

Structures of Higher Genus

Gromov - Witten Invariants of

Quintic 3-folds

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I. Background: A series of predications
from physics

II. A long journal: Development of
several new theories.

III. Recent Breakthrough: A group of
exceptionally talented young people.

Felix Janda, Shuai Guo, Gile Chen

I. Background:

Quintic 3-fold:

$$X_5 = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subset \mathbb{CP}^4$$

- Simplest Compact Calabi-Yau (CY) 3-fold.

millions of compact CY 3-folds

Everything we said here should generalize

- Building block of string-theoretic model of universe.
- 14 one-parameter models:

③

Gromov-Witten Invariants

$n_{g,d}$ = "virtual" number of holomorphic
curves of genus g , degree d in X_g

- fundamental geometric inv of X_g
- fundamental physical inv of topological string
- Very difficult to compute for both mathematician and physicist !!!

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A holy Grail: Compute $n_{g,d}$

- The effort to compute $n_{0,d}$ led to the birth of 'mirror symmetry': (Givental, Liu-Liarg-Yau) %

- Very difficult: Another ten years to

compute $n_{1,d}$ (Zinger, 2006)

unknown for $n_{g,d}$ $g > 1$

- Deep structures: modularity, finite generation, ... HLE

predication for $n_{g,d}, g \leq 1$

from physics (Klemm's group)

A series of physical Predications:

$$F_g = \sum_{d \geq 0} n_{g,d} q^d$$

1991: Candelas' group: Explicit formula for $F_{g=0}$

(solved by Givental, Liu-Liang-Yau in 96)

Birth of mirror symmetry
as a mathematical subject

(BCOV)

1993: Berthadsky - Cecotti - Ooguri - Vafa: Explicit formula for

$$F_{g=1}, F_{g=2}$$

($F_{g=1}$ was solved by Zinger in 2006)

2004: Yamaguchi - Yau:

$$F_g = P_g(u_1, u_2, u_3, u_4, L) - \text{poly of}$$

five generators u_1, u_2, u_3, u_4, L

constructed explicitly from $g=0$ data

2004: BCOV/Yamaguchi-Yau: holomorphic anomaly equation

$$\frac{\partial P_g}{\partial u_i} = P(P_h, h < g)$$



determine the dependence of P_g on u_i up to an initial condition

$$f_g = P_g(c, c, c, c, L) \quad - \text{degree } 3g-3$$

1993: Conifold gap condition: t_{con} — appropriate coordinate at $g=1$, $F_g(t_{\text{con}}) = \frac{c}{t_{\text{con}}^{2g-2}} + dt_{\text{con}}^w$

1993: orbifold regularity: The analytic continuation of $F_g(g)$ at $g \sim \infty$ is regular.



determine f_g up to $[\frac{2g-2}{r}]$ constant

STRUCTURE

Initial condition:

• $g=2, 3, \Rightarrow \left[\frac{2g-2}{5} \right] = 0$ & $n_{g,0}$ can be computed

• 1999: Katz - Klemm - Vafa: Explicit formula
for $F_{g=3}$

• Castelnuovo bound:

$$\begin{aligned} \overline{F}(\lambda, g) &= \sum_{g \geq 0} \lambda^{2g-2} F_g(g) \\ &= \frac{c(g)}{\lambda^2} + \ell(g) + \sum_{g \geq 0} \sum_{d \geq 0} \sum_{m=1}^{\infty} n_{\beta}^g \frac{1}{m} (2 \sin \frac{md}{2})^{\beta m} \lambda^{2g-2} \end{aligned}$$

formally defines Gopakumar - Vafa inv n_{β}^g
(formally count embedded curve (Lorenz - Barter))

Then. $n_{\beta}^g = 0$ for $g \geq N(\beta)$

Known: Katz : n_1^g, n_2^g, n_3^g

2006: Klenm's group: combine everything
to derive explicit formula for

$$\sum_{g \leq 1}$$

$\Rightarrow$

A record far beyond what mathematician
can prove ($g \leq 1$) !!

What is the difficulty?

Definition of Gromov-Witten inv.

$$\overline{\mathcal{M}}_{g,d}(X_5) = \left\{ \begin{array}{c} \text{degree} = d \\ \text{satisfies certain stability cond} \end{array} \right\} \xrightarrow{\quad} X_5$$

- $\overline{\mathcal{M}}_{g,d}(X_5)$ is compact. (Good)
- $\overline{\mathcal{M}}_{g,d}(X_5)$ is VERY VERY singular (Bad)
- Works of many mathematicians over ten years

$$[\overline{\mathcal{M}}_{g,d}(X_5)]^{\text{vir}} \in H_0(\overline{\mathcal{M}}_{g,d}(X_5), \mathbb{Q})$$
$$n_{g,d} = \int_{\overline{\mathcal{M}}_{g,d}(X_5)} \frac{1}{\dots}$$

Remarks:

• In $g=0$, $\overline{\mathcal{M}}_{0,d}(X_5)$ is not too bad $\Downarrow$

Givental, Liu-Liarg-Yau's solution

• In $g=1$, $\overline{\mathcal{M}}_{1,d}(X_5)$ is already bad. $\Downarrow$

A great deal of hard works
(Zinger)

• In $g \geq 2$, $\overline{\mathcal{M}}_{g,d}(X_5)$ is hopeless!

II. A long journal:

Starting point of my "recent" adventure

(ii)

↪ Search for alternative definition

FJRW - theory (2002 - 2009)

Landau - Ginzburg model:

$$W = X_1^5 + \dots + X_5^5: \mathbb{C}^5 \longrightarrow \mathbb{C}$$

orbifold line
↪ bundle on $\mathbb{C}$

$N_{g,n}$ = "virtual" counting of $(L_1, L_2, L_3, L_4, L_5)$

↑ with $L_i^5 \cong \omega_{\text{log}, \mathbb{C}}$ C-orbifold curve
of orbifold pts

$$F_g^{LG}(t) = \sum_{h \geq 0} N_{g,n} t^n \sim \text{FJRW generating funct.}$$

(12) Comparison Between GW & FJRW - theory

GW - theory:

solve: $\bar{\partial} f = 0$ $f: C \rightarrow X_S$

↑ linear PDE

↑ non linear target

FJRW - theory:

solve: $\bar{\partial} s_i + \overline{\partial_i W(s_1 \cdots s_5)} = 0$ for $s_i \in \Omega^0(L_i)$

↑ semi-linear PDE

↑ linear target

FJRW - theory
is Easier:

2f we can solve it for $W = X^S$

we can solve it for $W = \sum_{i=1}^5 X_i$

5-spin virtual cycle

LG | CY Correspondence

(13) $\Rightarrow$ imply orbifold regularity

$\mathcal{F}_g^{\text{GW}}(2)$ equiv upto $\mathcal{F}_g^{\text{LG}}(t)$
analytic continuation

$$g = t^{-5}$$

expansion
at $g=0/t=\infty$

expansion
at $g=\infty/t=0$

Results

(1) $g=0$, Chiodo-Ruan (2009),

(2) $g=1$, Guo-Ross (2017)

(3) $g \geq 2$, unknown

(4) computation of $\mathcal{F}_g^{\text{LG}}$ for $g \geq 2$ via 5-spin virtual cycle

Janda?

Recent advance: GLSM

↑
{ gauged linear σ -model }

GLSM: (Fan-Jarvis-Ruan)

An new curve counting theory for

LG-model of GIT quotient

$$W: \mathbb{A}^n_{\theta} / G \rightarrow \mathbb{C}$$

- generalize both GW and FJRW theory
- depends on a stability parameter $\epsilon > 0$

Example:

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$$W: \mathbb{C}^5 \times \mathbb{C}^* \xrightarrow{\theta} \mathbb{C} \quad W = P(x_1^5 + \dots + x_5^5)$$

$$\begin{aligned} & \uparrow \\ & (1, 1, 1, 1, 1, -5) \end{aligned} \quad \theta: \mathbb{C}^* \rightarrow \mathbb{C}^* \rightarrow \mathbb{C}^n$$

Chamber structure:

- ① $\theta > 0$, recover GW-theory of X_5
- ② $\theta < 0$, recover FJRW theory of $x_1^5 + \dots + x_5^5$

LG/CY correspondence $\Leftrightarrow$ Wall crossing



(II) Recent Breakthrough (Thanks to a group of exceptionally talented young people)

- BCOV $\mathcal{F}_{g=2}$ formula: Done ✓
 - Katz - Klemm - Vafa $\mathcal{F}_{g=3}$ formula: Sure if we are motivated
 - Yamaguchi - Yau finite generation: Done ✓
 - Holomorphic anomaly equation: Done ✓
 - LG/CY correspondence / orbifold regularity: Needs to write up ✓
 - Conifold gap condition: Have some idea, not enough
- + Katz's computation $n_0^g, n_1^g, n_2^g, n_3^g \Rightarrow \mathcal{F}_g$

Geometric Part: A Logarithmic Compactification
 (cf with Felix Janda)
 (Gile Chen)

A motivating example: r -spin virtual cycle

$$\mathcal{U}_g = \{ (C, L, s) \}. \quad C\text{-curve}, \quad L^{\vee} \cong \omega_{C/g}, \quad s \in H^0(L) \}$$

$$\bigcup \overline{\mathcal{M}}_{g,n}^{\frac{1}{r}} = \{ (C, L, L^{\vee} \cong \omega_{C/g}) \}$$

Assume all the markings are narrow:

$$\begin{array}{ccc} \mathbb{R}^* L & R^* L & \mathbb{R}^* R^* L = E \\ \downarrow \pi^* & \downarrow \pi^* & \downarrow \pi^* \\ \overline{\mathcal{M}}_{g,n}^{\frac{1}{r}} & \overline{\mathcal{M}}_{g,n}^{\frac{1}{r}} & \overline{\mathcal{M}}_{g,n}^{\frac{1}{r}} \end{array}$$

$\downarrow \pi^* = \mu$ section
 $\mu^* \omega = \pi^* \omega_{g,n}^{\frac{1}{r}}$

$\overline{m}_{g,n}^{\text{vir}} = \left(\begin{array}{c} \varphi_0^* R' \pi_{*} \mathcal{L} \\ \downarrow \gamma_{\mu} \\ \pi_{*} \mathcal{L} \end{array} \right)$ defines a local Euler class

Chang-Li-Li: $\sigma_i = S^i$: $R' \pi_{*} \mathcal{L} \rightarrow \mathbb{C}$ - cosection

$$\llbracket \mathcal{U}_{g,n} \rrbracket_{\text{loc}}^{\text{vir}}$$

A naive idea: (i) compactify $\mathcal{U}_{g,n} \rightsquigarrow \overline{\mathcal{U}}_{g,n}$

(ii) extends E and μ s.t. μ does not pick up additional zero

$$\begin{array}{ccc} E & \subset & \\ \downarrow \gamma_{\mu} & & \\ \mathcal{U}_{g,n} & & \end{array} \quad \begin{array}{c} \downarrow \mu \\ \overline{\mathcal{U}}_{g,n} \end{array} \quad \Rightarrow \quad \chi(\bar{E}) = [\mathcal{U}_{g,n}]_{\text{loc}}^{\text{vir}}$$

Problem: $\mathcal{U}$ always pick up poles!

Solution: A delicate compactification

$\overline{\mathcal{U}}_{g,n}$ equipped with a line bundle $\mathcal{L}$
with a section s s.t

$\bar{\sigma}: \text{obs} \rightarrow \mathcal{L}$

$\downarrow$
 $\overline{\mathcal{U}}_{g,n}$

admit $\mathbb{C}^*$ action $s \rightarrow \lambda s$

$\Downarrow$



$[\overline{\mathcal{U}}_{g,n}]^{\text{vir}}, [\overline{\mathcal{U}}_{g,n}]^{\text{vir, red}}$

Two virtual
cycles:

Theorem: $[\overline{\mathcal{U}}_{g,n}]^{\text{vir, red}} = [\mathcal{U}_{g,n}]^{\text{vir}}_{\text{loc}}$

GLSM of quintic

p-field (Chang - Li)

$$\mathcal{U}_g = \{ (C, L, (s_0 \dots s_4) \in H^0(L)^{\oplus 5} ; \quad p \in H^0(L^{\otimes 5} \otimes \omega_C) \}$$

stability

- $(s_0 \dots s_4)$ - stable map $\leadsto$ Chang - Li wall crossing
- $(s_0 \dots s_4)$ - quasi-map. $\Leftrightarrow$ we need Clader-J-R
F-Kim

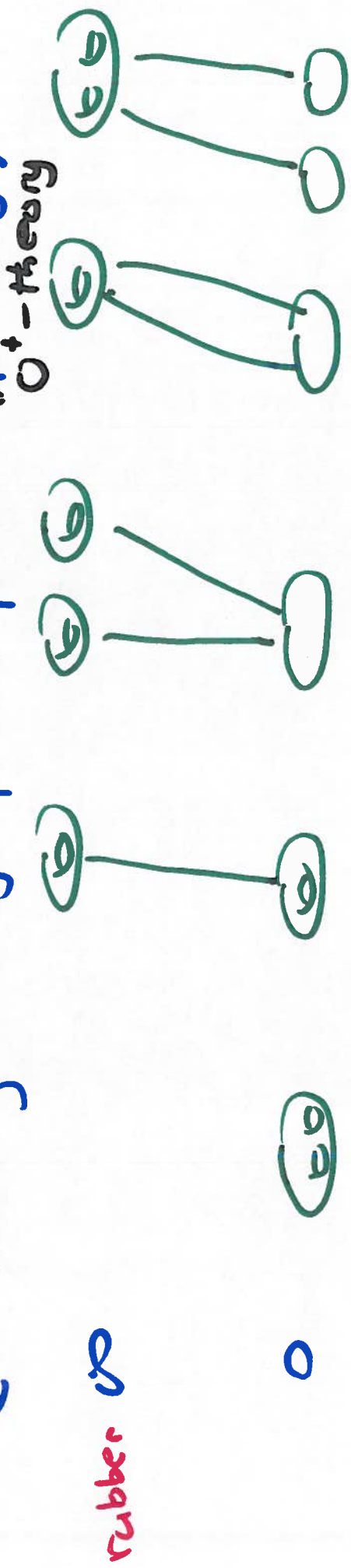
Apply machinery $\Rightarrow \overline{\mathcal{U}}_g$ - logarithmic compactification

with $[\overline{\mathcal{U}}_g]^{\text{vir}} ; \quad [\overline{\mathcal{U}}_g]^{\text{vir, red}} \approx [\mathcal{X}_3]^{\text{vir}}$

A test of new technique: localization for

(Formula part: (Guo-Janda-Ruan) case $g=2$)

$\mathcal{C}^*$ - localization graph (quasi-map theory)



rubber vertex $\Rightarrow$ "effective inv" $C_{g,d}$ $g \geq d$

$$d \leq \left\lceil \frac{2g-2}{5} \right\rceil$$

$$\mathcal{O}(1-5)$$

$\mathcal{O}$ -vertex $\Rightarrow$ twisted theory of $\mathbb{P}^4$

edge $\Rightarrow$ insertion to the twisted theory

$$F_{g=2}^{\mathcal{O}^+} = \text{Sum of five terms}$$

Each term $\in \mathbb{C}$ [standard 5 generators + extra gen]

Cancellation of extra generators $\Rightarrow$ BCOV formula

Get ride of extra generator $\uparrow$ caused by wrong $\mathbb{C}^*$ -action $t \mapsto (1 \ 1 \ 1 \ 1 \ 1 \cdot t)$

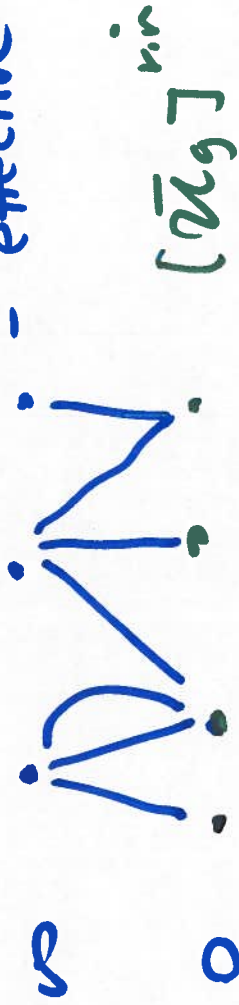
formal quintic: $\lambda \mapsto (1 \cdot \lambda_3 \cdot \lambda_5^2 \cdot \lambda_5^3 \cdot \lambda_5^4 \cdot 1)$
 $\sum \lambda_5 = 1$

Zagier - Zinger: formal quintic theory are generated by standard 5 generators

Comparison formula:

$$[\bar{\mathcal{U}}_g]^{vir, red} = \text{Sum of bi-parti graph}$$

- effective in G_n



$[\bar{\mathcal{U}}_g]^{vir}$

(t, λ) - equivariant

(t, λ) -localization: A pretty structure theorem

$$F_g^{u^+} = \sum_{g, t + g_k \leq g} c_{g, d_1} \cdots c_{g_k, d_k} F_{g, (g, d_1) \cdots (g_k, d_k)}$$



formal quintic theory

Corollary: Yamaguchi-Yau finite generation

Comparison to as much older theorem:

X -4-manifold, $\mathcal{Q}_X$ - Donaldson poly

Kroheimer - Mrowka (92):

$$\mathcal{Q}_X = \sum_i a_i e^{k_i}$$



Seiberg - Witten theory

k_i - Seiberg Witten class

a_i - $\dots$ $\dots$ inv

Speculation: $\exists$ a version of Seiberg-Witten theory for GW-theory?

Seiberg-Witten inv $\rightsquigarrow$ $\langle g_i, d_i \rangle$

Holomorphic anomaly equation:

- rewrite twisted theory using Givental - Teleman theorem
- clever re-package the formula

An interesting feature: same holomorphic anomaly equation for family of theories containing constants c_{g,d_i}

Independent approaches by

- Chang - Li - Li - Liu
- Maulik - Pandharipande
- Fan - Lee

THANK YOU

