Singularity theorems and the stability of compact extra dimensions

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Outline

Introduction: Penrose on extra dimensions

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To reach such conclusion he used the celebrated **singularity theorems**.

The classical Hawking-Penrose theorem

Theorem (Hawking and Penrose 1970)

If the convergence, causality and generic conditions hold and if there is one of the following:

- *a closed achronal set without edge,*
- *a closed trapped surface,*
- *a point with re-converging light cone*

then the space-time is causal geodesically incomplete.

Penrose's argument

- To use the singularity theorems, Penrose starts with a $(4+n)$-dimensional direct product $M_4 \times \mathcal{Y} = \mathbb{R} \times \mathbb{R}^3 \times \mathcal{Y}$ with metric as in e.g.

$$g = -dt^2 + dx^2 + dy^2 + dz^2 + g_{\mathcal{Y}}$$

and perturbs initial data given on a slice $\mathbb{R}^3 \times \mathcal{Y}$ (say $t = 0$) such that they do not 'leak out' into the $\mathbb{R}^3$-part: they only disturb the $\mathcal{Y}$-geometry.

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- He then forgets about the 3-dimensional typical space (in red) and considers a $(1+n)$-dimensional "reduced spacetime" $(\mathcal{Z}, g_{red})$ whose metric g_{red} is the evolution (e.g. Ricci-flat solution) of the initial data specified at $\mathcal{Y}$ $(t = 0)$.

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- the entire spacetime would be given by $\mathbb{R}^3 \times \mathcal{Z}$ with direct product metric

$$g_{pert} = g_{red} + dx^2 + dy^2 + dz^2$$

Penrose's argument (continued)

- But then, the H-P singularity theorem applies to $(\mathcal{Z}, g_{red})$ as it contains a compact slice and satisfies the convergence condition (because $R_{\mu\nu} = 0$).

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- He concluded that *"if we wish to have a chance of perturbing $\mathcal{Y}$ in a finite generic way so that we obtain a* non-*singular perturbation of the full (4+n)-spacetimes $M_4 \times \mathcal{Y}$, then we must turn to consideration of disturbances that significantly spill over into the M_4 part of the spacetime"*.

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- However, he claimed that such general disturbances are even more dangerous (due to the large approaching Planck-scale curvatures that are likely to be present in $\mathcal{Y}$).
- He defended that there is good reason to believe that these general perturbations will also result in spacetime singularities using again the H-P singularity theorem, but now using the *point with reconverging light cone* condition.

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"I believe that it is possible to show that with a generic but small perturbation (...) this saving property will be destroyed, so that the (...) singularity theorem will indeed apply, but a fully rigorous demonstration (...) is lacking at the moment. Details of this argument will be presented elsewhere in the event that it can be succinctly completed".

Other arguments

- Almost simultaneously Carroll *et al* argued that (large) extra dimensions must be dynamically governed by classical GR
[S.M.Carroll, J. Geddes, M.B.Hoffman, and R.M.Wald, Classical stabilization of homogeneous extra dimensions, Phys. Rev. D **66** (2002) 024036]

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- Since then, there have been several works analyzing this potential problem. For instance, Steinhardt and Wesley discussed how accelerated expansion imposes strong constraints on compact extra dimensions. [P.J.Steinhardt and D. Wesley, Dark energy, inflation, and extra dimensions, Phys. Rev. D **79** (2009) 104026]

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- Their conclusions were criticized by Koster and Postma, where one can find references to many other no-go and instability theorems. [R. Koster and M. Postma, A no-go for no-go theorems prohibiting cosmic acceleration in extra dimensional models, JCAP 12 (2011) 015]

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- I want to concentrate here on the arguments based on the existence of singularities.

The classical Hawking-Penrose theorem (again)

Theorem (Hawking and Penrose 1970)

If the convergence, causality and generic conditions hold and if there is one of the following:

- *a closed achronal set without edge, (co-dimension 1)*
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What about co-dimensions $3, \ldots, D-1$ — for instance, closed spacelike curves?

The Penrose singularity theorem

Theorem (The 1965 Penrose singularity theorem)

If the spacetime contains a non-compact Cauchy hypersurface and a closed trapped surface, and if the null convergence condition holds, then there exist incomplete null geodesics.

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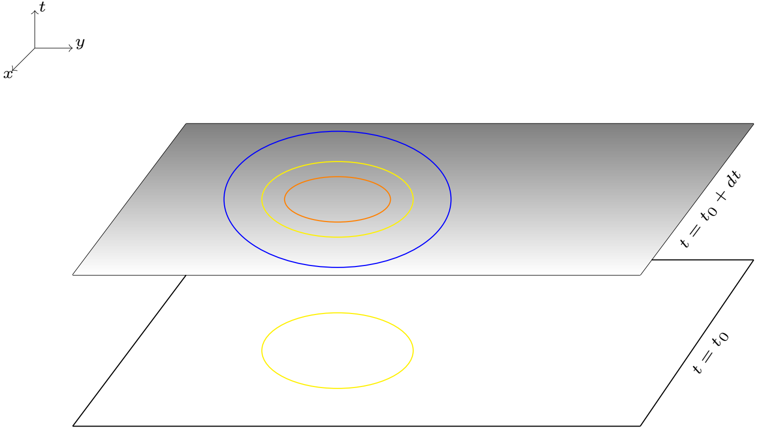
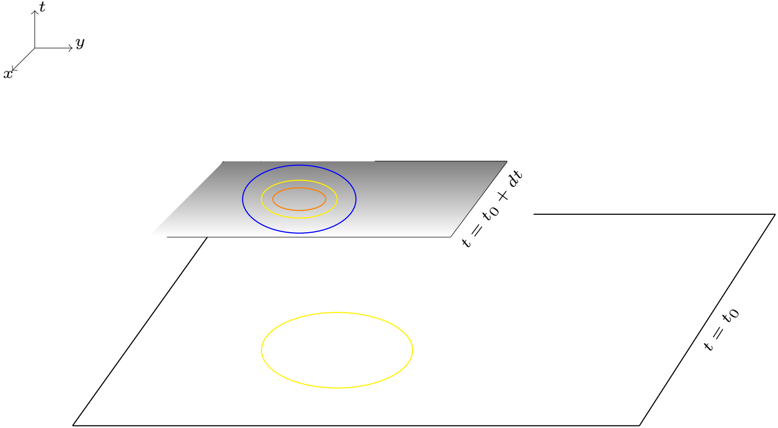
These are closed surfaces (that is, compact without boundary) such that their area tends to decrease locally along any possible *future* direction. (There is a dual definition to the past).

“Normal situation”



Possible trapping in contracting worlds



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Some time ago, Galloway and I started to analyze the reasons behind the absence of other co-dimensions in the H-P singularity theorem, and we realized that the three conditions (on the point with reconverging light cone, on the closed trapped surface, and on the spacelike compact slice) can be unified into one single criterion of geometrical basis.

And that this criterion is valid for any other co-dimension!

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And that this criterion is valid for any other co-dimension!

The unification concept of trapping for arbitrary co-dimension:
$\Longrightarrow$ The mean curvature vector $\vec{H}$!

Mathematical interlude: trapped submanifolds

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- Decomposing the derivatives of tangent vector fields $\{\vec{e}_A\}$ into its tangent and normal parts we have

$$e^{\rho}_{A}\nabla_{\rho}e^{\mu}_{B} = \overline{\Gamma}^{C}_{AB}e^{\mu}_{C} - K^{\mu}_{AB}$$

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- If they correspond to (future) null normals, they are called (future) null expansions.

Future-trapped subamanifolds: $\vec{H}$ is future on ζ

Definition (Trapped submanifold)

A spacelike submanifold ζ is said to be ***future trapped*** *(f-trapped from now on) if $\vec{H}$ is timelike and future-pointing everywhere on ζ, and similarly for past trapped.*

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Now that we have trapped submanifolds of any dimension, can we still get singularity theorems based on them?

Closed trapped submanifolds at work:

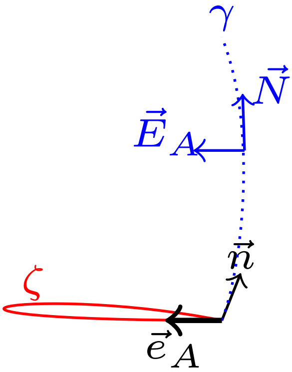
XXI-century singularity theorems

The parallel propagated projector $P^{\mu\nu}$

Notation

- n_μ: *future-pointing* normal to the spacelike submanifold ζ,
- γ: geodesic curve tangent to n^μ at ζ
- u: affine parameter along γ ($u = 0$ at ζ).
- N^μ: geodesic vector field tangent to γ ($N^\mu|_{u=0} = n^\mu$).
- $\vec{E}_A$: vector fields defined by parallel propagating $\vec{e}_A$ along γ ($\vec{E}_A|_{u=0} = \vec{e}_A$)
- By construction $g_{\mu\nu} E^\mu_A E^\nu_B$ is independent of u, so that $g_{\mu\nu} E^\mu_A E^\nu_B = g_{\mu\nu} e^\mu_A e^\nu_B = \gamma_{AB}$
- $P^{\nu\sigma} \equiv \gamma^{AB} E^\nu_A E^\sigma_B$ (at $u = 0$ this is the projector to ζ).

Notation on a picture



Generalized Hawking-Penrose singularity theorem

Theorem (Generalized Hawking-Penrose singularity theorem)

If the chronology, generic and convergence conditions hold and there is a closed f-trapped submanifold ζ of arbitrary co-dimension such that

$$\boxed{R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} \geq 0} \quad (1)$$

along every null geodesic emanating orthogonally from ζ then the spacetime is causal geodesically incomplete.

(G.J. Galloway and J.M.M. Senovilla, Singularity theorems based on trapped submanifolds of arbitrary co-dimension. *Class. Quantum Grav.* **27** (2010) 152002)

Remarks:

$$\boxed{R_{\mu\nu\rho\sigma}N^\mu N^\rho P^{\nu\sigma} \geq 0} \tag{1}$$

1. Spacelike hypersurfaces: $m = 1$, there is a unique timelike orthogonal direction n_μ. Then $P^{\mu\nu} = g^{\mu\nu} - (N_\rho N^\rho)^{-1} N^\mu N^\nu$ and (1) reduces to

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(the *timelike convergence condition* along γ).
2. Spacelike 'surfaces': $m = 2$, there are two independent null normals on ζ, say n_{μ} and ℓ_{μ}. (Define L_{μ} parallelly propagating ℓ_{μ} on γ). Then, $P^{\mu\nu} = g^{\mu\nu} - (N_{\rho}L^{\rho})^{-1}(N^{\mu}L^{\nu} + N^{\nu}L^{\mu})$ and again (1) reduces to
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3. points: $m = D$, (1) could be rewritten as a 'generic' condition $R_{\mu\nu\rho\sigma} N^\mu N^\rho > 0$.

The generalized Penrose singularity theorem

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If (M, g) contains a non-compact Cauchy hypersurface Σ and a closed f-trapped submanifold ζ of arbitrary co-dimension, and if

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holds along every future-directed null geodesic emanating orthogonally from ζ, then (M, g) is future null geodesically incomplete.

(G.J. Galloway and J.M.M. Senovilla, *ibid.*)

2nd generalized Penrose singularity theorem

No need for trapped submanifold!

The conclusion of the generalized Penrose theorem remains valid if the curvature condition and the trapping condition assumed there are **jointly replaced** by

$$\int_0^a R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} du > \theta(\vec{n}) \, ,$$

along each future inextendible null geodesic $\gamma : [0, a) \to M$ emanating orthogonally from ζ with initial tangent n^μ.

2nd generalized Penrose singularity theorem

Theorem

If (M, g) contains a non-compact Cauchy hypersurface Σ and is null geodesically complete, then for every closed spacelike submanifold ζ there exists at least one null geodesic γ with initial tangent n^μ orthogonal to ζ along which

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Observe that there is no restriction on the sign of $\theta(\vec{n})$.

Higher-dimensional spacetimes:

(warped) products

Direct product: “it just fails”

- Consider a spacetime $M = M_1 \times M_2$, $x^\mu = (x^a, x^i)$, with direct product metric

$$g_{\mu\nu} dx^\mu dx^\nu = \hat{g}_{ab}(x^c) dx^a dx^b + \bar{g}_{ij}(x^k) dx^i dx^j$$

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- Let $\varsigma \subset M_2$ be compact and dim $\varsigma = k$, and let e^i_A be tangent vectors to ς. Then, their parallel transports along geodesics are such that $\vec{E}_A = (0, \overline{E}^i_A)$

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- Geodesics decompose too, tangent vectors are $\vec{N} = (\hat{N}^a, \overline{N}^i)$, with $\hat{N}^a$ and $\overline{N}^i$ geodesic in $(M_1, \hat{g})$ and $(M_2, \bar{g})$, respectively.

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Direct product: “it just fails”

- Consider a spacetime $M = M_1 \times M_2$, $x^\mu = (x^a, x^i)$, with direct product metric
$$g_{\mu\nu}dx^\mu dx^\nu = \hat{g}_{ab}(x^c)dx^a dx^b + \bar{g}_{ij}(x^k)dx^i dx^j$$
- $R^\alpha{}_{\beta\mu\nu} = \left(\hat{R}^a{}_{bcd}, \overline{R}^i{}_{jkl}\right)$
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- But, there are $\perp$ ς-null geodesics with $\overline{n}^i = \overline{N}^i(0) = 0$, and for these $\overline{N}^i(u) = 0$, and $\theta(\vec{n}) = 0$, so that any of the two conditions would read

$$0 > 0 \qquad \text{(just fails)}$$

Perturbations: warped products

- Consider perturbing the previous spacetime. The simplest way to do it (geometrically) is by breaking the direct product structure and letting one of the two pieces influence the other:

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  2. Dynamical: the base $(M_1, \hat{g})$ is Lorentzian.
- They imply very different physical consequences! Actually, case 1 does not lead to singularities.

Extra-dimension spreading: "just fails" too

For case ❶, extra-dimension spreading over the Lorentzain part, either the latter is geodesically incomplete by itself or not, the extra dimensions being unable to turn it into null geodesically incomplete.

This follows for instance from a known result that if the Riemannian base of a warped product is complete —which is always the case for compact base— then the spacetime is geodesically complete if and only if the fiber so is.

[A. Romero and M. Sánchez, On completeness of certain families of semi-Riemannian manifolds, *Geom. Dedicata* **53** (1994) 103-117]

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Thus, Penrose's suggestion that "disturbances that significantly spill over into the 4-dimensional part of the spacetime" would be more dangerous and will result in singularities does not seem to sustain —at least in this warped-product situation.

Warped products: Curvature

- Recall $g_{\mu\nu}dx^\mu dx^\nu = \hat{g}_{ab}(x^c)dx^a dx^b + f^2(x^c)\bar{g}_{ij}(x^k)dx^i dx^j$

  $a, b, \ldots, h$ indices on 4-dimensional M_1; i, j, k, l indices on n-dimensional M_2. Total dimension $D := 4 + n$

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- $R^a{}_{ijk} = 0, \qquad R^i{}_{abc} = 0, \qquad R^i{}_{jab} = 0$
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- $R^i{}_{jkl} = \overline{R}^i{}_{jkl} - \hat{\nabla}^a f\hat{\nabla}_a f\left(\delta^i_k\bar{g}_{jl} - \delta^i_l\bar{g}_{jk}\right)$
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- $R_{ab} = \hat{R}_{ab} - n\dfrac{1}{f}\hat{\nabla}_a\hat{\nabla}_b f$
- $R_{ai} = 0$
- $R_{ij} = \bar{R}_{ij} - \bar{g}_{ij}\left(f\hat{\nabla}^b\hat{\nabla}_b f + (n-1)\hat{\nabla}_b f\hat{\nabla}^b f\right)$

Warped products: null geodesics

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- This tells us that the acceleration of the M_1-projected curve is always parallel to the gradient of f.

Warped products: null geodesics (2)

- From the above one knows that

$$\bar{g}_{ij}\bar{\mathrm{N}}^i\bar{\mathrm{N}}^j = \frac{C}{f^4}, \qquad C = \text{const.}$$

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- $C = 0$ means that the null geodesic lives exclusively in the Lorentzian part $(M_1, \hat{g})$ of the warped product.

Parallel transport along null geodesic $\perp \varsigma$, case ❷

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$$E^\mu_A = (0, \bar{E}^i_{A\|}/f)$$

where $\bar{E}^i_{A\|}$ are the parallel transports of $\vec{e}^{\,i}_A$ along the projected curve $\overline{\gamma} : x^i(u)$: $\bar{N}^j \overline{\nabla}_j \bar{E}^i_\| = 0, \qquad \bar{E}^i_\|(0) = \vec{e}^{\,i}.$

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- $g_{\mu\nu} N^\mu E^\nu_A = 0 \quad \Longrightarrow \quad \bar{g}_{ij} \bar{N}^i \bar{E}^j_{A\|} = 0.$

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- $g_{\mu\nu} N^\mu E^\nu_A = 0 \quad \Longrightarrow \quad \bar{g}_{ij} \bar{N}^i \bar{E}^j_{A\|} = 0.$
- $g_{\mu\nu} E^\mu_B E^\nu_A = \delta_{BA} \quad \Longrightarrow \quad \bar{g}_{ij} \bar{E}^i_{A\|} \bar{E}^j_{B\|} = \delta_{AB}.$

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- In this case the tensor $P^{\mu\nu} = \gamma^{AB} E^\mu_A E^\nu_B$ reads

$$P^{ab} = 0, \quad P^{ia} = 0, \quad P^{ij} = \frac{1}{f^2} \delta^{AB} \bar{E}^i_{A\|} \bar{E}^j_{B\|}$$

Expression (1), case ❷

- $R_{\mu\nu\rho\sigma}N^{\mu}N^{\rho}P^{\nu\sigma} = \delta^{AB}\bar{R}_{ijkl}\bar{N}^{i}\bar{N}^{k}\bar{E}^{j}_{A\|}\bar{E}^{l}_{B\|} - (D-m)\frac{1}{f}\frac{d^2 f}{du^2}|_{\gamma}$

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- A simple computation gives, for the initial expansion along $\vec{n}$:

$$\theta(\vec{n}) = \bar{\theta}_{\bar{n}} + (D-m)\frac{1}{f_0}\frac{df}{du}(0)$$

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- The integrated condition in the singularity theorem reads then

$$\int_0^\infty \left(\delta^{AB}\overline{R}_{ijkl}\bar{N}^i\bar{N}^k\bar{E}^j_{A\|}\bar{E}^l_{B\|} - (D-m)\frac{1}{f}\frac{d^2f}{du^2}|_\gamma\right) du > \bar{\theta}_{\bar{n}} + (D-m)\frac{1}{f_0}\frac{df}{du}(0)$$

Singularity theorems in warped products

Theorem

Let $M = M_1 \times_f M_2$ be a null geodesically complete D-dimensional warped product spacetime with Riemannian fiber $(M_2, \bar{g})$ and metric

$$g_{\mu\nu} dx^\mu dx^\nu = \hat{g}_{ab}(x^c) dx^a dx^b + f^2(x^c) \bar{g}_{ij}(x^k) dx^i dx^j$$

containing a non-compact Cauchy hypersurface. Then, every compact submanifold $\zeta \subset M_2$, of any possible co-dimension m, launches at least one future-directed null geodesic emanating orthogonally to ζ satisfying the inequality

$$\int_0^\infty \left(\delta^{AB} \overline{R}_{ijkl} \bar{N}^i \bar{N}^k \bar{E}^j_{A\|} \bar{E}^l_{B\|} - (D-m) \frac{1}{f} \frac{d^2 f}{du^2}|_\gamma \right) du \leq \bar{\theta}_{\bar{n}} + (D-m) \frac{1}{f_0} \frac{df}{du}(0) \, .$$

Analysis of the inequality condition

The negation of the condition:

$$\int_\gamma \left(\delta^{AB} \overline{R}_{ijkl} \bar{N}^i \bar{N}^k \bar{E}^j_{A\|} \bar{E}^l_{B\|} - (D-m) \frac{1}{f} \frac{d^2 f}{du^2} \right) du$$
$$> \bar{\theta}_{\bar{n}} + (D-m) \frac{1}{f_0} \frac{df}{du}(0)$$

(There is also a version to the past).

- For any $\zeta \subset M_2$, there are always ζ-orthogonal null geodesics with $\bar{n}^i = 0$ and thus with $\bar{N}^i(u) = 0$ (those with $C = 0$).

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- in more geometrical terms this is

$$-\int_\gamma \frac{1}{f}\hat{N}^a\hat{N}^b\hat{\nabla}_a\hat{\nabla}_b f > \left(\frac{1}{f}\hat{N}^a\hat{\nabla}_a f\right)(0)$$

Analysis of the inequality condition (continued)

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- The last is, therefore, a necessary condition (along all null geodesics $\perp$ to ζ with $C = 0$) for the singularities to appear.
- If this condition actually holds for all null geodesics starting at a given "M_2" (i.e., for a choice of x^a =const.), then this M_2 is itself a compact submanifold leading to null geodesic incompleteness.
- Nevertheless, if this does not happen for any choice of "M_2", it can happen for an appropriate subset and one can still have null incompleteness if the corresponding null geodesics are orthogonal to particular submanifolds $\zeta \subset M_2$.

Analysis of the inequality condition (continued)

- The last is, therefore, a necessary condition (along all null geodesics $\perp$ to ζ with $C = 0$) for the singularities to appear.
- If this condition actually holds for all null geodesics starting at a given "M_2" (i.e., for a choice of x^a =const.), then this M_2 is itself a compact submanifold leading to null geodesic incompleteness.
- Nevertheless, if this does not happen for any choice of "M_2", it can happen for an appropriate subset and one can still have null incompleteness if the corresponding null geodesics are orthogonal to particular submanifolds $\zeta \subset M_2$.
- In this case, one still needs to check that the found inequality condition holds for the remaining null geodesics orthogonal to ζ, those with $C > 0$.

Analysis of the necessary condition for $C = 0$

- Recall: $-\int_\gamma \frac{1}{f}\hat{N}^a\hat{N}^b\hat{\nabla}_a\hat{\nabla}_b f > \left(\frac{1}{f}\hat{N}^a\hat{\nabla}_a f\right)(0)$

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- Observe that, from the expression of the Ricci tensor and as $N^\mu = (\hat{N}^a, 0)$ for these null geodesics, the null energy condition (NEC) on them reads

$$R_{\mu\nu}N^\mu N^\nu = \hat{R}_{ab}\hat{N}^a\hat{N}^b - n\frac{1}{f}\hat{N}^a\hat{N}^b\hat{\nabla}_a\hat{\nabla}_b f \geq 0$$

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- This immediately implies

$$-\int_\gamma \frac{1}{f}\hat{N}^a\hat{N}^b\hat{\nabla}_a\hat{\nabla}_b f \geq -\frac{1}{n}\int_\gamma \hat{R}_{ab}\hat{N}^a\hat{N}^b \leq 0$$

where the last inequality follows if the NEC holds on average in the noticeable, observed, 4-dimensional spacetime.

Back to the inequality condition

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$$\int_\gamma \left(\delta^{AB} \overline{R}_{ijkl} \bar{N}^i \bar{N}^k \bar{E}^j_{A\|} \bar{E}^l_{B\|} - (D-m) \frac{1}{f} \frac{d^2 f}{du^2} \right) du > \bar{\theta}_{\bar{n}} + (D-m) \frac{1}{f_0} \frac{df}{du}(0)$$

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- Hence, we need an analysis of the behaviour of $d^2 f / du^2$ along these null geodesics.

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- The general expression for this second derivative along the given geodesics is

$$d^2f/du^2|_\gamma = (C/f^3)\hat{\nabla}^b f \hat{\nabla}_b f + \hat{N}^a \hat{N}^b \hat{\nabla}_a \hat{\nabla}_b f$$

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- Actually, keeping the values of the coupling constants (and the Planck mass) independent of position in space implies f should depend only on time and thus $\hat{\nabla}^b f \hat{\nabla}_b f < 0$.
- In consequence, $d^2 f/du^2|_\gamma$ will become negative in a large class of reasonable situations.

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$$X^2 := (D-m)\left(\frac{1}{f_0}\frac{df}{du}(0) + \int_\gamma \frac{1}{f}\frac{d^2 f}{du^2}du\right) > 0$$

can be proven to be strictly positive for the family of null geodesics orthogonal to a given compact ζ. It follows that the condition such that singularities arise according to the theorem becomes

$$\int_{\bar{\gamma}} \delta^{AB}\bar{R}_{ijkl}\bar{N}^i\bar{N}^k\bar{E}^j_{A\|}\bar{E}^l_{B\|}du > \bar{\theta}_{\bar{n}} - X^2$$

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- The importance of this form is that the lefthand side is a quantity relative to the extra-dimensional space $(M_2, \bar{g})$ exclusively.

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- If the co-dimension is 5, that is dimension $n-1$, then $\delta^{AB}\bar{R}_{ijkl}\bar{N}^i\bar{N}^k\bar{E}^j_{A\|}\bar{E}^l_{B\|} = \bar{R}_{ij}\bar{N}^i\bar{N}^j$ (Ricci-flat M_2 OK!)

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- As M_2 is compact, the integral may be the sum of an infinite number of integrals on closed geodesics.
- Therefore, one can find many (physical) situations where this incompleteness arises.

Concluding remarks

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- In essence, *dynamical* perturbations can sometimes lead to the appearance of singularities, destroying the stationary classical stability of the extra-dimensional space.
- On a positive side, the condition as given involving quantities of only the extra-dimensional space may help in finding the stable possibilities, providing information on which classes of compact extra-dimensions may be viable and why —and for which warping functions $f(t)$.

Thank you for your attention!

あなたの注意のために大変ありがとう