

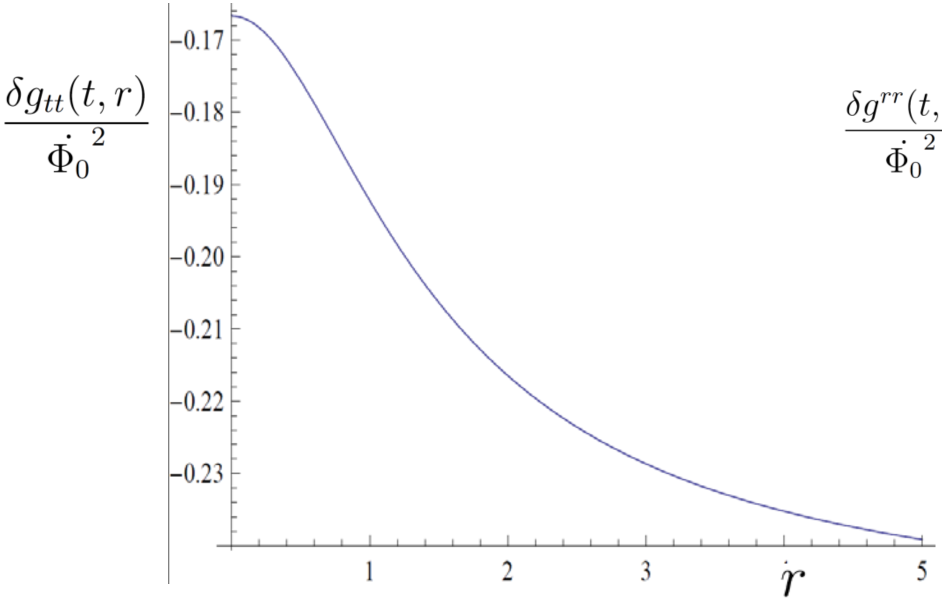


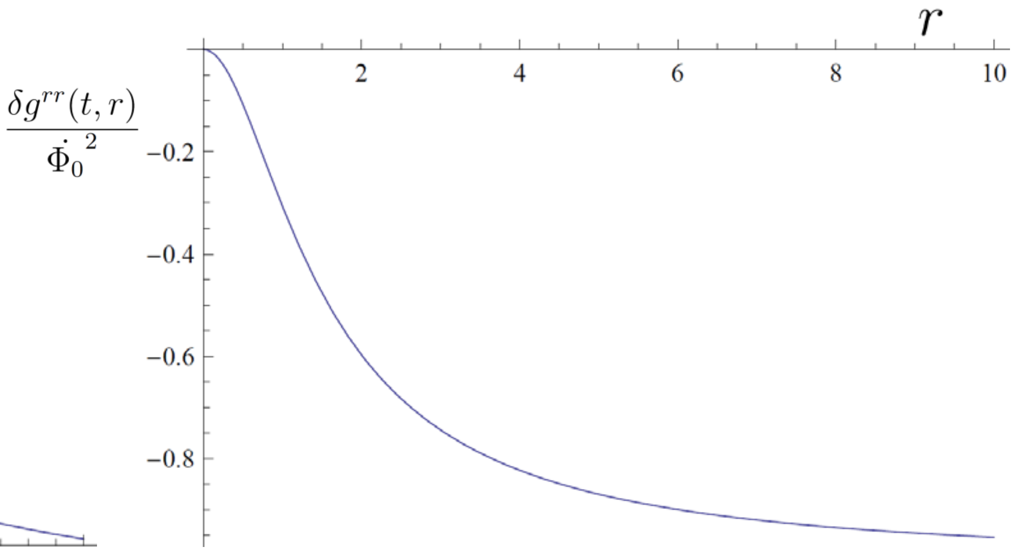
- We have used – and will keep using - $R=1$ units.

- The metric components are

$$g_{tt} = 1 + r^2 - \frac{1}{4}\dot{\Phi}_0^2 + \frac{1}{12}\dot{\Phi}_0^2\frac{\ln(1+r^2)}{r^2}$$

$$\frac{1}{g_{rr}} = 1 + r^2 - \frac{1}{12}\dot{\Phi}_0^2[1 - \frac{1}{r^2}\ln(1+r^2)]$$





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- When the variation becomes fast enough we expect black holes to form – staying in the supergravity approximation. (see e.g. *Bhattacharya and Minwalla* for an analysis with small amplitude dilaton).
- In the gauge theory this means that so long as the 't Hooft coupling is large – so that the supergravity calculation can be trusted – thermalization does not happen and energy is not dissipated.
- Furthermore, if we performed this analysis in Poincare coordinates – with a $\Phi_0(t)$ which depends only on *Poincare* time, a black hole will always form.

- Holographic RG calculation of the energy yields

$$E = - < T^t_t > V_{S^3} = \frac{3N^2}{16} + \frac{N^2 \dot{\Phi}_0^2}{32}$$

- While the expectation value of the operator dual to the dilaton is

$$< \hat{\mathcal{O}}_{l=0} >= -\frac{N^2}{16} \ddot{\Phi}_0$$

- The Noether relation is satisfied

$$\frac{dE}{dt} = -\dot{\Phi}_0 < \hat{\mathcal{O}}_{l=0} >$$

- Therefore, if we always stay in the supergravity regime – nothing dramatic happens : when the coupling gets back to a constant value – all the energy which was pumped into the system is extracted out and we have a perfect bounce.

- What else could have happened ?
- One could repeat the same exercise for a black 3-brane with a dilaton varying slowly compared to the temperature. This is like changing the coupling of the gauge theory on flat space at a finite temperature. In that case

$$< \hat{\mathcal{O}}_{l=0} >= c_1 \dot{\Phi}_0$$

- The rate of change of the temperature is

$$\frac{dT}{dt} = \frac{1}{12\pi} \dot{\Phi}_0^2$$

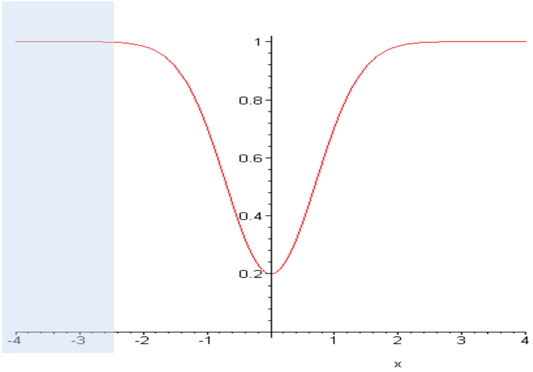
- The temperature therefore keeps increasing – the energy pumped into the system by a time-varying coupling gets dissipated.

*[Bhattacharya, Loganayagam, Minwalla, Nampuri, Trivedi ,Wadia (2008)]*

- A key feature of the above calculation is the decoupling of the various modes at leading order in $\epsilon$
- The equation for the dilaton decouples from the equations for the metric components.
- This happens because we have started with the vacuum and driving the system by banging on the boundary with a force with frequency $\epsilon$.
- The source on the boundary couples directly to the dilaton – other modes are excited by nonlinear couplings of the dilaton to these modes.
- Since these couplings necessarily involve derivatives of the dilaton, they are suppressed at small $\epsilon$ .

# The Stringy Regime

- We have been talking about the regime of large ‘t Hooft coupling,



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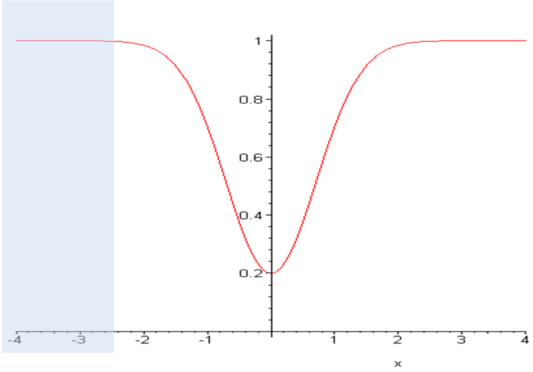
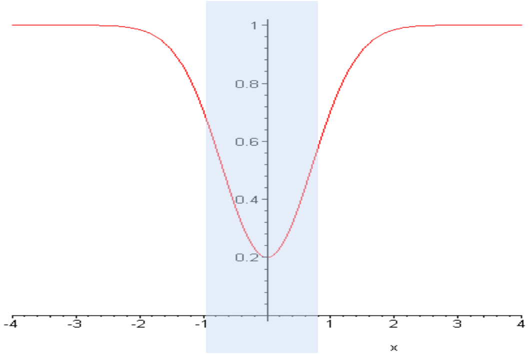


- But we really want to know what happens when the coupling has become small



- In this regime of weak coupling in the gauge theory, the curvature is large, and stringy effects cannot be ignored any more

$$\lambda l_s^2 \sim \frac{1}{\mathcal{R}}$$

We want to turn to the gauge theory to see what happens.

# Adiabatic Approximation

- The boundary gauge theory is a standard quantum mechanical system with

  (1) A slowly varying time dependent parameter – the coupling

  (2) The instantaneous Hamiltonian has a discrete spectrum with a gap above the ground state.

- This latter follows from the fact that the theory lives on $S^3$ and that the states form a unitary representation of the conformal algebra for any value of the coupling.
- **The appropriate approximation scheme is the Adiabatic Approximation**.

- Consider a hamiltonian $H(\zeta(t))$ which depends on a time dependent parameter $\zeta(t)$ . Consider the eigenstates of the instantaneous hamiltonian $H(\zeta)$

$$H(\zeta)|\phi_m(\zeta) >= E_m(\zeta)|\phi_m(\zeta) >$$

- The Adiabatic Theorem implies that if $\zeta \to \zeta_0$ in the far past, and we start with the ground state $|\phi_0 >$ of $H(\zeta_0)$ in the far past, the state at any time $t$ is well approximated by

$$|\psi^0(t) >\simeq |\phi_0(\zeta) > e^{-i\int_{-\infty}^{t} E_0(\zeta)dt}$$

where $|\phi_0(\zeta) >$ is the ground state of the instantaneous hamiltonian corresponding to $\zeta = \zeta(t)$ . $E_0(\zeta)$ is the value of the ground state energy for $\zeta = \zeta(t)$ .

- The leading corrections are given by

$$|\psi^1(t) >= \sum_{n\neq 0} a_n(t)|\phi_n(\zeta) > e^{-i\int_{-\infty}^{t} E_n dt}$$

- Where

$$a_n(t) = -\int_{-\infty}^{t} dt' \frac{< \phi_n(\zeta)|\frac{\partial H}{\partial \zeta}|\phi_0(\zeta) >}{E_0 - E_n} \dot{\zeta}\, e^{-i\int_{-\infty}^{t'} (E_0 - E_n)dt'}$$

- This correction is small provided

$$| < \phi_n|\frac{\partial H}{\partial \zeta}|\phi_0 > \dot{\zeta}| \ll (E_1 - E_0)^2$$

- Note that the quantity $(E_1 - E_0)$ is the energy gap between the ground state and the first excited state.

- In our case - Yang-Mills theory on $S^3$ of unit radius, and a time dependent coupling ,

$$\zeta(t) = \Phi_0(t)$$

- Furthermore $\dfrac{\partial H}{\partial \Phi_0} \sim \hat{\mathcal{O}}_{l=0}$
- Where $\hat{\mathcal{O}}_{l=0}$ is the operator dual to the spherically symmetric modes of the bulk dilaton. Thus the condition for validity of the adiabatic approximation is

$$| < \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 > \dot{\Phi}_0 | \ll (E_1 - E_0)^2$$

- It may be easily seen that $< \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 > \sim O(N)$, and since we are using $R = 1$ units, $(E_1 - E_0) \sim O(1)$ this condition becomes

$$N\epsilon \ll 1$$

- The adiabatic approximation of course has nothing to do with the value of the coupling constant – so this holds for weak ‘t Hooft coupling as well.
- If $\epsilon$ is so small that this condition holds, the adiabatic theorem ensures that at late times – when we again have a space-time interpretation of the gauge theory – we get back $AdS_5 \times S^5$ with exponentially small corrections.
- However this condition $N\epsilon \ll 1$ is much stronger than the condition $\epsilon \ll 1$ which we used in performing the supergravity analysis.
- We need to find a scheme which has an overlapping regime of validity with supergravity.

# Coherent States and Adiabaticity

- The adiabatic approximation described above is good for description of the system in terms of states of the gauge theory which are obtained by a finite number of operators on the vacuum – these are states containing a finite number of particles in the bulk.
- Classical solutions in the bulk are, however, described by coherent states.
- In the boundary theory these are coherent states of gauge invariant operators.

- A general coherent state has the form

$$|\Psi(t)>= \exp\left[i\chi(t) + \sum_I \lambda^I(t)\hat{\mathcal{O}}^I_{(+)}\right]|0>_A$$

- Where $\hat{\mathcal{O}}^I_{(+)}$ are the creation parts of gauge invariant operators in the theory. For example, the operator dual to the spherically symmetric dilaton is

$$\hat{\mathcal{O}}_{l=0} = \int d\Omega_3 \, \mathrm{Tr} F_{\mu\nu}F^{\mu\nu}$$

- The adiabatic vacuum is denoted by $|0>_A$
- The algebra of the operators $\hat{\mathcal{O}}^I$, together with the Schrodinger equation determines the evolution of the coherent state parameters $\chi(t)$ and $\lambda^I(t)$ .
- At $N=\infty$ these states go over to classical configurations – which have a good description in terms of local fields in the large ‘t Hooft coupling regime.

- Each operator of the gauge theory can be associated with a field in the bulk.
- The dynamics of these fields is in general given by a horrible non-local collective field theory.
- Only in special situations this collective field theory becomes local and useful, e.g.

(i) For a single matrix quantum mechanics – here the collective field theory is the string field theory of the two dimensional string.

(ii) Strong ‘t Hooft coupling limit of N=4. Here the collective field theory is classical supergravity.

- Usually this is an impossible plan to implement. The algebra of the operators is complicated and they all couple to each other.
- In our situation, however, the dynamics of these modes are driven entirely by a time dependent coupling constant. This directly drives the dynamics of the mode which comes with the operator $\hat{\mathcal{O}}_{l=0}$
- Other modes are excited due to non-trivial 3-point functions

  $$< \hat{O}_1 \hat{O}_2 \hat{O}_3 > \ \sim \ 1/N$$ (Normalized operators)

  so that the corresponding probability goes as $1/N^2$. However, as we will see soon, a coherent state produced by this slow driving has roughly $O(N^2\epsilon^2)$ quanta, so that the effective 3 point coupling in such states is suppressed relative to the 2 point function by a factor of $\epsilon$
- Thus for $\epsilon \ll 1$ these operators can be considered to be independent of each other with small non-linearities.

- To lowest order in $\epsilon$ it is therefore sufficient to consider a coherent state of the operator dual to the s-wave dilaton.
- Express this operator as a sum over oscillators

$$\hat{\mathcal{O}}_{l=0} = N \sum_{n=1}^{\infty} F(2n)[A_{2n} e^{-i2nt} + A_{2n}^{\dagger} e^{i2nt}]$$

$$|F(2n)|^2 = \frac{A\pi^4}{3} n^2(n^2 - 1)$$

Fixed by 2 point function

- Construct the coherent state

$$|\psi >= \hat{N}(t) e^{(\sum_n \lambda_n A_{2n}^{\dagger})} |\phi_0 >$$

Adiabatic Vacuum

- Then the Schrodinger equation implies

$$i\frac{d\lambda_n}{dt} = -i\frac{F(2n)}{2n}\dot{\Phi}_0 + 2n\ \lambda_n$$

The initial conditions are $\lambda_n(-\infty) = 0$, and the boundary dilaton has the property that $\dot{\Phi}_0(-\infty) = 0$

- This equation can be solved exactly。
- However, we want to write this solution somewhat differently – by successively integrating by parts

$$\frac{1}{N}\lambda_n(t) = \frac{F(2n)}{2n}\left[\frac{\dot{\Phi}_0}{(2in)} + \frac{\ddot{\Phi}_0}{4n^2} + \cdots\right]$$

This is an expansion in time derivatives – the adiabatic approximation we are seeking.

As promised the coherent state parameter is $O(N\epsilon)$ - so that the average number of quanta is $O(N^2\epsilon^2)$

- Note that we have assumed that the oscillators are independent – this is valid for small $\dot{\Phi}_0$ , $\ddot{\Phi}_0$ . This means that only the first two terms in this expansion are significant.

- This adiabatic approximation is valid provided

$$|\frac{\ddot{\Phi}_0}{n\dot{\Phi}_0}| \ll 1 \qquad \forall n$$

- It is clearly sufficient to have

$$|\frac{\ddot{\Phi}_0}{\dot{\Phi}_0}| \sim \epsilon \ll 1$$

- Note that 2n is the characteristic frequency, which is quantized since the theory lives on $S^3$ . If there was no gap in the spectrum, the frequency could be arbitrarily small and the adiabatic approximation would not hold.
- The condition for validity is exactly what we had in our supergravity analysis.

- However , for this to describe nice coherent states which become classical in the $N = \infty$ limit, we must also have

$$\lambda_n \gg 1$$

- This condition can be seen to be equivalent to

$$|N\dot{\Phi}_0| \sim N\epsilon \gg 1$$

- In this case we can compare our gauge theory answers with supergravity. They agree upto numerical factors

$$< \hat{\mathcal{O}}_{l=0} > \sim N^2\ddot{\Phi}$$

$$< E > \sim N^2(\dot{\Phi})^2 \sim O(N^2\epsilon^2)$$

# Small ‘t Hooft coupling

- The framework developed above applies to all values of the ‘t Hooft coupling – therefore can be extended to the regime of small couplings as well.
- Now, however, we have an infinite tower of string modes – whose duals are gauge invariant operators which become as important as the ones which are dual to supergravity modes.
- This is because for large $\lambda$ the dimensions of operators dual to higher stringy modes - and hence the frequencies of the corresponding oscillators - are $O(N)$ as opposed to supergravity modes whose frequencies are $O(1)$ .
- For small $\lambda$ , however, the dimensions of all these modes are comparable.

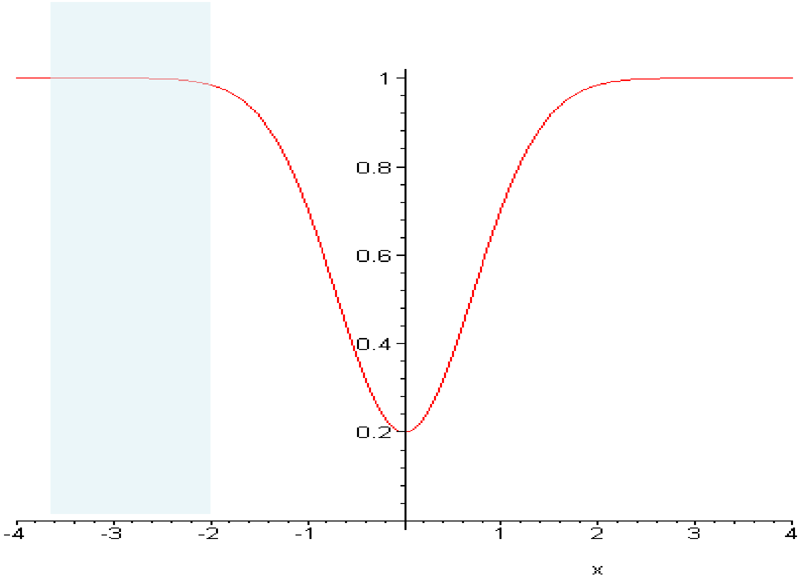
- Nevertheless, the basic ingredients which went into our coherent state adiabatic approximation are still in place

  (1) The couplings between different oscillators are still suppressed by $\epsilon$ .

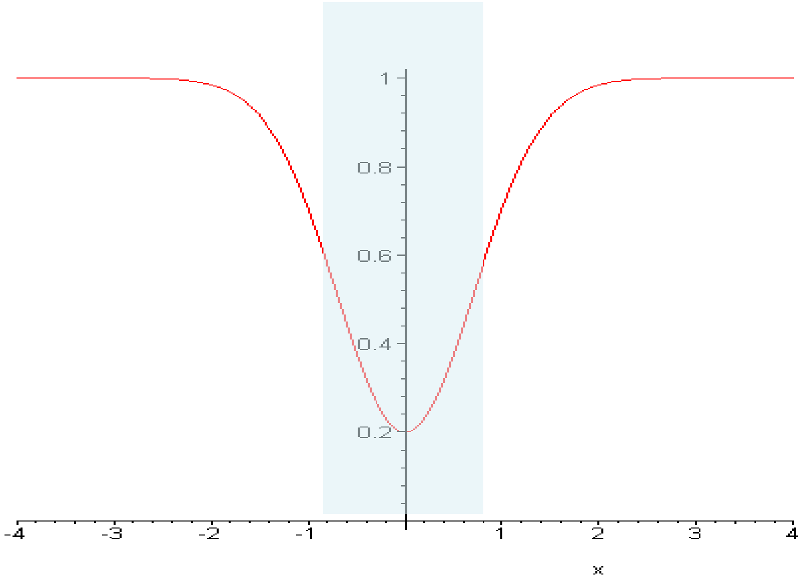
  (2) The frequencies are still $O(1)$ for any value of $\lambda$ , so that the system is always far from resonance.

  (3) For $N\epsilon \gg 1$ the states are still classical.

- It would therefore appear that the adiabatic approximation still holds.
- If this is really true……



After passing through the stringy region of high curvature, one would essentially have $AdS_5 \times S^5$ with exponentially small corrections.



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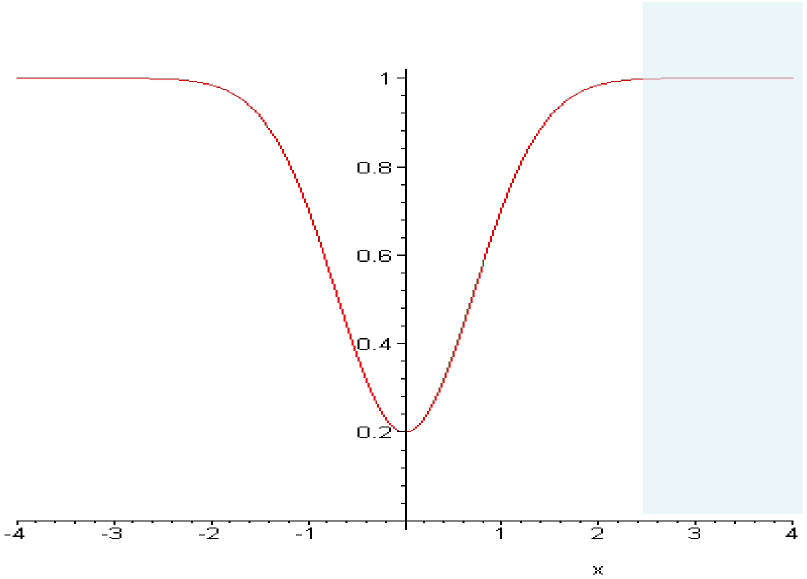


After passing through the stringy region of high curvature, one would essentially have $AdS_5 \times S^5$ with exponentially small corrections.

- **However, there is another possibility.**
- There are $O(N^2)$ stringy modes (non-chiral operators) , while there are only $O(1)$ supergravity modes.
- While individual couplings are suppressed by $\epsilon$ , there is a possibility that whatever energy is transferred to these modes may thermalize.
- If thermalization does happen – the energy is dissipated and cannot be extracted back when the coupling rises again to large values.
- **At late times one would have $O(N^2\epsilon^2)$ thermalized energy in the system**.

- At late times, the ‘t Hooft coupling is again large and we can use known results of AdS/CFT to guess the outcome.
- This depends on how small $\epsilon$ is.
- For $\epsilon \ll (g_{YM}^2 N)^{5/4}/N$ the result would be a gas of supergravity modes.
- For $(g_{YM}^2 N)^{5/4}/N < \epsilon \ll (g_{YM}^2 N)^{-7/8}$ one would have a gas of higher string modes.
- For $(g_{YM}^2 N)^{-7/8} < \epsilon \ll 1$ one would have small black holes, i.e. Black holes whose size is much smaller than $R_{AdS}$
- This is the worst that can happen – large black holes require an energy $O(N^2)$ which is much larger than the energy we have.

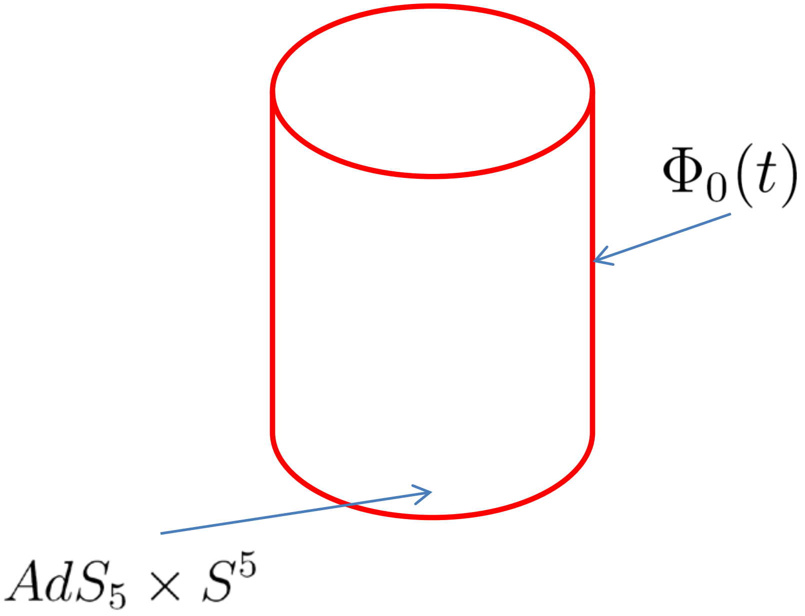
  They will not form.
- In any case *most* of space-time would be close to $AdS_5 \times S^5$

- It is difficult to determine whether thermalization would indeed occur – the time scale involved in interactions is the same as the time scale by which the system is driven – and there is no obvious answer to this question.
- Perhaps the most significant result of our analysis is that in this case of slowly varying coupling, a big black hole is never formed.
- In the far future one might be left with small black holes. They will evaporate, but that time scale is much larger - $O(N^2)$
- In any case, the formalism developed can be, in principle, used to provide a smooth description of time evolution through what *appears* as a singularity from the gravity viewpoint.

# NOW – SOME DETAILS

# I: The Supergravity Solution

- Initial condition : In the asymptotic past the gauge theory is in its vacuum state – the dual space-time is $AdS_5 \times S^5$
- Boundary condition : The boundary value of the dilaton field is specified to be a function of time $\Phi_0(t)$



- We need to solve the equations of motion in a power series expansion in $\epsilon$

$$R_{AB} = -\frac{4}{R^2} g_{AB} + \frac{1}{2}\partial_A \Phi \partial_B \Phi \qquad\qquad \nabla^2 \Phi = 0$$

- Expand the fields

$$O(\epsilon)$$

$$\Phi(t) = \Phi_0(t) + \Phi_1(r,t) + \Phi_2(r,t) \cdots$$

$$g_{ab} = g_{ab}^{(0)} + g_{ab}^{(1)} + g_{ab}^{(2)} + \cdots$$

- Derivatives with respect to $r$ are not small.
- To the lowest nontrivial order in $\epsilon$ the solution is very simple.

- We need to solve the equations of motion in a power series expansion in $\epsilon$

$$R_{AB} = -\frac{4}{R^2} g_{AB} + \frac{1}{2} \partial_A \Phi \partial_B \Phi \qquad \nabla^2 \Phi = 0$$

- We need to solve the equations of motion in a power series expansion in $\epsilon$

$$R_{AB} = -\frac{4}{R^2} g_{AB} + \frac{1}{2}\partial_A \Phi \partial_B \Phi \qquad\qquad \nabla^2 \Phi = 0$$

- Expand the fields

$O(\epsilon)$

$$\Phi(t) = \Phi_0(t) + \Phi_1(r,t) + \Phi_2(r,t) \cdots$$

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$$R_{AB} = -\frac{4}{R^2} g_{AB} + \frac{1}{2}\partial_A \Phi \partial_B \Phi \qquad\qquad \nabla^2 \Phi = 0$$

- Expand the fields

$O(\epsilon^2)$

$$\Phi(t) = \Phi_0(t) + \Phi_1(r,t) + \Phi_2(r,t) \cdots$$

$$g_{ab} = g_{ab}^{(0)} + g_{ab}^{(1)} + g_{ab}^{(2)} + \cdots$$

- Derivatives with respect to $r$ are not small.
- To the lowest nontrivial order in $\epsilon$ the solution is very simple.

- First let us look at the corrections to the metric. Define

$$E_{AB} = R_{AB} + \frac{4}{R^2} g_{AB}$$

- Then Einstein's equations become

$$\begin{aligned} E^{(0)}_{AB} + \overset{O(\epsilon)}{E^{(1)}_{AB}} + \overset{O(\epsilon^2)}{E^{(2)}_{AB}} + \cdots = \frac{1}{2}\big[&\partial_A\Phi_0\partial_B\Phi_0 + \partial_A\Phi_0\partial_B\Phi_1 \\ &+ \partial_A\Phi_1\partial_B\Phi_0 + \partial_A\Phi_1\partial_B\Phi_1 + \cdots\big] \end{aligned}$$

- Each term on the RHS is at least of order $O(\epsilon^2)$.
- This immediately means that

$$g^{(1)}_{AB} = 0$$

- Now consider the dilaton equation of motion

$$\underset{\underset{O(\epsilon^2)}{\uparrow}}{\nabla_0^2\Phi_0} + \underset{\underset{O(\epsilon)}{\nwarrow}}{\nabla_0^2\Phi_1} + \underset{\underset{O(\epsilon)}{\nearrow}}{\nabla_1^2\Phi_0} + \underset{\underset{O(\epsilon^2)}{\nwarrow}}{\nabla_1^2\Phi_1} + \underset{\underset{O(\epsilon^2)}{\nearrow}}{\nabla_0^2\Phi_2} = 0.$$

- Since the $O(\epsilon)$ correction to the metric vanishes, $\nabla_1^2 = 0$ and the equation now becomes

$$\underset{\underset{O(\epsilon^2)}{\uparrow}}{\nabla_0^2\Phi_0} + \underset{\underset{O(\epsilon)}{\uparrow}}{\nabla_0^2\Phi_1} + \qquad\qquad\qquad\qquad + \underset{\underset{O(\epsilon^2)}{\uparrow}}{\nabla_0^2\Phi_2} = 0.$$

- Clearly, we must have $\Phi_1 = 0$ so that the equation becomes

$$\nabla_0^2\Phi_0 + \nabla_0^2\Phi_2 = 0$$

This is simply the linearized equation for a scalar field in the AdS background in the presence of a source.

- Since $\Phi_2$ is already of order $O(\epsilon^2)$ and each time derivative costs a power of $O(\epsilon)$, to lowest order any time derivative on $\Phi_2$ can be ignored.
- Furthermore the boundary value of the dilaton $\Phi_0(t)$ is a function of time only, we have now reduced the equation to an ordinary differential equation.

$$\frac{1}{r^3}\frac{d}{dr}\left[r^3(1+r^2)\frac{d\Phi_2}{dr}\right] = \frac{\ddot{\Phi}_0}{1+r^2}$$

- We need to solve this with the boundary condition

$$\Phi_2(\infty) = 0$$

- We also need to impose boundary conditions at the origin
- We will look for solutions which are regular at the origin.
- **This is not guaranteed**. But let's see.

- The solution to the ODE for

Integration constants

$$\begin{aligned}\Phi_2(r,t) \quad &= \quad \frac{1}{4}\ddot{\Phi}_0(t)\left[\frac{1}{r^2}\log(1+r^2) - \frac{1}{2}(\log(1+r^2))^2 - \mathrm{dilog}(1+r^2)\right] \\ &+a_1(t)\frac{1}{2}\left[\log(1+r^2) - \frac{1}{r^2} - 2\log\ r\right] \ + a_2(t).\end{aligned}$$

- The first line is regular at $r = 0$

$$\frac{1}{r^2}\log(1+r^2) - \frac{1}{2}[\log(1+r^2)]^2 - \mathrm{dilog}(1+r^2) \sim 1 + \frac{r^2}{2} + \cdots$$

- However, the factor multiplying $a_1(t)$ diverges

$$\log(1+r^2) - \frac{1}{r^2} - 2\log r \sim -\frac{1}{2r^4} + \cdots$$

- Therefore we indeed have solutions regular at the origin by choosing

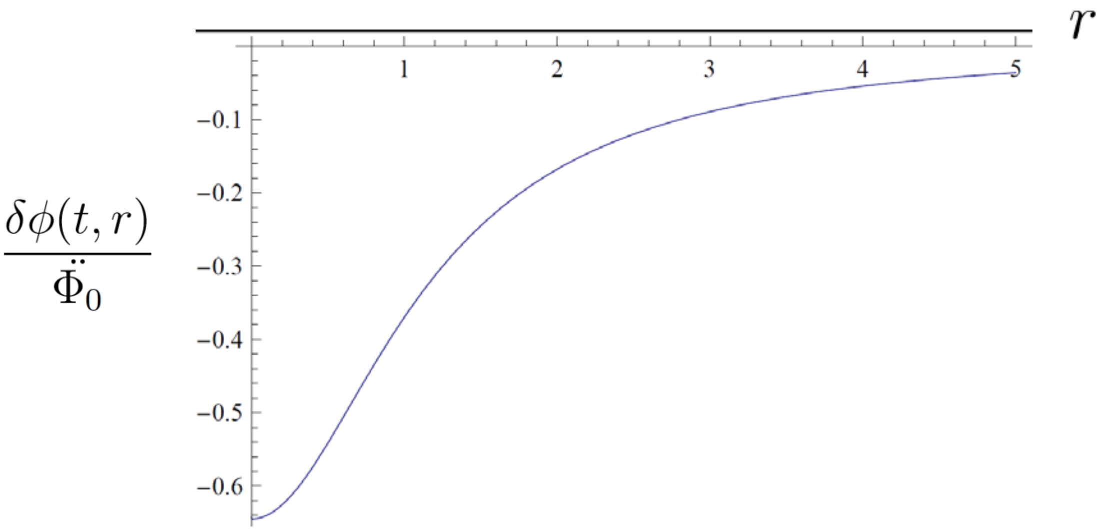
$$a_1(t) = 0$$

- Finally, the remaining integration constant is determined by imposing the boundary condition $\Phi_2(\infty) = 0$ at all times, setting

$$a_2(t) = -\frac{\pi^2}{24}\ddot{\Phi}_0(t)$$

- The final solution is

$$\Phi(t,r) = \Phi_0(t) + \frac{1}{4}\ddot{\Phi}_0(t)\left[\frac{1}{r^2}\log(1+r^2) - \frac{1}{2}(\log(1+r^2))^2 - \mathrm{dilog}(1+r^2) - \frac{\pi^2}{6}\right]$$



- The backreaction on the metric is also calculated in this derivative expansion. The general form of a spherically symmetric metric is

$$ds^2 = -g_{tt}(t,r)dt^2 + g_{rr}(t,r)dr^2 + 2g_{tr}(t,r)drdt + R^2(t,r)d\Omega_3^2$$

- Note that

$$g_{tt}, g_{rr}, R \sim O(1) + O(\epsilon^2)$$

$$g_{tr} \sim O(\epsilon^2)$$

- Using this, it is possible to perform coordinate transformations such that

$$g_{tr} = 0 \qquad R = r$$

- The final form of the metric we will work with is

$$ds^2 = -e^{2A(t,r)}dt^2 + e^{2B(t,r)}dr^2 + r^2 d\Omega_3^2$$

- We now need to solve Einstein’s equations

$$R_{AB} + 4g_{AB} = \frac{1}{2}\partial_A\Phi\partial_B\Phi$$

- Since $\Phi = \Phi_0 + O(\epsilon^2)$ we have

$$\partial_A\Phi\partial_B\Phi = \partial_A\Phi_0\partial_B\Phi_0 + O(\epsilon^3)$$

- Therefore, upto $O(\epsilon^2)$ the dilaton can be set to its boundary value $\Phi_0(t)$.
- Furthermore, as argued above, $g_{AB} = g^{(0)}_{AB} + O(\epsilon^2)$ - this means that in the calculation of the components of the Ricci tensor we can ignore all time derivatives.
- Therefore, to $O(\epsilon^2)$ ,Einstein’s equations have the form

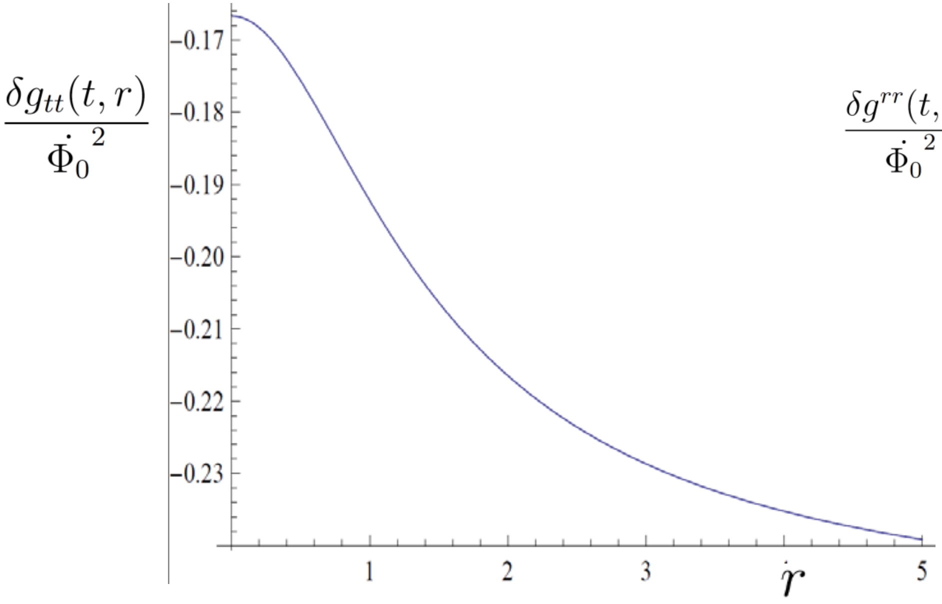
$$\hat{O}(r)g^{(2)}_{ab} = f_{ab}(r)\dot{\Phi}_0^2$$

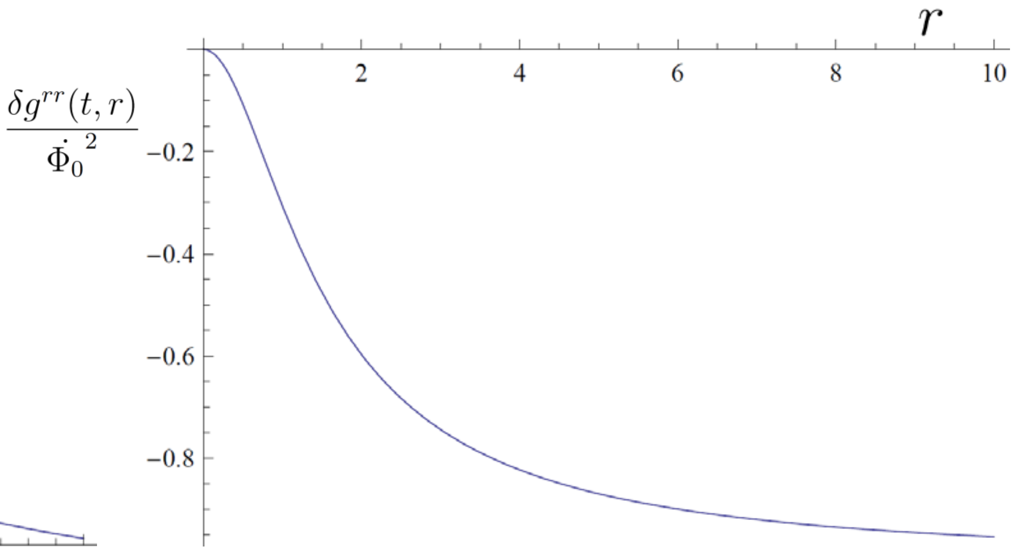
- Where $\hat{O}(r)$ is a differential operator in the radial variable. The time dependence of the solution is therefore entirely given by the boundary value of the dilaton,

$$g^{(2)}_{ab} = \mathcal{F}(r)_{ab}\dot{\Phi}_0^2$$

- Once again the solution with given initial and boundary conditions is completely smooth – there are no horizons

$$
\begin{aligned}
g_{tt} &= 1 + r^2 - \frac{1}{4}\dot{\Phi}_0^2 + \frac{1}{12}\dot{\Phi}_0^2\frac{\ln(1+r^2)}{r^2} \\
\frac{1}{g_{rr}} &= 1 + r^2 - \frac{1}{12}\dot{\Phi}_0^2[1 - \frac{1}{r^2}\ln(1+r^2)]
\end{aligned}
$$





- An important feature of this calculation is that the equations for the fluctuations of the metric $\delta g^{rr}(t,r)$ , $\delta g_{tt}(t,r)$ and fluctuations of the dilaton $\delta\phi(t,r)$ decouple to this lowest non-trivial order.
- We will see that this feature will persist beyond the supergravity regime.

- Holographic RG calculation of the energy yields

$$E = - < T_t^t > V_{S^3} = \frac{3N^2}{16} + \frac{N^2 \dot{\Phi}_0^2}{32}$$

- While the expectation value of the operator dual to the dilaton is

$$< \hat{\mathcal{O}}_{l=0} >= -\frac{N^2}{16} \ddot{\Phi}_0$$

- The Noether relation is satisfied

$$\frac{dE}{dt} = -\dot{\Phi}_0 < \hat{\mathcal{O}}_{l=0} >$$

- Therefore, if we always stay in the supergravity regime – nothing dramatic happens : when the coupling gets back to a constant value – all the energy which was pumped into the system is extracted out and we have a perfect bounce.

- In the small_derivative expansion it is always possible to find a solution which is regular at the origin.
- However, this would not be possible if the variation of the boundary field is fast.
- Consider, e.g. a complementary regime,

(1) The amplitude of the dilaton is small, $\eta$

(2) However $\Phi_0(t)$ is nonzero only for

$$0 < t < (\delta t)$$

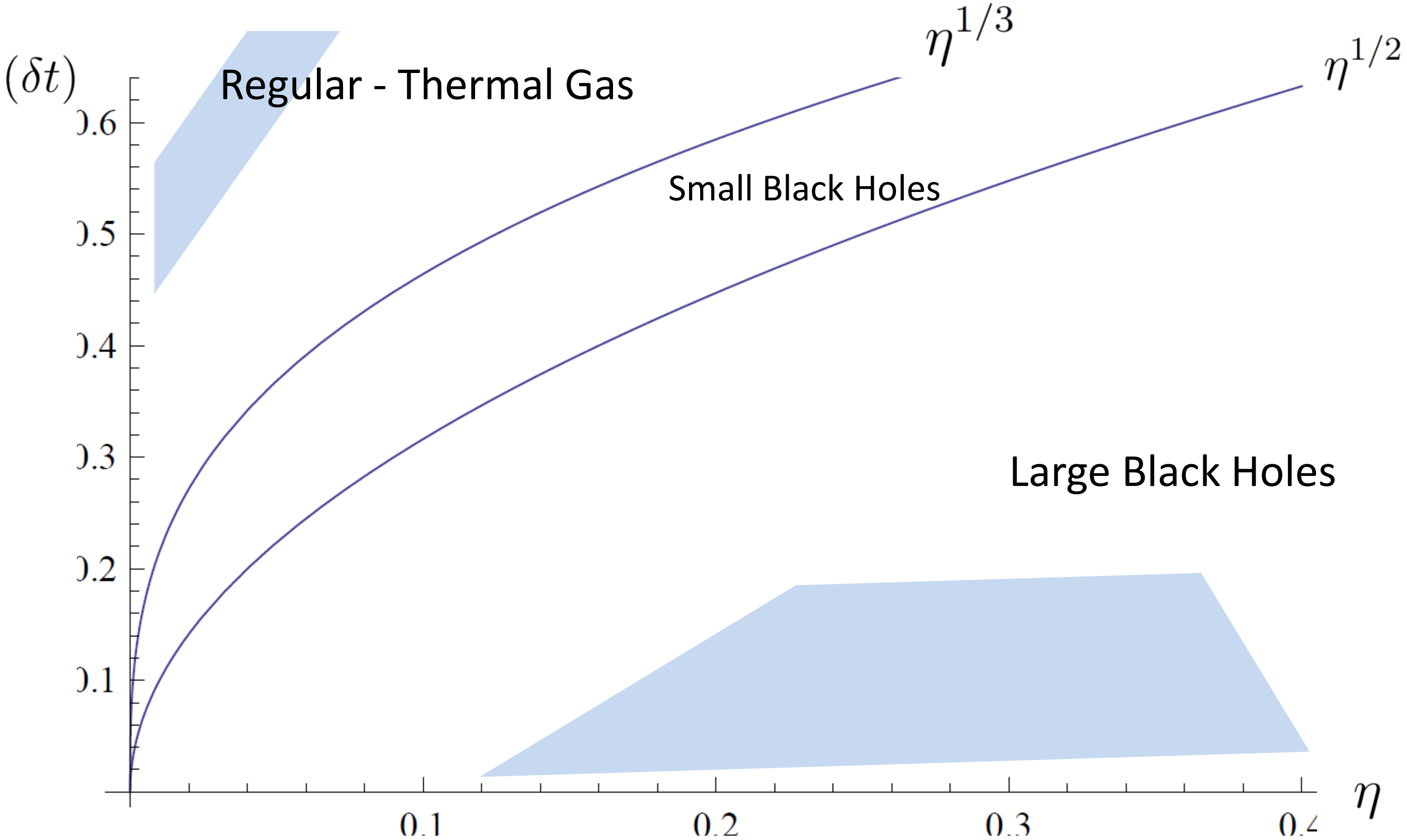
*Bhattacharya and Minwalla* studied this problem in an expansion in $\eta$ ..

They found that a regular solution can be found only when

$$(\delta t) \gg \eta^{1/3}$$

On the other hand, they could also solve the problem in the regime $(\delta t) \ll \eta^{1/3}$ consistently – and found that a horizon is formed.

# Phase Diagram of Solutions of *Bhattacharyya and Minwalla*



# II :Usual Adiabatic Approximation

- We argued that the standard form of the Adiabatic approximation holds provided

$$| < \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 > \dot{\Phi}_0 | \ll (E_1 - E_0)^2$$

- Here $\hat{\mathcal{O}}_{l=0}$ is the gauge theory operator which is dual to the bulk dilaton, and $(E_1 - E_0)$ is the energy gap.
- The states $|\phi_n >$ for which the LHS is non-zero are those which are created by the operator $\hat{\mathcal{O}}_{l=0}$ acting on the vacuum. This is because the N=4 theory has a state-operator correspondence.
- Since $< \hat{\mathcal{O}}_{l=0} \ \hat{\mathcal{O}}_{l=0} > \sim N^2$ the normalized state is $|\phi_n > \sim \frac{1}{N} \hat{\mathcal{O}}_{l=0} |0 >$
- Therefore $< \phi_n | \hat{\mathcal{O}}_{l=0} | \phi_0 > \sim N$
- Since $(E_1 - E_0) \sim O(1)$ this leads to the condition

$$N \epsilon \ll 1$$

# III: Driven Harmonic Oscillator

- Consider a driven harmonic oscillator

$$H = \frac{1}{2}\dot{X}^2 + \frac{1}{2}\omega_0^2(X + \frac{J(t)}{\omega_0^2})^2 \qquad \frac{\dot{J}}{J} \ll \omega_0$$

- The adiabatic vacuum is the ground state of the instantaneous Hamiltonian – the hamiltonian where $J(t)$ is regarded as a time-independent constant. In this case it is trivial to write this down

$$|\phi_0 >= N_\alpha e^{\alpha a^\dagger}|0 >$$

- Where $a^\dagger$ is the standard creation operator and

$$\alpha = -\frac{J}{\sqrt{2\omega_0^3}}$$

- $N_\alpha$ is a normalization constant.

- Define the shifted annihilation/creation operators

$$\tilde{a} = a - \alpha, \tilde{a}^\dagger = a^\dagger - \alpha$$

- The adiabatic vacuum is annihilated by $\tilde{a}$ , $\tilde{a}|\phi_0 >= 0$
- In terms of these the Hamiltonian is

$$H = \omega_0(\tilde{a}^\dagger \tilde{a}) + \frac{1}{2}\omega_0$$

- While its time derivative is

$$\frac{\partial H}{\partial t} = \dot{J}(\frac{\tilde{a} + \tilde{a}^\dagger}{\sqrt{2\omega_0}})$$

- The usual adiabatic approximation would require

$$< n|\frac{\partial H}{\partial t}|\phi_0 >\ll (E_n - E_0)^2$$

- Since $(E_n - E_0) \sim \omega_0$ and $< n|$ can only be the single oscillator state, this condition becomes

$$\dot{J} \ll \omega_0^{5/2}$$

- Let us now solve this problem classically. With the vacuum initial condition ( $X(-\infty), P(-\infty) = 0$ ) the solution is

$$X(t) = \int d\omega \frac{J(\omega)}{(\omega + i\epsilon)^2 - \omega_0^2} e^{-i\omega t}$$

- When the source is slowly varying,

$$\frac{\ddot{J}}{J} \ll \omega_0^2$$

Instantaneous minimum

- Expand the denominator and perform fourier transform

$$X = -\frac{J(t)}{\omega_0^2} + \frac{\ddot{J}}{\omega_0^4} + \cdots$$

- This is the adiabatic approximation to the classical solution.
- The condition $\dot{J} \ll \omega_0^{5/2}$ which we encountered in the quantum adiabatic expansion does not appear.

- Classical solutions correspond to coherent states. So we need to formulate a version of adiabatic approximation for such coherent states.
- In fact – as we have seen – the instantaneous minimum itself corresponds to the adiabatic vacuum – which is a coherent state

$$|\phi_0 >= N_\alpha e^{\alpha a^\dagger}|0 > \qquad \alpha = -\frac{J}{\sqrt{2\omega_0^3}}$$

- When the source is turned on, the state of the system which began initially as the vacuum (which is the same as the adiabatic vacuum at early times) would be a coherent state of the form

$$|\psi(t) >= N(t)e^{\lambda(t)a^\dagger}|\phi_0 >$$

- Where $N(t)$ is a normalization factor. We need to determine this and the coherent state parameter $\lambda(t)$ by imposing the Schrodinger equation on this state.

- This leads to the following equations

$$i\dot{N}(t) = [\frac{1}{2} - \frac{J(t)}{\sqrt{2\omega_0^3}}]\omega_0 \; N(t)$$

$$i\dot{\lambda} = i\frac{\dot{J}}{\sqrt{2\omega_0^3}} + \omega_0 \lambda$$

- With initial condition $\lambda(-\infty) = 0$ the solution for $\lambda(t)$ is

$$\lambda(t) = \frac{e^{-i\omega_0 t}}{\sqrt{2\omega_0^3}} \int_{-\infty}^{t} \dot{J}(t')e^{i\omega_0 t'} dt'$$

- By successive integration by parts this may be written as

$$\lambda(t) = \frac{1}{\sqrt{2\omega_0^3}} \left[ \frac{\dot{J}}{i\omega_0} + \frac{\ddot{J}}{\omega_0^2} - \frac{\partial_t^3 J}{i\omega_0^3} + \cdots \right]$$

- This will become the classical adiabatic expansion for $|\lambda| \gg 1$

- For this adiabatic expansion we only need

$$\frac{\ddot{j}}{\dot{j}\omega_0} \ll 1$$

In fact the condition that this reproduces the classical solution, i.e. $|\lambda| \gg 1$ becomes

$$\dot{j} \gg \omega_0^{5/2}$$

Which is exactly the opposite of what the usual quantum mechanical adiabatic approximation required.

# Large-N Coherent States

- The set of all gauge-invariant operators in a large-N gauge theory correspond to an infinite set of collective fields.
- The coupling constant of the "collective field theory" is 1/N.
- That does not mean, of course, we can ignore all interactions when $N = \infty$ – this depends on the kind of states we have.
- For example, in any field theory with a coupling $g$ , field configurations which are themselves of $O(1/g)$ certainly contribute at weak coupling.
- These field configurations correspond to coherent states whose coherent state parameters are of $O(1/g)$.
- Such coherent states have $O(1/g^2)$ quanta – this cancels the effect of large-N suppression of couplings.

- Consider for example a coherent state of the form

$$|\Psi(t) >= \exp\left[i\chi(t) + \sum_I \lambda^I(t)\hat{\mathcal{O}}^I_{(+)}\right] |0>_A$$

- When the parameters $\lambda^I$ are $O(N)$ we expect these to describe classical configurations in the $N = \infty$ limit.
- However, in our present context this state is the result of the time dependence of the boundary dilaton. In fact we will self-consistently find that the coherent state parameter is $O(N\epsilon)$
- This means that there are $O(N^2\epsilon^2)$ quanta in such a state.
- Which implies that while there is no large-N suppression of interaction terms in the collective field theory, there is a suppression by powers of $\epsilon$
- The $m$ -point coupling is in fact suppressed by $\epsilon^{m-2}$ .
- Therefore to lowest non-trivial order in $\epsilon$ we can treat the collective fields as free fields – collections of harmonic oscillators.
- In our case it is sufficient to consider the operator dual to dilaton.

- To leading order in $\epsilon$ the s-wave dilaton operators can be written as a sum of harmonic oscillators

$$\hat{O}_{l=0} = N \sum_{n=1}^{\infty} F(2n)[A_{2n}e^{-i2nt} + A_{2n}^{\dagger}e^{i2nt}]$$

$$[A_m, A_n] = [A_m^{\dagger}, A_n^{\dagger}] = 0 \qquad [A_m, A_n^{\dagger}] = \delta_{m,n}$$

$$[H, A_{2n}^{\dagger}] = (2n)A_{2n}^{\dagger} \qquad [H, A_{2n}] = -(2n)A_{2n}$$

- In the strong 't Hooft coupling regime $A_{2n}^{\dagger}$ creates a single particle dilaton state in the bulk with zero $S^3$ angular momentum
- The integer $n$ is a "radial" quantum number, conjugate to the extra dimension. The energy of this single particle state is $2n$.
- The factor $F(2n)$ can be determined by requiring that the above expansion leads to the correct 2 point function

$$|F(2n)|^2 = \frac{A\pi^4}{3} n^2(n^2 - 1)$$

- The oscillators $A_{2n}^\dagger$ are the analogs of the shifted oscillators of the driven Harmonic Oscillator problem, $\tilde{a}^\dagger$
- In fact,

$$\frac{\partial H}{\partial t} = -\hat{\mathcal{O}}_{l=0}\dot{\Phi}_0 = -N\sum_n F(2n)[A_{2n} + A_{2n}^\dagger]\dot{\Phi}_0$$

- Compare this with the expression in the driven oscillator

$$\frac{\partial H}{\partial t} = \dot{J}(\frac{\tilde{a} + \tilde{a}^\dagger}{\sqrt{2\omega_0}})$$

- Thus

$$\dot{J}_n = -NF(2n)\sqrt{4n}\dot{\Phi}_0$$

- We can now translate all our results for the driven harmonic oscillator to the present problem.

- Consider a coherent state of the form

$$|\psi >= \hat{N}(t)e^{(\sum_n \lambda_n A^\dagger_{2n})}|\phi_0 >$$

- Where $\hat{N}(t)$ is a normalization factor.
- The equation satisfied by $\lambda_n$ is

$$i\frac{d\lambda_n}{dt} = -i\frac{F(2n)}{2n}\dot{\Phi}_0 + 2n\ \lambda_n$$

- The initial conditions are $\lambda_n(-\infty) = 0$, and the boundary dilaton has the property that $\dot{\Phi}_0(-\infty) = 0$
- This equation can be of course solved exactly

$$\lambda_n(t) \ \ = \ \ -\frac{F(2n)\ e^{-2int}}{2n}\int_{-\infty}^{t}\dot{\Phi}_0(t')\ e^{2int'}\ dt'$$

- However, we want to write this solution somewhat differently – by successively integrating by parts

$$\begin{aligned}
\lambda_n(t) &= -\frac{F(2n)\ e^{-2int}}{2n}\int_{-\infty}^{t}\dot{\Phi}_0(t')\ e^{2int'}\ dt' \\
&= \frac{F(2n)}{2n}\left[\frac{\dot{\Phi}_0}{(2in)} - \frac{e^{-2int}}{(2in)}\int_{-\infty}^{t}\ddot{\Phi}_0(t')e^{2int'}\right] \\
&= \frac{F(2n)}{2n}\left[\frac{\dot{\Phi}_0}{(2in)} + \frac{\ddot{\Phi}_0}{4n^2} + \cdots\right]
\end{aligned}$$

- This is an expansion in time derivatives – the adiabatic approximation we are seeking.

- This adiabatic approximation is valid provided

$$|\frac{\ddot{\Phi}_0}{n\dot{\Phi}_0}| \ll 1 \qquad \forall n$$

- It is clearly sufficient to have

$$|\frac{\ddot{\Phi}_0}{\dot{\Phi}_0}| \sim \epsilon \ll 1$$

- Note that n is the characteristic frequency, which is quantized since the theory lives on $S^3$ . If there was no gap in the spectrum, the frequency could be arbitrarily small and the adiabatic approximation would not hold.
- The condition for validity is exactly what we had in our supergravity analysis.

- However for this to be applicable to coherent states which behave classically, we must also have

$$\lambda_n \gg 1$$

- Recall

$$\lambda_n(t) = \frac{F(2n)}{2n}\left[\frac{\dot{\Phi}_0}{(2in)} + \frac{\ddot{\Phi}_0}{4n^2} + \cdots\right]$$

So we must have $|F(2n)\dot{\Phi}_0| \gg n^2$

Since for large n, $F(2n) \sim n^2$ so we have the condition

$$|N\dot{\Phi}_0| \sim N\epsilon \gg 1$$

In this regime we can compare our answers with supergravity.

They agree upto numerical factors -

$$< \hat{\mathcal{O}}_{l=0} > \sim N^2\ddot{\Phi}$$

$$< E > \sim N^2(\dot{\Phi})^2$$

# Small ‘t Hooft Coupling

- The framework developed above applies to all values of the ‘t Hooft coupling – therefore can be extended to the regime of small couplings as well.
- The basic ingredients which went into our coherent state adiabatic approximation are still in place

  (1) The couplings between different oscillators are still suppressed by $\epsilon$ .

  (2) The frequencies are still $O(1)$ for any value of $\lambda$ , so that the system is always far from resonance.

  (3) For $N\epsilon \gg 1$ the states are still classical.
- It would therefore appear that the adiabatic theorem still holds.

- Now, however, we have an infinite tower of string modes – whose duals are gauge invariant operators which become as important as the ones which are dual to supergravity modes.
- This is because for large $\lambda$ the dimensions of higher stringy modes - and hence the frequencies of the corresponding oscillators - are $O(N)$ as opposed to supergravity modes whose frequencies are $O(1)$ .
- For small $\lambda$ , however, the dimensions of all these modes are comparable.

- **So, there is another possibility.**
- There are $O(N^2)$ stringy modes (non-chiral operators) , while there are only $O(1)$ supergravity modes.
- While individual couplings are suppressed by $\epsilon$ , there is a possibility that whatever energy is transferred to these modes may thermalize.
- If thermalization does happen – the energy is dissipated and cannot be extracted back when the coupling rises again to large values.
- At late times one would have $O(N^2\epsilon^2)$ thermalized energy in the system.

- At late times, the ‘t Hooft coupling is again large and we can use known results of AdS/CFT to guess the outcome using entropic arguments.
- For $\epsilon \ll (g_{YM}^2 N)^{5/4}/N$ the result would be a gas of supergravity modes.
- For $(g_{YM}^2 N)^{5/4}/N < \epsilon \ll (g_{YM}^2 N)^{-7/8}$ one would have a gas of higher string modes.
- For $(g_{YM}^2 N)^{-7/8} < \epsilon \ll 1$ one would have small black holes, i.e. Black holes whose size is much smaller tl $R_{AdS}$
- This is the worst that can happen – large black holes require an energy $O(N^2)$ which is much larger than the energy we have.

  They will not form.
- In any case most of space-time would be close to $AdS_5 \times S^5$

- The time scale of interactions between the various collective fields (the string modes) is $\epsilon$.
- The time scale for variation of the gauge theory coupling $\lambda$ is also $\epsilon$ .
- Since there is no separation of these time scales, it is not easy to determine whether thermalization indeed takes place.

# Epilogue

- The region of strong curvature which we studied in this work using gauge theory dual corresponds to a stringy regime in the bulk.
- Quantum corrections are still suppressed.
- Can worldsheet string theory tell us something about this region – particularly about the question of thermalization ?
- String theory in $AdS_5 \times S^5$ is notoriously hard. However in our case some approximate methods may lead to some insight – currently being investigated with *Simeon Hellerman*.

# Concluding Remark

- Most of recent work in this area aims to arrive at toy models of cosmology where the meaning and physics of singularities can be studied in a controlled fashion.
- This is clearly a *caricature* of cosmology – and the investigation is in its early stages.
- Hopefully *(in Sidney Coleman's words)* this is a recognizable carricature.

ありがとうございました