Classifying 5d SCFTs using Calabi-Yau geometry

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The subject of this talk is recent progress in classifying 5d SCFTs with interacting UV fixed points, and is based on:

- 1705.05836 [PJ, Hee-Cheol Kim, Cumrun Vafa, Gabi Zafrir]
- 1801.04036 [PJ, Sheldon Katz, Hee-Cheol Kim, Cumrun Vafa]
- 1809.01650 [Lakshya Bhardwaj, PJ]
- 1811.10616 [Lakshya Bhardwaj, PJ]
- (to appear) [Lakshya Bhardwaj, Hee-Cheol Kim, PJ, Houri Tarazi, Cumrun Vafa]

and references therein.

A bit of history

In the 1990s, it was found that string theory predicts UV complete SQFTs in 5d and 6d. Investigation of 5d $\mathcal{N} = 1$ theories was initiated by series of papers from ‘96-‘98. [Seiberg] [Morrison-Seiberg] [Douglas-Katz-Vafa] [Ganor-Morrison-Seiberg] [Intriligator-Morrison-Seiberg (IMS)] [Diaconescu-Entin]

These authors used string compactifications and SUSY to identify many 5d theories, notably leading to some partial classifications:

1. Geometric classification of rank one SCFTs in terms of del Pezzo surfaces dP_n, $n = 0, \ldots, 8$ and $\mathbb{P}^1 \times \mathbb{P}^1$
2. Gauge theory classification in terms of $(\mathfrak{g}, \mathbf{R}, k)$
3. (p, q) 5-brane webs in type IIB

Status of the problem

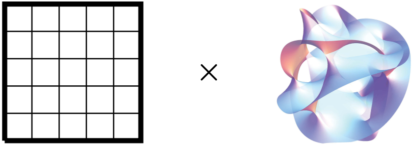
Since then, there have been numerous papers produced studying various aspects of 5d SCFTs such 5-brane webs in type IIB string theory, global symmetry enhancements, counting BPS degeneracies, geometric singularities, and more: [Bergman-Rodriguez-Gomez-Zafrir '14][Tachikawa '15][Bergman-Zafrir '15][Hayashi-Kim-Lee-Taki-Yagi '15][Yonekura '15][Zafrir '15][Kim-Taki-Yagi '15][Hwang-Kim-Kim-Park '16][PJ-Kim-Vafa-Zafrir '17][Xie-Yau '17][Hayashi-Kim-Lee-Yagi '18][Apruzzi-Lin-Mayrhofer '19][Closset-Del Zotto-Saxena '18][...and others!]

Full classification of 5d $\mathcal{N} = 1$ theories remains an open problem. Today I will discuss some current efforts to arrive at a more complete classification of 5d SQFTs with UV fixed points.

Old wisdom...

M-theory compactified on a local CY 3-fold X is described at low energies by an effective abelian 5d $\mathcal{N} = 1$ QFT:

$\mathbb{R}^{1,4}$ $\quad\quad\quad$ X



Dynamics, massive deformations, and vacuum moduli are completely encoded in the topology and complex geometry of X. Also, the abelian theory has UV superconformal fixed point if X admits a suitable singular degeneration.

Problem of classifying 5d SCFTs is equivalent to the problem of classifying 3-folds X which admit such singularities.

...New perspective

So far, I have not described anything new. What's crucial to our program is the following observation: a local 3-fold X can be described as a neighborhood of a union of holomorphic surfaces

$$S = \cup S_i$$

intersecting along holomorphic curves

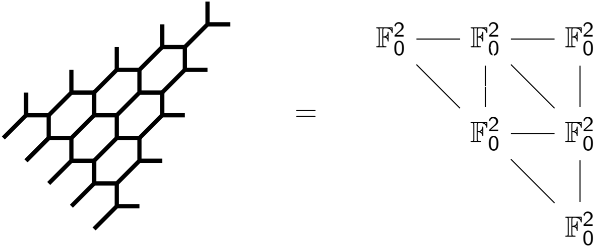
$$C_{ij} = S_i \cap S_j$$

In order to check if X has a UV fixed point, we conjecture we only need the following data:

1. Topological intersection numbers $c_{ijk} = \int_X S_i \cdot S_j \cdot S_k$
2. Cone of holomorphic curves $M(S_i)$ for each S_i

Physically the above data corresponds to BPS central charges, effective gauge couplings, and Chern-Simons couplings of the theory.

There are several advantages to this approach. For one, the description of X in terms of $S = \cup S_i$ is **uniform** and quite **general**. Example, 5d T_5 theory (toric)



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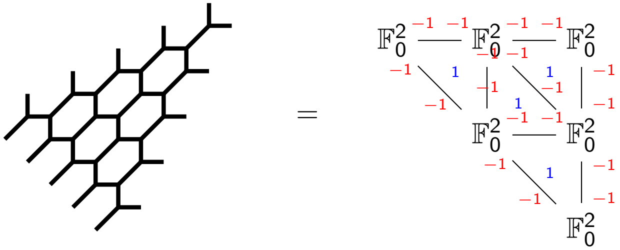
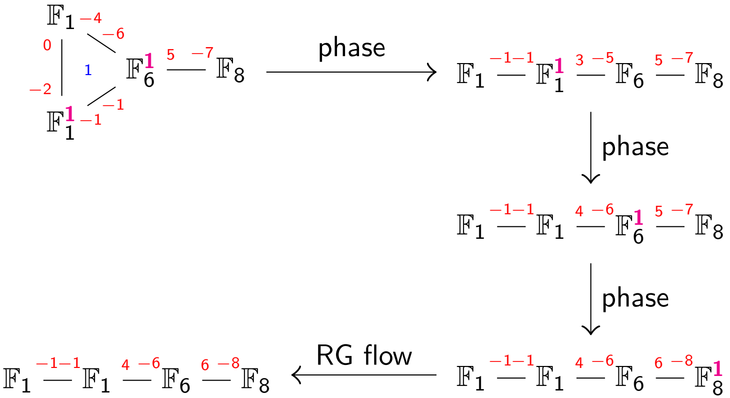


The data c_{ijk} is explicit, and $M(S_i)$ are implicit in the righthand diagram.

Another example, F_4 gauge theory with one hypermultiplet transforming in the **26** (non-toric):



Another advantage—mass deformations/RG flows of theories can easily be read off from geometry.

Biggest advantage: condition for X to admit elliptic fibration (i.e. 6d UV fixed point theory) easily identified! This establishes a geometric link between 6d and 5d fixed points—**can we use this link to classify 5d SCFTs?**

Current results:

- 5d SCFTs up to rank 2; all descend from elliptic 3-folds
    1. Rank 1:
        - $\mathbb{F}_0^8 \to \mathbb{F}_0^7 \to \cdots$
    2. Rank 2 [PJ-Katz-Kim-Vafa '18]:
        - $\mathbb{F}_0^5 — \mathbb{F}_0^5 \to \mathbb{F}_0^4 — \mathbb{F}_0^5 \to \cdots$
        - $\mathbb{F}_4^9 — \mathbb{F}_0 \to \mathbb{F}_4^8 — \mathbb{F}_0 \to \cdots$
        - $\mathbb{F}_0^6 — \mathbb{F}_2 \to \mathbb{F}_0^6 — \mathbb{F}_2 \to \cdots$
        - $\mathbb{F}_0^3 — \mathbb{F}_6 \to \mathbb{F}_0^2 — \mathbb{F}_6 \to \cdots$
- Elliptic 3-folds associated to 6d SCFTs compactfied on a circle
- Elliptic 3-folds associated to twisted compatifications [in progress]

Goal and plan for talk

In this talk I will explain how RG flows of 5d SCFTs are encoded in geometry, and how this idea can be used to more exhaustively classify 5d SCFTs starting from circle compactifications of 6d SCFTs.

Plan for the talk:

1. Review of 5d SCFTs
2. M-theory construction
3. RG flows
4. Classifying 5d SCFTs via 6d SCFTs
5. Conclusion and future prospects

5d $\mathcal{N} = 1$ field theories

The 5d $\mathcal{N} = 1$ superconformal algebra has bosonic subgroup $SO(2,5) \times SU(2)_R \times F$. No exact marginal deformations, and only SUSY-preserving relevant deformations are mass deformations. There is a Coulomb branch of moduli $\mathcal{C}$ parametrized by **real** scalars $\phi_{i=1,\ldots,r}$ which is not lifted by mass deformations.

At a generic point the low energy EFT is a $U(1)^r$ gauge theory where r is the rank (i.e. number of photon fields):

$$\mathcal{L} = (\partial_i \partial_j \mathcal{F}) d\phi^i \wedge \star d\phi^j + (\partial_i \partial_j \mathcal{F}) F^i \wedge \star F^j + \frac{1}{24\pi^2} (\partial_i \partial_j \partial_k \mathcal{F}) A^i \wedge F^j \wedge F^k + \cdots$$

The massive BPS spectrum consists of electric particles and magnetic monopole strings with central charges:

$$Z_{\text{elec}} = n_i \phi^i + s_f m^f , \quad Z_{\text{mag}} = n^i \partial_i \mathcal{F}$$

The abelian theory is described by a one-loop exact prepotential at most cubic in ϕ^i:

$$6\mathcal{F} = c_{lmn} \phi^l \phi^m \phi^n , \quad \phi^{l=r+f} \equiv m^f$$

The prepotential controls (for $i, j, k = 1, \dots, r$):

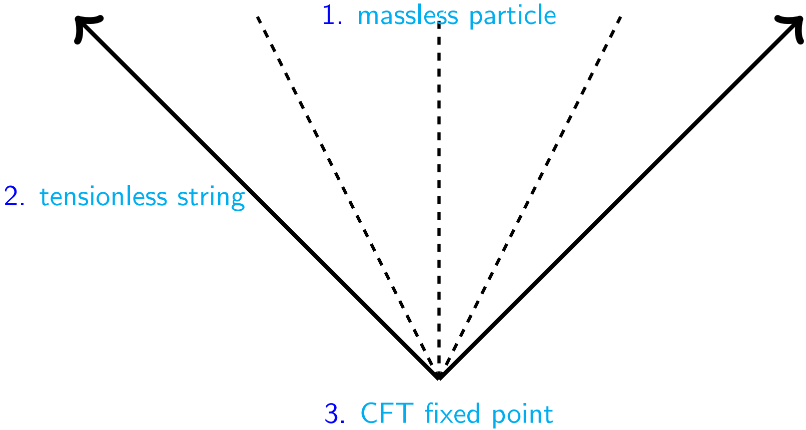
- Monopole string tensions $\partial_i \mathcal{F} = \frac{1}{2} c_{imn} \phi^m \phi^n$
- U(1) couplings $\partial_i \partial_j \mathcal{F} = c_{ijn} \phi^n$
- Chern-Simons couplings $\partial_i \partial_j \partial_k \mathcal{F}$

Coulomb branch moduli ϕ^i have mass dimension 1, define a scale for the theory. For the theory to have a UV completion, there must be a consistent EFT at all energy scales. Hence, ϕ^i are unbounded $\implies$ Coulomb branch is **non-compact**.

The phase structure is dictated by three possibilities for the massive BPS spectrum:

1. A particle becomes massless, along hyperplane.
2. Monopole string becomes tensionless (also along hyperplanes.)
3. Particles and strings become massless and tensionless—this is the CFT fixed point.

This gives the Coulomb branch the structure of a fan in $\mathbb{R}^r$:

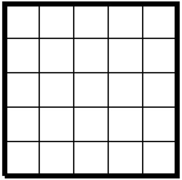


We assume above structure is common to generic 5d $\mathcal{N} = 1$ theories.

M-theory on a CY 3-fold

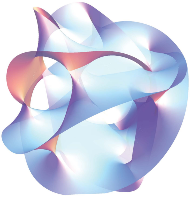
M-theory compactified on a smooth local CY 3-fold X is described at low energies by a 5d $\mathcal{N} = 1$ QFT on the Coulomb branch [Witten '97]:

$\mathbb{R}^{1,4}$



$\times$

X



We have the following dictionary:

- M2 branes wrapping holomorphic curves (volume) are BPS particles (mass)
- M5 branes wrapping holomorphic surfaces (volume) are BPS strings (tension)

Let $[J] = \phi^i [S_i]$ be an expansion of Kähler class $[J]$ in a basis of surface classes $[S_i]$. The volumes of holomorphic p-cycles C_p are controlled by Kähler moduli:

$$\mathrm{vol}(C_p) = \frac{1}{p!} \int_{C_p} [J]^p = \frac{1}{p!} \int_X [J]^p \cdot [C_p]$$

where $\cdot$ denotes the intersection product. (I will drop the bracket notation.) In particular, for fixed ϕ, we get the prepotential for an EFT:

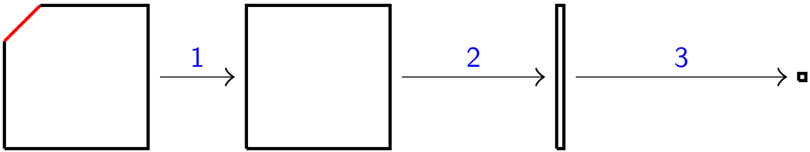
$$\mathrm{vol}(X) = \frac{1}{3!} \int_X J^3 = \frac{1}{3!} c_{ijk} \phi^i \phi^j \phi^k = \mathcal{F}, \quad c_{ijk} = \int_X S_i \cdot S_j \cdot S_k.$$

Phase structure

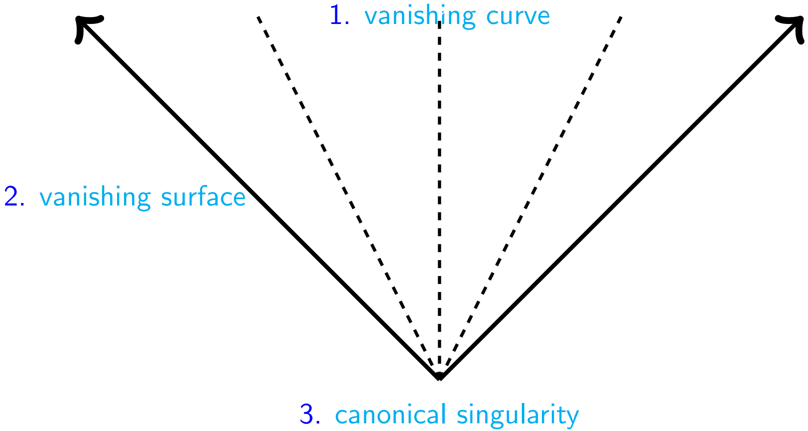
Let's talk about the full moduli space using a string dual **cartoon**: type IIB (p, q) 5-brane diagrams.

Three types of singularities can occur:

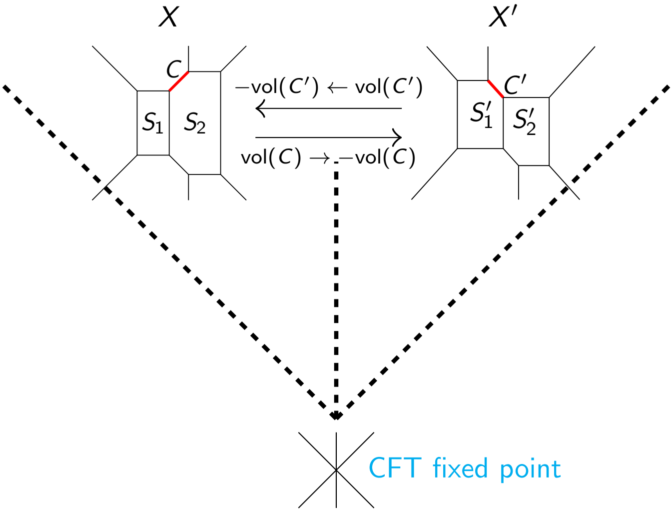
1. A curve can collapse to a point (massless particle.)
2. A surface can collapse to a curve (tensionless string.)
3. A surface can collapse to a point (CFT fixed point.)



When these singularities occur at finite distance in moduli space, the (extended) Kähler moduli space of X represents the Coulomb branch of the 5d theory:



Ex.: Zoom in on singular phase transition (1) due to a massless particle/collapsing curve:



5d $\mathcal{N} = 1$ theory is captured by a collection of surfaces $S = \cup S_i$ intersecting each other in X

This captures two pieces of data:

1. Prepotential (triple intersection numbers), for **EFT description**
2. Central charges of elementary BPS spectrum (volumes of hol. p-cycles), to **check UV completion**

The volumes of the full set of holomorphic curves C_i (i.e. the Mori cone) **cannot** be extracted from non-abelian gauge theory presentation:

$$6\mathcal{F} = \frac{3}{g_{\mathrm{YM}}^2} h_{ij}\phi^i\phi^j + \frac{1}{2}\left[\sum_{\alpha,\mathrm{roots}} |\langle \phi, \alpha \rangle|^3 - \sum_{\mathbf{R}_f,\mathrm{irreps}} \sum_{w \in \mathbf{R}_f} |\langle \phi, w \rangle + m_f|^3\right]$$

Full BPS spectrum is visible in geometry!

Criteria for UV fixed point

The basic consistency test that at theory must pass is that the massive BPS spectrum and U(1) couplings are positive at a generic point in $\mathcal{C} \implies$

$$\text{vol}(C_i) = \int_X J \cdot C_i > 0$$

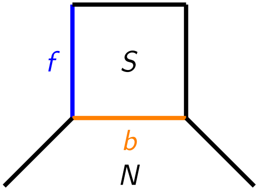
on the interior of $\mathcal{C}$, where C_i are generators of the Mori cone.

The above ensures volumes of **all** holomorphic cycles will be positive [Kleiman]. This also seems to be sufficient for positivity of $\partial_i \partial_j \mathcal{F}$, but a I am not aware of a proof of why this is true.

Conjecture: the above conditions imply the existence of a canonical singularity $\implies$ S_i are rational or ruled surfaces [Reid].

(Notation: $\mathbb{F}_n^p \equiv \text{Bl}_p \mathbb{F}_n$, i.e. blowup of the Hirzebruch surface $\mathbb{F}_n$ at p points.)

As an example, consider the rank 1 theory $S = \mathbb{F}_0$—N:



We expand a Kähler class as $J = \phi S + mN$. Using

$$\int_X S^3 = \int_{\mathbb{F}_0} K_{\mathbb{F}_0}^2, \quad \int_X S^2 \cdot N = \int_N b^2, \quad \int_X S \cdot N^2 = \int_{\mathbb{F}_0} b^2$$

we get

$$6\mathcal{F} = \int_X J^3 = 8\phi^3 - 6m\phi^2.$$

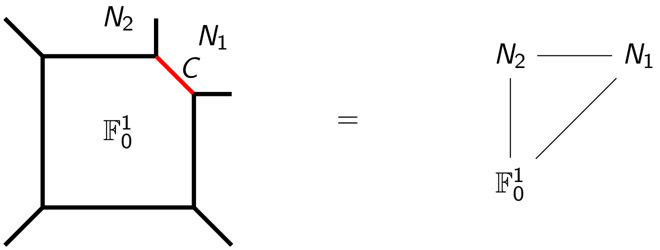
Positivity of the hol. curves $c = c_1 f + c_2 b, c_i \geq 0$ (i.e. BPS charges) implies

$$\text{Coulomb branch} \;:\; \text{vol}(b) = 2\phi \geq 0, \quad \text{vol}(f) = 2\phi + m \geq 0.$$

RG flows

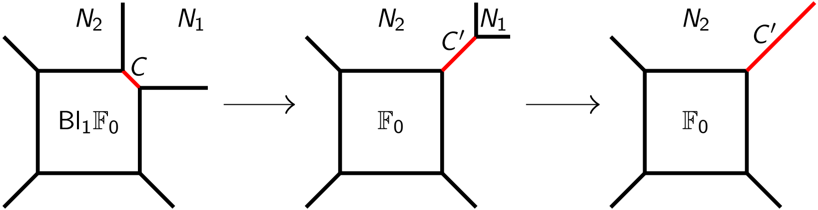
Different 5d theories can be related by RG flows triggered by large mass deformations.

Example: $SU(2) + 1\mathbf{F}$. We describe X as:



Using $J = \phi[\mathbb{F}_0^1] + m_f N_f$, we have $\text{vol}(C) = \phi + m_1 - m_2 > 0$.

We mass deform this theory by sending $m_1 \to -\infty$, which drives $\mathrm{vol}(C) \to -\infty$:



This is the geometric version of integrating out matter:

$$\overset{\mathbb{F}_0^1}{SU(2)} + 1\mathbf{F} \quad \xrightarrow{\text{RG flow}} \quad \overset{\mathbb{F}_0}{SU(2)}_{\theta=0}$$

Geometrically, this looks like a blowdown of a curve in $\mathbb{F}_0^1$.

What happens in the other direction, i.e. integrating in matter?
Conjecture: this can be done finitely many times, ending with a theory which is not a 5d SCFT.

In fact this appears to classify 5d SCFTs:

1. Rank 1:
   - $\mathbb{F}_0^8 \to \mathbb{F}_0^7 \to \cdots$
2. Rank 2 [PJ-Katz-Kim-Vafa '18]:
   - $\mathbb{F}_0^5 — \mathbb{F}_0^5 \to \mathbb{F}_0^4 — \mathbb{F}_0^5 \to \cdots$
   - $\mathbb{F}_4^9 — \mathbb{F}_0 \to \mathbb{F}_4^8 — \mathbb{F}_0 \to \cdots$
   - $\mathbb{F}_0^6 — \mathbb{F}_2 \to \mathbb{F}_0^6 — \mathbb{F}_2 \to \cdots$
   - $\mathbb{F}_0^3 — \mathbb{F}_6 \to \mathbb{F}_0^2 — \mathbb{F}_6 \to \cdots$

We can also flop to different phases and study mass deformations there!

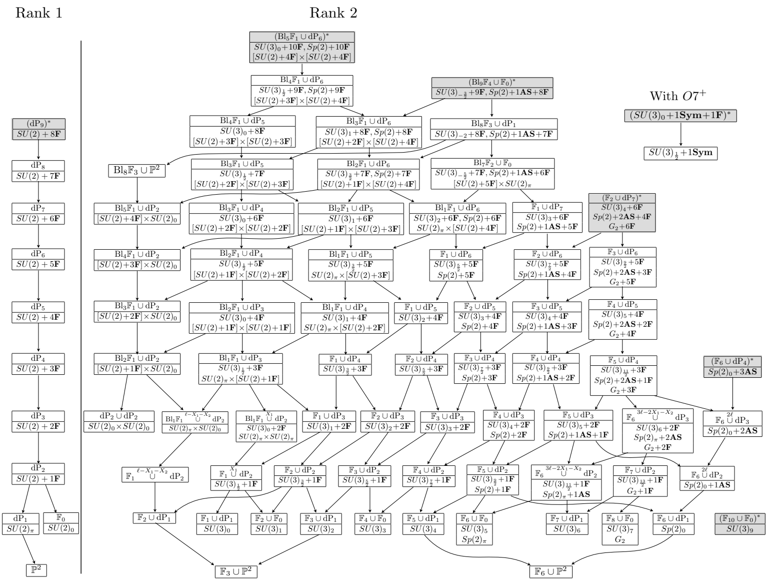


Figure: RG-flow diagram of rank 1 and 2 SCFTs

5d KK Theories

The theories at the top are 5d Kaluza-Klein (KK) theories. They have 6d UV fixed points, and can be viewed as 6d SCFTs compactified on S^1 (more on this later!)

5d KK theories correspond to **elliptically fibered** 3-folds. If X is elliptic, we can invoke duality between M-theory and F-theory:

$$\overset{\text{M-theory}}{X} \overset{\text{(string duality)}}{=} \overset{\text{F-theory}}{X \times S^1_R}$$

and check that X corresponds to a 6d SCFT on S^1, where

$$\text{vol}(F) = \frac{1}{R}.$$

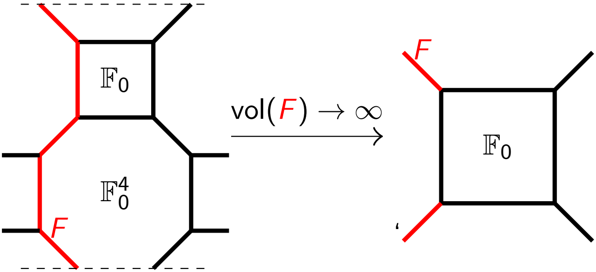
Criterion for elliptic 3-fold: $\exists\ J^3(\phi_*) = 0$ s.t. $J^2(\phi_*) \neq 0$.

5d SCFTs can be found by studying RG flows of 5d KK theories $\cup S_i$. The one requirement is that such a deformation removes the KK scale,

$$\mathrm{vol}(F) \to \infty.$$

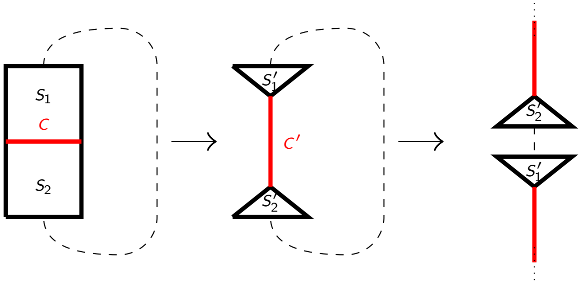
In principle, there are many ways to do this [del-Zotto-Heckman-Morrison '17] which may lead to 5d SCFTs of various rank.

Example: $\overset{\mathrm{I}_2^s}{2} \overset{S^1}{\to} \mathbb{F}_0 = \mathbb{F}_0^4$. The red line is the elliptic fiber. We send the volume of F to infinity to flow to $SU(2)_0$.



To make this procedure systematic, we study a subset of RG flows corresponding to sending the volume of -1 curves C to infinity, which preserve the rank of the 5d theory [PJ-Bhardwaj '18]. Three cases:

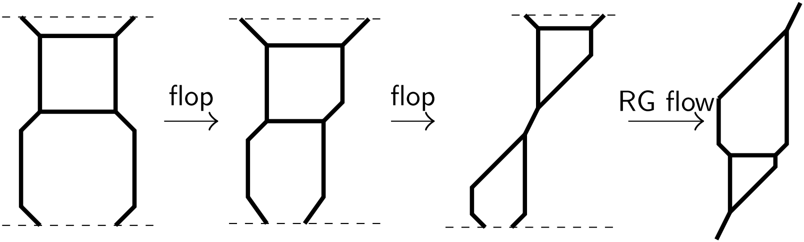
1. C does not intersect any junction $S_i \cap S_j$. (See RG flow cartoon.)
2. C intersects a point on a junction $S_i \cap S_j$. Only flop transitions.
3. C is part of some junction $S_i \cap S_j$. (S_i are joined in a loop):



Let's see an example of the third case. Again, consider $\mathbb{F}_0 = \mathbb{F}_0^4$. We can perform two flop transitions:

$$\underset{\color{red}{\text{5d KK}}}{\mathbb{F}_0 = \mathbb{F}_0^4} \xrightarrow{\text{flop(s)}} \mathbb{F}_1 = \mathbb{F}_0^3 \xrightarrow{\text{RG flow}} \underset{\color{cyan}{\text{5d SCFT}}}{\mathbb{F}_1 - \mathbb{F}_0^3}.$$

Schematically,



Classifying 5d SCFTs via 6d SCFTs

The tensor branch of a 6d SCFT $\mathfrak{T}$ can be constructed by compactifying F-theory on a singular elliptically fibered CY 3-fold $X_{\mathfrak{T}}$,

$$X_{\mathfrak{T}} \to B$$

where B is a non-compact complex surface. The locus $\Delta \subset B$ carrying singular fibers is a collection of rational curves Σ_i such that

$$A_{ij} = \int_B \Sigma_i \cdot \Sigma_j$$

is negative-definite.

$\mathfrak{T}$ is the worldvolume theory of 7-branes wrapping Σ_i, and D3 branes wrapping Σ_i are BPS strings with tension proportional to $\text{vol}(\Sigma_i)$. At the conformal point, BPS strings become tensionless.

Consider $\mathfrak{T}$ compactified on S^1_R with generic holonomies for all symmetries around S^1_R. This is engineered by F-theory on $\widetilde{X}_{\mathfrak{T}} \times S^1_R$

$$
\begin{array}{ccc}
\text{F-theory} & & \text{M-theory} \\
\widetilde{X}_{\mathfrak{T}} \times S^1_R & \cong & \widetilde{X}_{\mathfrak{T}} \\
\Big\downarrow & & \Big\downarrow \\
\text{6d } \mathfrak{T} \text{ on } S^1_R & \cong & \text{5d } \mathfrak{T}_{\mathrm{KK}}
\end{array}
$$

where $\widetilde{X}_{\mathfrak{T}} \to X_{\mathfrak{T}}$ is a resolution. The circle compactification of $\mathfrak{T}$ on the tensor branch is a 5d KK theory $\mathfrak{T}_{\mathrm{KK}}$ on the Coulomb branch.

Thus given a 6d SCFT $X_{\mathfrak{T}}$, we can identify many 5d SCFTs by studying RG flows using geometry of a smooth 3-fold $\widetilde{X}_{\mathfrak{T}}$!

Example: 6d $SU(2) + 4\mathbf{F}$ on S^1

The 3-fold X is a curve $\Sigma \subset B$ with $\Sigma^2 = -2$, carrying type I_2^s Kodaira singular fibers. A resolution $\widetilde{X} \to X$ is given by the hypersurface

$$y_1^2 z + a_1 x_1 y_1 z + a_{3,1}\sigma_1 y_1 z^2$$
$$- (e_1 x_1^3 + a_{2,1}\sigma_1 e_1 x_1^2 z + a_{4,1}\sigma_1 x_1 z^2 + a_{6,2}\sigma_1^2 z^3) = 0$$

of a projective bundle $f : \widetilde{Y} \to B$ whose fibers have homogeneous coordinates

$$[e_1 x_1 : e_1 y_1 : z][x_1 : y_1 : \sigma_1]$$

with certain projective symmetries. Here, $\sigma \equiv \sigma_1 e_1 = 0$ is the location of the pullback of Σ and $e_1 = 0$ describes the exceptional divisor E_1 of the blowup.

We need to describe $\widetilde{X}$ as a collection of surfaces $\cup S_i$, from which we extract:

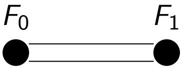
1. Triple intersection numbers c_{ijk}
2. Mori cone $M(\widetilde{X})$

First, let us expand a Kähler class J in the following basis:

$$J = \phi_0 S_0' + \phi_1 S_1', \quad S_i' \equiv S_i|_{\widetilde{X}}, \quad S_0 = f^*(\Sigma) - S_1, \quad S_1 = E_1$$

Above, S_0, S_1 are (respectively) the classes of the divisors $\sigma_1 = 0$ and $e_1 = 0$ in $\widetilde{Y}$.

Why the above choice of basis? This basis is compatible with the total transform of the singular fiber $F = F_0 + F_1$:



Since each irreducible component $F_i \cong \mathbb{P}^1$ is a rational curve, as it moves over Σ it sweeps out a ruled surface S_i.

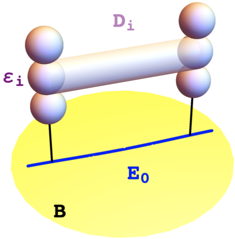


Figure: (Source: arXiv:1504.01387)

This information can be used to determine the birational equivalence class and Mori cone of each S_i.

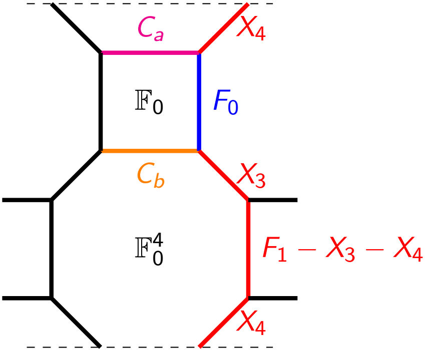
The triple intersection numbers c_{ijk} can straightforwardly be evaluated as pushforwards of intersection products in the intersection ring of the ambient space $\widetilde{Y}$ to the base B:

$$c_{ijk} = \int_{\widetilde{X}} S'_i \cdot S'_j \cdot S'_k = f_* \int_{\widetilde{Y}} S_i \cdot S_j \cdot S_k \cdot [\widetilde{X}].$$

The above strategy allows us to use the intersection data of $\Sigma^2, K_B \cdot \Sigma = 2g(\Sigma) - 2 - \Sigma^2$ to evalute c_{ijk}:

$$\begin{aligned}
c_{000} &= \int_{S_0} K_{S_0}^2 = -4 \int_B (\Sigma \cdot K_B + \Sigma^2) = 8 \\
c_{111} &= \int_{S_1} K_{S_1}^2 = 2 \int_B (2\Sigma \cdot K_B - \Sigma^2) = 4 \\
c_{011} &= \int_{S_0} (S_0 \cdot S_1)^2 = -4 \int_B \Sigma \cdot K_B = 0 \\
c_{001} &= \int_{S_1} (S_0 \cdot S_1)^2 = 2 \int_B (2\Sigma \cdot K_B + \Sigma^2) = -4.
\end{aligned}$$

The above computations show $\widetilde{X}$ is described by
$S = S_0 \cup S_1 = \mathbb{F}_0 \;\overline{\underline{\quad}}\; \mathbb{F}_0^4$,



where $\mathbb{F}_0 \cap \mathbb{F}_0^4 = C_a + C_b$ satisfies

$$\int_{\mathbb{F}_0} {C_a}^2 = \int_{\mathbb{F}_0} {C_b}^2 = 0, \quad \int_{\mathbb{F}_0^4} {C_a}^2 = \int_{\mathbb{F}_0^4} {C_b}^2 = -2.$$

Summary and current status

5d SCFTs, associated to local 3-folds X, are efficiently described as collection of rational or ruled surfaces $S = \cup S_i$ intersecting in some pattern—this picture has led to a **classification** of **rank one** and **two** theories

RG flows are encoded in geometry as blowdowns of -1 curves in S

Using M/F theory duality, we can use rank-preserving RG flows from smooth elliptically fibered 3-folds $\widetilde{X}_{\mathfrak{T}}$ to (partially?) **classify** 5d SCFTs

A method for identifying a collection of surfaces $S_{\mathfrak{T}}$ associated to some resolution $\widetilde{X}_{\mathfrak{T}} \to X_{\mathfrak{T}}$ has been described for **all 6d SCFTs** $\mathfrak{T}$

Future directions

There is still much work to be done for classification:

1. Explicit description of Mori cones for $S_{\mathfrak{T}}$ (currently implicit)
2. Mapping out RG flows
3. Twisted compactifications $X_{\mathfrak{T}} \times S^1_R$ (in progress)
4. Frozen singularities

Nekrasov partition function?

Flavor symmetry enhancements at the UV fixed point?

Higgs branch?

T-dual pairs of LSTs admitting F-theory construction (without frozen singularities)?

Thank you!