Renormalisation group interfaces

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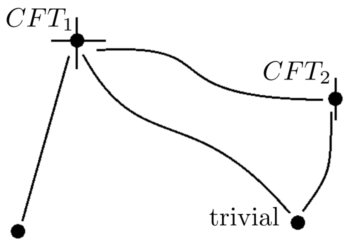
Renormalisation . . .

- CFTs fixed points of renormalisation process of a QFT
- Perturbation $\delta\mathcal{S} = \lambda \int d^d x \, \Phi^0_{UV}$ in UV triggers RG flow to IR



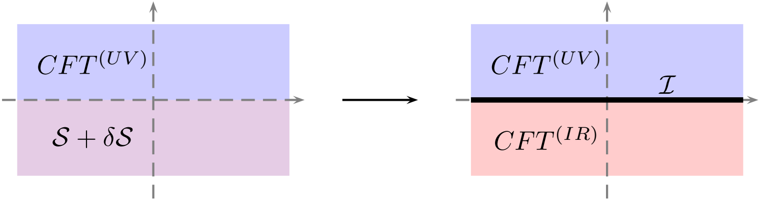
- If IR fixed point non-trivial: Relation degrees of freedom UV/IR

$$\Phi_{UV} \mapsto \Phi_{ren}(\lambda_{bare}) \longrightarrow \Phi_{ren}(\lambda_*) \mapsto \sum_{\Phi_{IR}} b_{\Phi_{UV},\Phi_{IR}} \, \Phi_{IR}$$

Zamolodchikov 86

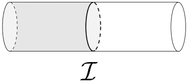
... as an interface

- Perturb only in a half-space: $\delta\mathcal{S} = \lambda \int_{x_1<0} d^d x \, \Phi^0_{UV}$



- Conformal renormalization group interface $\mathcal{I}$ separates $CFT^{(UV)}$ from $CFT^{(IR)}$

Brunner-Roggenkamp 07

- $\langle \Phi_{IR}|$  $|\Phi_{UV}\rangle = \langle \Phi_{IR}|\Phi_{UV}\rangle_{\mathcal{I}} = b_{\Phi_{IR},\Phi_{UV}}$

Outline

Conformal interfaces

Why RG interfaces?

RG interfaces for integrable flows between coset CFTs

Coset RG interfaces and boundary conditions

The 't Hooft limit of $W_{k,N}$ RG interfaces

Connection to holography

Conformal interfaces Affleck-Wong 94, Affleck-Oshikawa 96, Petkova-Zuber 01

- Codimension 1 junction between $CFT^{(1)}$ and $CFT^{(2)}$. Junction condition preserves an $SO(d,1)$ subgroup of $SO(d+1,1)$.
- Locally, for a planar junction: $T^{(1)}_{\perp\,\|} = T^{(2)}_{\perp\,\|}$.
- Transfer matrix $\perp$ and $\|$ to interface should yield same partition function ($d=2$: Cardy's condition).

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- Transfer matrix $\perp$ and $\|$ to interface should yield same partition function ($d=2$: Cardy's condition).
- Rough classification: Reflection and transmission coefficients. In $d=2$, reflection and transmission of energy and momentum:

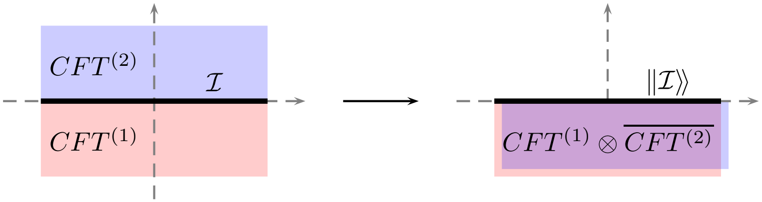
$$\mathcal{R} = \frac{2}{c^{(1)}+c^{(2)}}\left(\langle T^{(2)}\tilde{T}^{(2)}|0^{(1)}\rangle_{\mathcal{I}} + \langle 0^{(2)}|T^{(1)}\tilde{T}^{(1)}\rangle_{\mathcal{I}}\right),$$
$$\mathcal{T} = \frac{2}{c^{(1)}+c^{(2)}}\left(\langle T^{(2)}|T^{(1)}\rangle_{\mathcal{I}} + \langle \tilde{T}^{(2)}|\tilde{T}^{(1)}\rangle_{\mathcal{I}}\right).$$

Quella-Runkel-Watts 06

- $\mathcal{R}+\mathcal{T}=1$, and $0 \le \mathcal{R}, \mathcal{T} \le 1$ in unitary theories.
- $\mathcal{T}=1$: topological interface (symmetries, dualities, projections)
  $\mathcal{R}=1$: boundary condition

Conformal interfaces Affleck-Wong 94, Affleck-Oshikawa 96, Petkova-Zuber 01

- Special conf. trsfs. allow *folding trick* :



What is interesting about RG interfaces?

- Non-perturbative information about RG flows — Brunner-Roggenkamp 07
- "Minimal" interfaces? — Douglas 10, Bachas *etal* 13, Brunner-SC 15
- "Counting" of RG flows — Gukov 15
- Tractable examples of conformal interfaces; many ways to check:
  - From renormalization
  - From fusion with boundary states, or with each other
  - From holography

Some examples of RG interfaces

- $d = 2$ $\mathcal{N} = (2, 2)$ Minimal Models (LG description)

  Brunner-Roggenkamp 07

- (numeric) results in holography (SUSY; relation to holographic RG) Bobev-Pilch-Warner 14, Karndumri-Upathambhakul 17

- $d = 2$ compactified free boson

  Bachas-Gaberdiel 04, Bachas-Brunner 07, Konechny 15

- $d = 2$ theories corresponding to different points in CY moduli spaces Bachas *etal* 13

- $O(N)$ models (free/Wilson-Fisher) Gliozzi *etal* 15

- $d = 2$ flows between Ising and Lee-Yang Konechny 16

- Massive flows (boundary states of the UV) Cardy 17

- Generalised free theories (double trace deformations)

  Melby-Thompson-SC 17

- $d = 2$ coset CFTs (integrable flows) Gaiotto 12

RG interfaces for integrable flows between coset CFTs

Coset model CFTs in 2d

Goddard-Kent-Olive 85

- Representations: Branching spaces $M_{k,\ell} = \dfrac{\hat{a}_k \otimes \hat{a}_\ell}{\hat{a}_{k+\ell}}$
  (a simple algebra, $\hat{a}_k$ affine)

- Some quantities easily derived from individual WZW models:
  - $T^{\text{coset}} = T^{(k)} + T^{(l)} - T^{(k+l)}$
  - Central charge $c = \frac{\dim(a)\,\ell}{\ell + g^\vee}\left(1 - \frac{g^\vee(\ell+g^\vee)}{(k+g^\vee)(k+\ell+g^\vee)}\right)$.
  - Primary states labelled $(r_k, r_\ell, r_{k+\ell})$

- Consider modular A invariant.

- Examples: $a = su(2)$, $\ell = 1$: Virasoro minimal models,
  $\ell = 2$: $\mathcal{N} = 1$ minimal models,
  $a = su(N)$, $\ell = 1$: bosonic $W_{k,N}$ models.

RG interfaces for integrable flows between coset CFTs

Perturbed coset models

Ahn-Bernard-LeClair 90

- Relevant ("thermal") perturbation by $\Phi^{UV}_{(0,0,\mathrm{adj})}$ in $M_{k,\ell}$
- *Massive* and *massless* perturbation: Latter takes
$$M_{k,\ell} \to M_{k-\ell,\ell}$$
- Can view $\mathcal{H}_{k,\ell}$ as representation of algebra defined by (non-local) symmetry currents $J_{(\mathrm{adj},0,0)}$, or $J_{(0,\mathrm{adj},0)}$.
- Lead to currents conserved along the entire flow ⇒ "fractional supersymmetries"
- Perturbation yields integrable QFT.
- $k \gg 1$: Massless flows under perturbative control.

RG interfaces for integrable flows between coset CFTs

Gaiotto's RG interface proposal

Gaiotto 12

- Massless flow $\frac{\hat{a}_k \otimes \hat{a}_\ell}{\hat{a}_{k+\ell}} \to \frac{\hat{a}_{k-\ell} \otimes \hat{a}_\ell}{\hat{a}_k}$
- RG interface corresponds to **boundary state** in $\frac{\hat{a}_k \otimes \hat{a}_\ell}{\hat{a}_{k+\ell}} \otimes \frac{\hat{a}_{k-\ell} \otimes \hat{a}_\ell}{\hat{a}_k}$
- Fractional supersymmetries preserved to all orders in perturbation
- Suggestive: Boundary condition links generators of $\hat{a}_k$, $\hat{a}_l$
  $\Rightarrow$ lives in symmetry sector

$$\left(\frac{\hat{a}_k}{\hat{a}_k}\right) \otimes (\hat{a}_\ell \otimes \hat{a}_\ell) \otimes \left(\frac{\hat{a}_{k-\ell}}{\hat{a}_{k+\ell}}\right)$$

RG interfaces for integrable flows between coset CFTs

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- Behaviour of elementary topological defects under fusion $\Rightarrow$ **Selection rules** for overlap Φ^{UV}, Φ^{IR}: Gaiotto 12
  - Same $\hat{a}_k$ representation labels
  - Same $\hat{a}_\ell$ representation labels
- Perturbing field: $(0, 0; \text{adj}) \to (\text{adj}, 0; 0)$ Ahn-Bernhard-LeClair 90

RG interfaces for integrable flows between coset CFTs

- Ansatz for boundary state:
  - Projection in $\hat{a}_k$ sector: Implemented by topological defect
    Crnkovic etal 89, Gaiotto 12
  - Standard permutation brane in $\hat{a}_\ell$ sector Recknagel 03
  - Standard Cardy state in sector $\hat{a}_{k-\ell}/\hat{a}_{k+\ell}$ Cardy 89

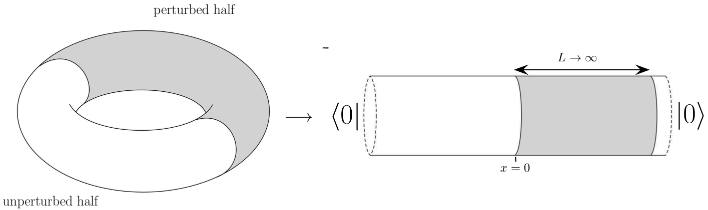
- $\frac{\hat{a}_k \otimes \hat{a}_\ell}{\hat{a}_{k+\ell}} \otimes \frac{\hat{a}_{k-\ell} \otimes \hat{a}_\ell}{\hat{a}_k} \xrightarrow{\mathcal{D}} \hat{a}_\ell \otimes \hat{a}_\ell \otimes \frac{\hat{a}_{k-\ell}}{\hat{a}_{k+\ell}} \| \text{permutation} \otimes \text{Cardy} \rangle\rangle$

- Such boundary states labelled by 4 representations:
$$\mathcal{D}(R_k) \| R_\ell, R_{k-\ell}, R_{k+\ell} \rangle\rangle,$$
  R_i representation label of $\hat{a}_i$.

- Perturbation theory fixes these representation labels.

RG interfaces for integrable flows between coset CFTs

- One way to go: Calculate boundary entropy g Affleck-Ludwig
- Start with UV theory, perturb only on half-space:



- $\log Z^{pert}(L) = \frac{Lc^{UV}}{6} + \sum_n \lambda^n (L f_n + g_n)$
- Read off boundary entropy g of the interface

RG interfaces for integrable flows between coset CFTs

- $\mathcal{D}(R_k)\|R_{k-\ell}, R_\ell, R_{k+\ell}\rangle\rangle = g|0\rangle\rangle + \ldots \;\Rightarrow\;$ fixes $R^i = 0$: Brunner-SC 16

$$\mathcal{D}(0_k)\|0_\ell, 0_{k-\ell}, 0_{k+\ell}\rangle\rangle = \sum_{\{r\}} \frac{\sqrt{S^{(k-\ell)}_{0,r}\,\bar{S}^{(k+\ell)}_{0,r}}}{S^{(k)}_{0,r}} P_{r_k}|r_{k-\ell}, r_\ell, r_\ell, r_{k+\ell}\rangle\rangle_{\mathbb{Z}_2}$$

RG interfaces for integrable flows between coset CFTs

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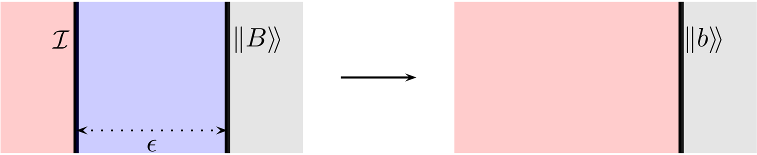
"Gaiotto's recipe": For operators $\Phi^{UV}_{(r_k,r_\ell,r_{k+\ell})}$, $\Phi^{IR}_{(r_{k-\ell},r'_\ell,r'_k)}$, can compute $b_{\Phi_{IR},\Phi_{UV}}$ explicitly as a disc one-point correlator.

Coset RG interfaces and boundary conditions

Important cross check for RG interface: Does it implement the correct flows of boundary conditions from CFT_{UV} to CFT_{IR}?

Fusion of conformal RG interfaces to boundary conditions:



- Simple for topological interfaces: $\mathcal{D}_a\|b\rangle\rangle = N_{ab}{}^c\|c\rangle\rangle$.
- RG interfaces: Divergence for $\epsilon \to 0$ because $\mathcal{I}$ is reflective (Casimir energy) Bachas *etal.* 02

Coset RG interfaces and boundary conditions

In general: Calculation of fusion very hard. Konechny 15

However, Gaiotto interface suggests superficial checks: Selection rules.

First check: Use interplay with topological defects.

- Fractional supersymmetries $\Rightarrow$ there are $(\mathcal{D}_D, \mathcal{D}_d)$:

$$\mathcal{D}_d \cdot \mathcal{I} = \mathcal{I} \cdot \mathcal{D}_D$$

($e.g.$ Virasoro Minimal Models: $D = (\delta, 1)$, $d = (1, \delta)$). Then

$$\mathcal{D}_d \cdot \mathcal{I} \cdot \|A\rangle\rangle = \mathcal{I} \cdot \mathcal{D}_D \cdot \|A\rangle\rangle = \mathcal{I} \cdot {N_{DA}}^B \|B\rangle\rangle \, .$$

Coset models: RG cannot trigger boundary changing perturbation between different $\|B\rangle\rangle \Rightarrow \mathcal{I}$ acts linearly:

$${N_{da}}^b = \sum_{B:B\to b} {N_{DA}}^B \, .$$

Coset RG interfaces and boundary conditions

Second check: Use that

- $\|A\rangle\rangle = \sum_I C_{AI}|I\rangle\rangle$, $\qquad \|a\rangle\rangle = \sum_I C_{ai}|i\rangle\rangle$

Suggests

$$\mathcal{I} \cdot |I\rangle\rangle = \sum_i I_{Ii}|i\rangle\rangle \,,$$

where the I_{Ii} satisfy same selection rules as the b_{Φ_I,Φ_i}.

For Virasoro Minimal Models:

$$\|A_1, A_2\rangle\rangle \to \begin{cases} \|a_1, a_2\rangle\rangle = \|1, A_1\rangle\rangle \,, & A_2 = 1 \\ \|a_1, a_2\rangle\rangle = \|A_2 - 1, A_1\rangle\rangle \,, & 1 < A_2 < k+2 \end{cases}$$

From this, find indeed that Roggenkamp 12

$$I_{Ii} = f(I,i)\delta(I,i) \,, \quad \delta(I,i) = 1 \text{ iff } b_{\Phi_I,\Phi_i} \neq 0 \,.$$

Puzzle: Sometimes $\delta(I,i) = 1$, but $f(I,i) = 0$ — Ishibashi states become massive. Why only in certain special cases?

The 't Hooft limit of $W_{k,N}$ RG interfaces

Melby-Thompson-SC 17

$$\frac{su(N)_k \otimes su(N)_1}{su(N)_{k+1}} \to \frac{su(N)_{k-1} \otimes su(N)_1}{su(N)_k}$$

- N fixed, $k \to \infty$: Limit is trivial interface.
  Limit theories form *continuous orbifolds*. Gaberdiel-Suchanek 12
  Perturbing operator a (marg. irrelevant) current-current deformation.
- N, $k \to \infty$, $\lambda = N/(k+N)$ fixed: 't Hooft limit
  Limit theories are *generalised free CFTs*. Greenberg 61
  Perturbing operator is the double trace of fundamental scalar;

$$\Delta = 2 - 2\lambda\,.$$

The 't Hooft limit of $W_{k,N}$ RG interfaces

Example: g factor

$$g^2 = \frac{S_{00}^{(k-1)} S_{00}^{(k+1)}}{(S_{00}^{(k)})^2} \overset{\text{'tHooft}}{=} \exp\left[\pi \int_0^\lambda \nu^2 \cot(\pi\lambda)\, d\nu \ + \ \frac{\lambda^2}{N} \ + \mathcal{O}(N^{-2})\right] .$$

The 't Hooft limit of $W_{k,N}$ RG interfaces

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Another example: $\langle \Phi^{IR}_{\text{adj},0} | \Phi^{UV}_{0,0} \rangle_{\mathcal{I}}$.
Representative of $\phi^{IR}_{\text{adj},0}$ in numerator of IR coset:

$$|\phi^{IR}_{(\text{adj},0)}\rangle \otimes |0^{(k)}\rangle \propto \left(J_{-1}^{(k-1)a} - (2N+k-1)\, J_{-1}^{(1')a} \right) |J_a^{(k-1)}\rangle \otimes |0^{(1')}\rangle \,,$$

$\mathbb{Z}_2$ overlap replaces $J_{-1}^{(1')a} \mapsto J_{-1}^{(1)a}$:

$$\langle (\phi^{UV}_{0,0}\, \tilde{\phi}^{IR}_{\text{adj},0})\, \mathbb{Z}_2 (\tilde{\phi}^{UV}_{0,0}\, \phi^{IR}_{\text{adj},0}) \rangle = \frac{1-\lambda}{1+\lambda} \frac{1}{k} \,.$$

Prefactor

$$\frac{\sqrt{S_{\text{adj}0}^{(k-1)} S_{00}^{(k+1)}}}{S_{00}^{(k)}} = g\, \frac{k \sin(\pi\lambda)}{\pi(1-\lambda)} + \mathcal{O}(N^0)$$

Together: $\qquad b_{\Phi^{UV}_{00}, \Phi^{IR}_{\text{adj}0}} = \dfrac{\sin(\pi\lambda)}{\pi(1+\lambda)}$

The 't Hooft limit of $W_{k,N}$ RG interfaces

Last example: Reflection and transmission

$$\langle T^{(UV)}\tilde{T}^{(UV)}|0^{(IR)}\rangle_{\mathcal{I}} = \tfrac{1}{2}\lambda^2(1+\lambda),$$
$$\langle 0^{(UV)}|T^{(IR)}\tilde{T}^{(IR)}\rangle_{\mathcal{I}} = \tfrac{1}{2}\lambda^2(1-\lambda),$$
$$\langle T^{(UV)}|T^{(IR)}\rangle_{\mathcal{I}} = N \times \tfrac{1}{2}(1-\lambda^2).$$

$$\mathcal{R} = \tfrac{2}{c^{(UV)}+c^{(IR)}}\left(\langle T^{(2)}\tilde{T}^{(2)}|0^{(1)}\rangle_{\mathcal{I}} + \langle 0^{(2)}|T^{(1)}\tilde{T}^{(1)}\rangle_{\mathcal{I}}\right) = 0\,,$$
$$\mathcal{T} = \tfrac{2}{c^{(UV)}+c^{(IR)}}\left(\langle T^{(2)}|T^{(1)}\rangle_{\mathcal{I}} + \langle \tilde{T}^{(2)}|\tilde{T}^{(1)}\rangle_{\mathcal{I}}\right) = 1\,.$$

So interface seems topological — but only in a sense, since

$$c^{(IR)},\, c^{(UV)} \sim N \to \infty\,.$$

All higher spin symmetry currents are broken.

Connection to holography

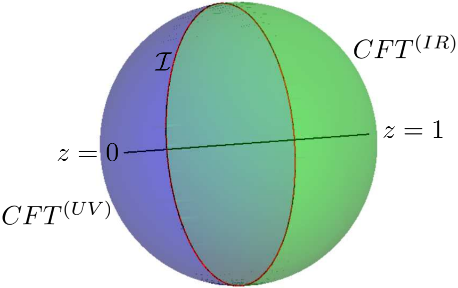
- 't Hooft limit of $W_{k,N}$ models dual to Higher Spin Theory on AdS_3 (H^3). Gaberdiel-Gopakumar 12
- Perturbing operator $\Phi^{(UV)}_{0,\mathrm{adj}}$ is the double trace of the scalar $\Phi^{(UV)}_{0,f}$.
- $\Phi^{(UV)}_{0,f}$ corresponds to massive scalar field φ in the bulk.
- In the IR, scalar is $\varphi \leftrightarrow \Phi^{(IR)}_{f,0}$: Classically field of same mass in bulk; dimension $\Delta = 2 \pm 2\lambda$ determined by boundary condition (2 consistent quantisations). Klebanov-Witten 99
- Can compute two-point functions of φ in bulk by QFT methods.

Connection to holography

Bulk Janus coordinates $X = (z, x)$, Bak *etal* 03

$$ds^2_{H^{d+1}} = \frac{dz^2}{4z^2(1-z)^2} + \frac{ds^2_{H^d}(x)}{4z(1-z)}, \qquad z \in (0,1)$$

slice H^3 into copies of H^2.



Then compute propagator $G(X, X')$ for scalar φ of mass $m^2 = \Delta_\pm(2 - \Delta_\pm)$, with corresponding boundary conditions.

Connection to holography

Standard AdS/CFT procedure gives the 2-point correlation functions (flat frame coordinates)

$$\left\langle \Phi^{IR}_{f,0}(x)\Phi^{IR}_{f,0}(x')\right\rangle = \frac{1}{|x-x'|^{2\Delta_{IR}}}\left(1 + B\,\xi^{\Delta_{IR}}{}_2F_1\left(\begin{matrix}1,\Delta_{IR}\\ \Delta_{IR}+1\end{matrix}\,\middle|\,-\xi\right)\right)$$

with $B = \frac{\sin(\pi\lambda)}{\pi(1+\lambda)} = b_{\Phi^{IR}_{\mathrm{adj},0},\Phi^{UV}_{0,0}}$,

$$\left\langle \Phi^{UV}_{0,f}(x)\Phi^{IR}_{f,0}(x')\right\rangle = \sqrt{\frac{\sin(\pi\lambda)}{\pi\lambda}}\,\frac{1}{\sqrt{\Gamma(\Delta_{IR})\Gamma(\Delta_{UV})}}\,\frac{(-\xi)^{-1}}{(2y')^{\Delta_{IR}}(2y)^{\Delta_{UV}}}$$

y, y' distance from interface, $\xi = -(x-x')^2/(4yy')$ conf. cross ratio.

- Constants match with interface prediction.
- Can also compute g factor (contribution of interface to free energy).
- Analysis of bulk generalises easily to any dimension.

Summary & Outlook

- Conformal RG interfaces capture universal (non-perturbative) data of RG flows
- Examples of RG interfaces can be explicitly constructed
- RG interfaces allow various cross-checks:
  (perturbtive) RG calculations, fusion with boundary conditions, Ishibashi-states, checks from holography

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- Fusion of (Gaiotto's) RG interfaces
- Does RG interface really always minimise g within symmetry class?
- Distance in phase space from properties of RG interface?
- Holography: Can we give a prescription for RG interface setup for every holographic RG flow?
- What can interface quantities tell us about the RG flow / relation of RG flows?
- Entanglement across interface