Increased complexity by surgery
Foulon Hasselblatt Vaugon

Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Increasing the complexity of free particle motion by breaking its mechanical nature

Patrick Foulon[1] Boris Hasselblatt[2] Anne Vaugon[3]

Tufts University School of Arts and Sciences
Graduate School of Mathematical Sciences, The University of Tokyo
KAVLI IPMU Institute for the Physics and Mathematics of the Universe

[1] CIRM & CNRS

[2] Tufts University & University of Tokyo

[3] Université Paris Sud

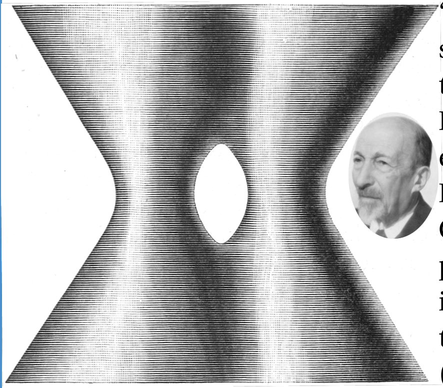
November 1, 2018

Hadamard and negatively curved surfaces (1898)

The set E of vectors at a base point whose geodesics are bounded

Les surfaces à courbures opposées et leurs lignes géodésique. J. Math. Pures Appl. 4 (**1898**): 27–73.

Increased complexity by surgery
Foulon Hasselblatt Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap



"The manner in which these sets arise clearly recalls that of the sets encountered by Mr. Poincaré, introduced more explicitly to the subject by Mr. Bendixson, then studied by Mr. Cantor and which, while being perfect, are not *dense* in any interval. Here the angles λ play the role of the intervals called (a_ν, b_ν) by Mr. Cantor."

Picture from Hadamard

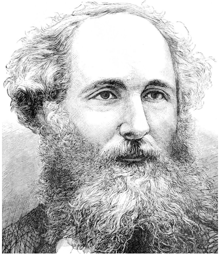
The Maxwell–Boltzmann Ergodic Hypothesis

Increased complexity by surgery

Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

James Clerk Maxwell (1831–1879) and Ludwig Boltzmann (1844–1906) aimed to give a rigorous formulation of the kinetic theory of gases and statistical mechanics.



Boltzmann's H-theorem says that the time and space (ensemble) averages of an observable (a function on the phase space) agree.

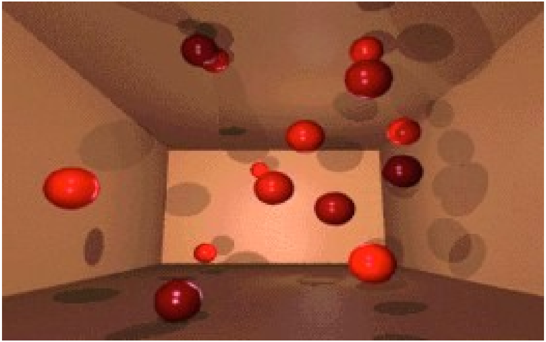
Vorlesungen über Gastheorie I & II. Leipzig: Ambrosius Barth, **1896–1898**, or Gesamtausgabe, Graz: Akademische Druck und Verlagsanstalt, 1981, 1.

The Maxwell–Boltzmann Ergodic Hypothesis

Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Boltzmann's H-theorem assumed the Ergodic Hypothesis:

The trajectory of the point representing the state of the system in phase space passes through every point on the constant-energy hypersurface of the phase space.



Increased complexity by surgery

Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples

Summary

Foulon surgery
Definitions
Statement of results

Proofs
Surgery
Local computations

Periodic orbit growth

Complexity of contact *structure*

Many flows

Mind the gap

Poincaré and many physicists doubted the Ergodic Hypothesis since no example satisfying it had been exhibited.
Poincaré: Sur la théorie cinétique des gas. Revue Générale des Sciences pures et appliqueés 5 (**1894**), 513–521

Paul and Tatiana Ehrenfest proposed the alternative **Quasi-Ergodic Hypothesis**: *The trajectory of the point representing the state of the system in phase space is dense on the constant energy hypersurface of the phase space.*
Begriffliche Grundlagen der statistischen Auffassung in der Mechanik.
In Encyklopaedie der Mathematischen Wissenschaften, Leipzig **1912**: Teubner. 4: Art. 32, 1–90.

The quest was on. . .

Ein mechanisches System mit quasiergodischen Bahnen

Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Emil Artin:
Ein mechanisches System mit quasiergodischen Bahnen
Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, vol. 3 (**1924**), pp. 170-175.

"May it be permitted to point to a simple mechanical system with 2 degrees of freedom and quasiergodic orbits upon which the author came in the course of a correspondence with Mr. G. Herglotz…

…From this, one already obtains that the "quasiergodic chains" have the cardinality of the continuum. More! According to results of Mr. Celestyn Burstin almost all numbers ξ have a "quasiergodic" continued-fraction expansion. Therefore, almost all of the geodesic lines through a point of the surface are quasiergodic."

Ein mechanisches System mit quasiergodischen Bahnen

Increased complexity by surgery

Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples

Summary

Foulon surgery
Definitions
Statement of results

Proofs
Surgery
Local computations

Periodic orbit growth

Complexity of contact *structure*

Many flows

Mind the gap

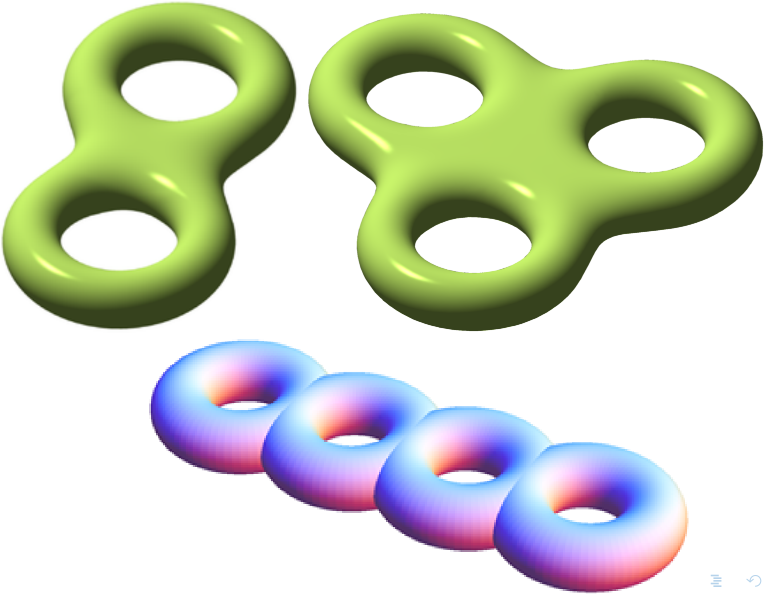
"Let us remark on the physical realizability. One obtains. . . the surface of rotation of a tractrix (curve of pursuit) of a string of length 1. It is known to have curvature $K = -1$, so our half-plane can be partially developed onto this surface. . . But with that we have our physical realization. . . Our mechanical system can be interpreted. . . as the force-free motion of a point particle (the point being constrained to remain on the surface)."

Free particle motion on…

Increased complexity by surgery

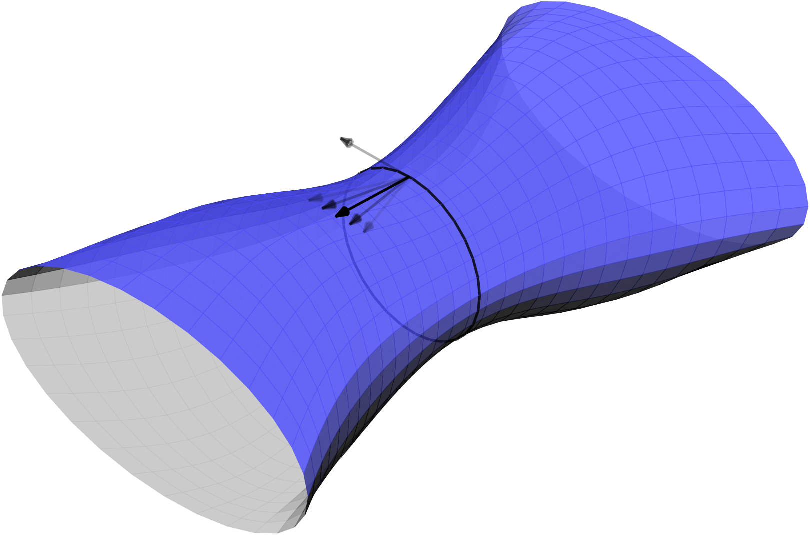
Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

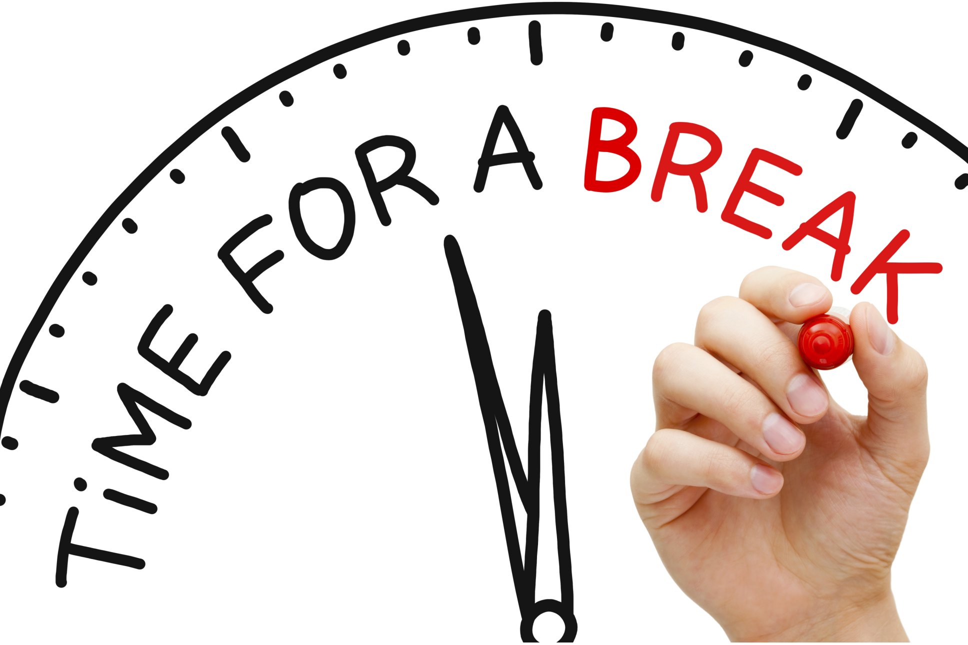


Breaking the configuration space…

Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

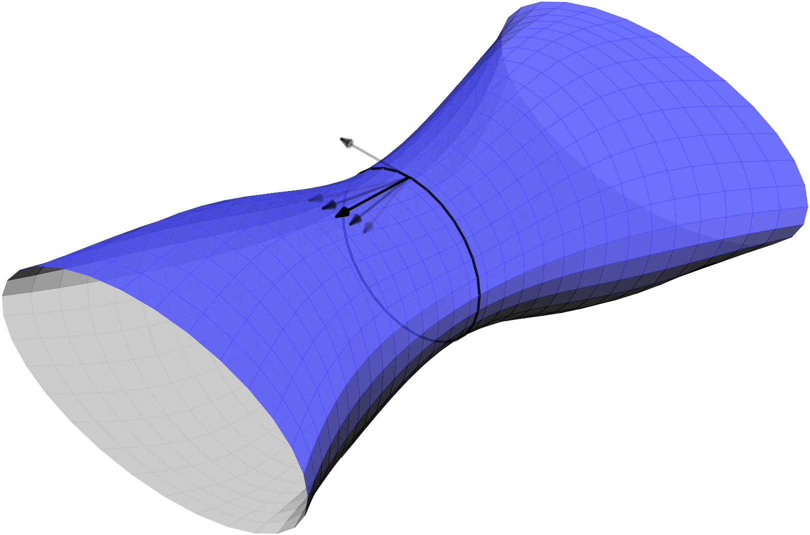


…preserves “contact,” increases complexity…


TIME FOR A BREAK

Foulon–Ding–Geiges–Weinstein–Handel–Thurston surgery

Increased complexity by surgery
Foulon Hasselblatt Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap



Agenda

Increased complexity by surgery

Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

- Foulon–Ding–Geiges–Weinstein–Handel–Thurston surgery is a contact surgery on the unit tangent bundle of a surface.
- This forces orbit complexity.
- It produces several contact 3-flows.
- From the fiber flow a flow with quadratic or exponential complexity.
- From the geodesic flow a contact structure whose every Reeb flow has positive entropy.
- If the surgered geodesic flow is hyperbolic, then it has increased orbit growth.
- There are manifolds with 2 contact flows, one exponentially complex, one polynomially.

The entire talk is in dimension 3.

Definition of contact Anosov flow

Increased complexity by surgery
Foulon Hasselblatt Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Definition (Contact, Anosov, algebraic)

A 1-form A is contact $:\Leftrightarrow A \wedge dA$ is a volume.
Its Reeb field R_A is defined by $R_A \in \ker dA$ & $A(R_A) = 1$.
Its flow $\varphi\colon \mathbb{R} \times M \to M$ preserves A (contact flow)

Φ is an Anosov flow if $TM = \underbrace{\mathbb{R}X}_{\text{flow}} \oplus \underbrace{E^+}_{\text{unstable}} \oplus \underbrace{E^-}_{\text{stable}}$,
$\exists\, C > 0,\ \lambda \in (0,1) \quad \forall\, t > 0$:

$$\|D\varphi^{-t} \restriction E^+\| \le C\lambda^t \quad \text{and} \quad \|D\varphi^{t} \restriction E^-\| \le C\lambda^t.$$

Algebraic $:\Leftrightarrow$ finitely covered by the geodesic flow of a surface [or the suspension of a diffeomorphism of the 2-torus].[1]

[1] not contact—hence out of scope!

Comments

Increased complexity by surgery
Foulon Hasselblatt Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Anosov flows are abundant (surgeries by Franks–Williams, Christy, Fried, Handel–Thurston, Goodman, Bonatti–Langevin, Barbot, Fenley, Béguin–Bonatti–Yu).

But, experts thought surgeries could not produce contact flows

The method for showing that this surgery yields contact flows was new: It involves a suitable deformation of the contact form and a time-change adapted to this deformation.

Nonalgebraic flows

Increased complexity by surgery
Foulon Hasselblatt Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Theorem (Exotic contact Anosov flows)

Geodesic flows of negatively curved surfaces admit smooth Dehn surgeries that produce new (analytic) contact Anosov flows. These surgeries include the Handel–Thurston surgery, and the resulting flow always has the following properties:

1. *It acts on a manifold that is not a unit tangent bundle.*
2. *It is not topologically orbit equivalent to an algebraic flow.*
3. *It has "exponentially more" closed orbits than its progenitor…*
4. *…even in any one homology class.*

“Exotic” properties

Increased complexity by surgery
Foulon Hasselblatt Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Theorem (Contact Anosov flows on *hyperbolic* manifolds)

∃ contact Anosov flows on hyperbolic 3-manifolds.[2] *For them*[3] *each closed orbit is isotopic to infinitely many others.*[4] *and any Reeb flow*[5] *with the same contact structure has exponential orbit growth, infinite free homotopy classes, and exponential growth in each homology class.*

---

[2] If $M \smallsetminus K$ is a hyperbolic manifold, *e.g.*, K projects to a filling geodesic, then all but finitely many of our Dehn surgeries produce a hyperbolic manifold.

[3] and for any topologically orbit equivalent contact Anosov flow

[4] Barthelmé–Fenley: Freely homotopic closed orbits are isotopic —so there is no knot theory of closed orbits in a free homotopy class.

[5] $A(R_A) = 1$ & $dA(R_A, \cdot) = 0$

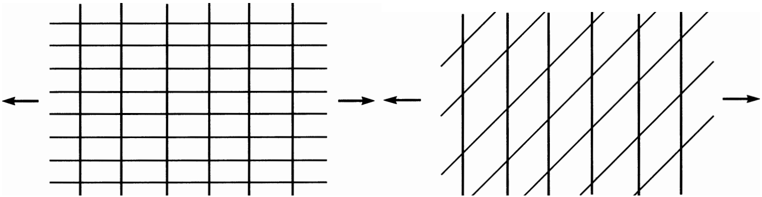
Mechanisms

due to Fenley and Barbot

Increased complexity by surgery
Foulon Hasselblatt Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

These flows are $\mathbb{R}$-covered, *i.e.*, on the universal cover the stable and unstable leaf spaces are $\mathbb{R}$.
That each orbit lies in exactly one (center-)un/stable leaf defines an embedding of $\mathcal{O}^\Phi$ in $\mathscr{L}^u \times \mathscr{L}^s$. Its image $\mathcal{O}$ is either

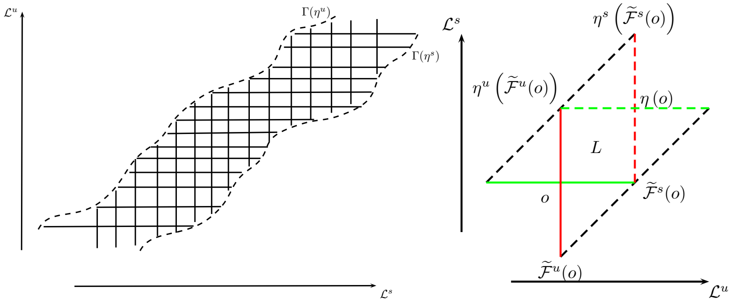
- $\mathscr{L}^u \times \mathscr{L}^s$—"product flow" (orbit-equivalent to suspension of toral automorphism), or
- the open set between graphs of homeomorphisms $\alpha, \beta \colon \mathscr{L}^u \to \mathscr{L}^s$—"skewed".



$\mathbb{R}$-covered cases [Picture by Fenley]

Mechanisms
due to Fenley and Barbot

Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

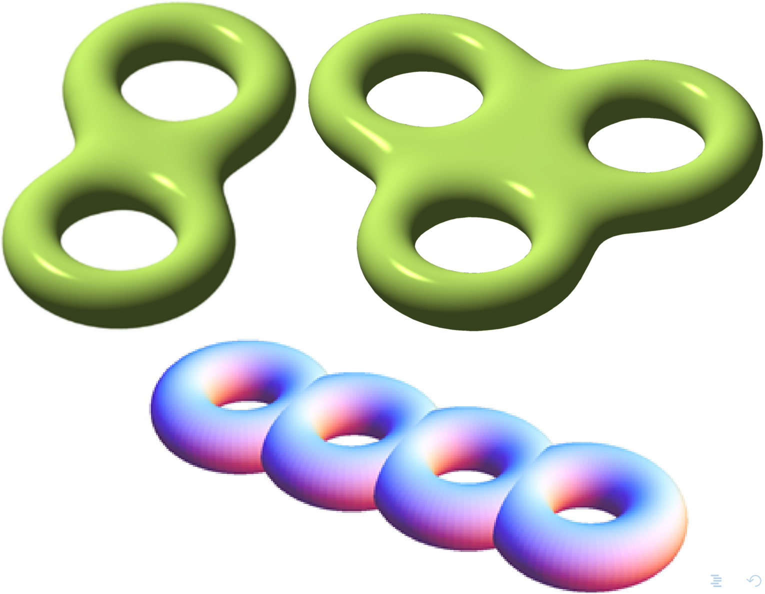


The orbit space of a skewed $\mathbb{R}$-covered flow [Pictures by Barthelmé]

Fenley: If Φ is skewed and M is a hyperbolic manifold, then every closed orbit is freely homotopic to infinitely many others.
Barthelmé: Actually, isotopic.

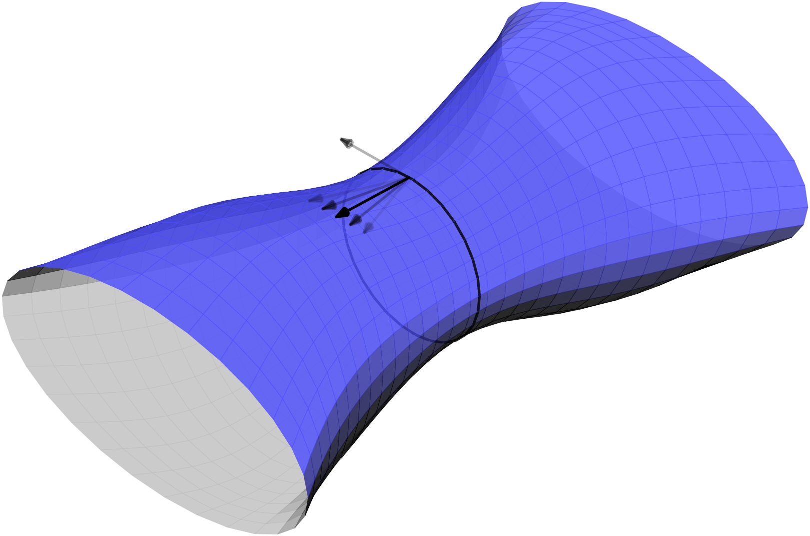
Handel–Thurston surgery

Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap



Handel–Thurston surgery

Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap



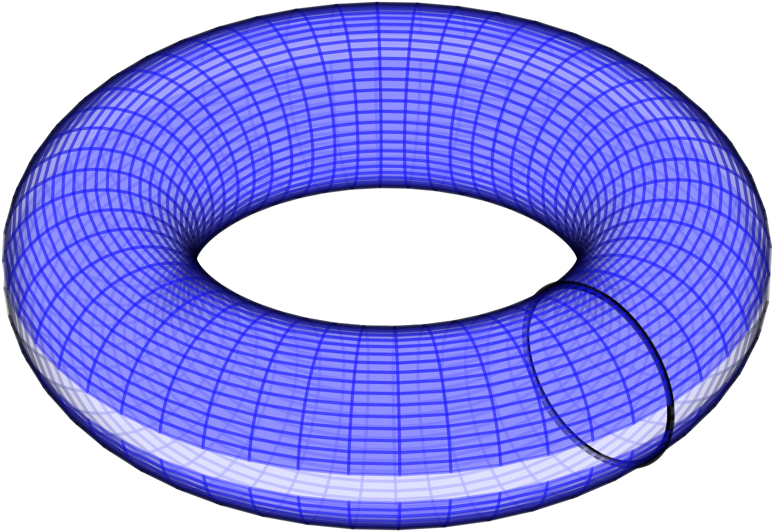
A closed geodesic and normal vectors

Surgery in phase space

Increased complexity by surgery

Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap



Annulus $\Sigma = S^1 \times (\pi/2 - \epsilon, \pi/2 + \epsilon) \subset S^1 \times S^1$ before surgery

Surgery in phase space

Increased complexity by surgery

Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples

Summary

Foulon surgery
Definitions
Statement of results

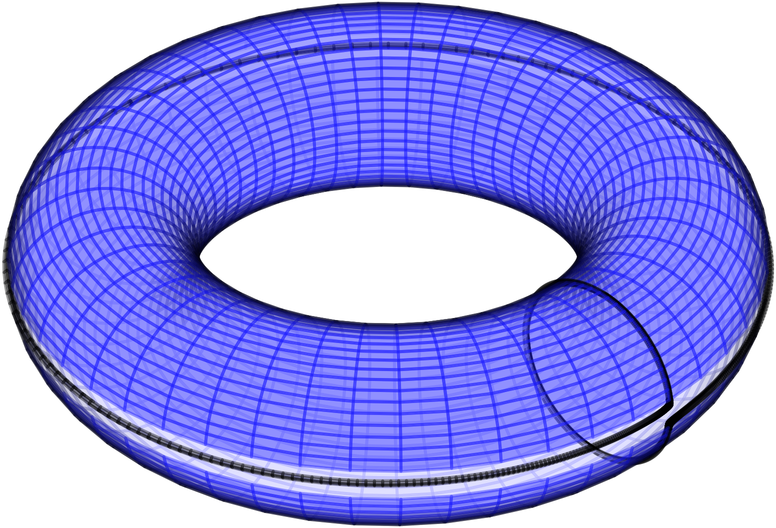
Proofs
Surgery
Local computations

Periodic orbit growth

Complexity of contact *structure*

Many flows

Mind the gap



Annulus $\Sigma = S^1 \times (\pi/2 - \epsilon, \pi/2 + \epsilon) \subset S^1 \times S^1$ after surgery

Darboux flow box

Increased complexity by surgery
Foulon Hasselblatt Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

γ: Normal vectors to closed geodesic = knot tangent to the contact structure (Legendrian, $A(\gamma') \equiv 0$) and transverse to both E^- and E^+.

Smooth annulus Σ: $\gamma \subset \Sigma \pitchfork X$, $\Sigma \pitchfork E^- \oplus E^+$ away from γ

$\exists$ coordinates (s, w) on Σ such that $S^1 \ni s =$ parameter for γ, and

on $\Lambda := \bigcup_{t \in (-\eta, \eta)} \varphi^t(\Sigma)$ [t = transverse parameter = flow time]

$$A = dt + w\,ds, \qquad dA = dw \wedge ds, \qquad A \wedge dA = dt \wedge dw \wedge ds.$$

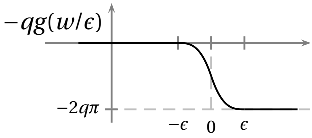
Surgery and volume-preservation

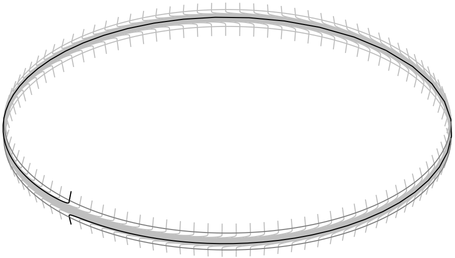
Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Define $(1,q)$-Dehn surgery ($q \in \mathbb{N}$) by the transition map

$$F\colon S^1 \times (-\epsilon,\epsilon) \to S^1 \times (-\epsilon,\epsilon), \quad \boxed{(s,w) \mapsto (s+f(w),w)}$$

with $$f\colon [-\epsilon,\epsilon] \to S^1, \quad w \mapsto \exp(-iqg(w/\epsilon)).$$





Then $F_* A \wedge dA = A \wedge dA$: the new flow X_F preserves volume.

Fixing the contact form

“Splitting the difference”

Increased complexity by surgery
Foulon Hasselblatt Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

But $F_*A = A + wf'(w)\,dw$. To fix, “split the difference”:

$$h(t,w) := \frac{1}{2}\underbrace{\lambda(t)}\int_{-\epsilon}^{w} xf'(x)\,dx \text{ on } (-\eta,\eta)\times(-\epsilon,\epsilon) \text{ and } h=0 \text{ outside.}$$

$\lambda\colon \mathbb{R}\to[0,1]$ is a smooth bump function

Then $dh_{|_{t=0}} = \frac{1}{2}wf'(w)\,dw$; $A_h := A \pm dh$ for $\pm t \geq 0$ gives

$$\boxed{F_*(A_h) = F_*(A-dh) = F_*A - F_*\,dh = (A+2\,dh) - dh = A + dh = A_h}$$

$\Rightarrow$ A_h defines a contact form on the surgered manifold.

Its Reeb field is a time-change $X_h := \boxed{\dfrac{X_F}{1 \pm dh(X_F)}}$.

Did I mention hyperbolicity?

Hyperbolicity

Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

The flow is hyperbolic (only) if $q \in \mathbb{N}$ by the cone criterion—the surgery transition kicks cones into cones…

…and indeed in such a way that:

Proposition (Lyapunov exponents)

The positive Lyapunov exponent of X_F is no less than the positive Lyapunov exponent of X, so

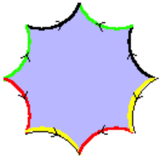
$$h_{Liouville}(X_F) \geq h_{Liouville}(X)$$

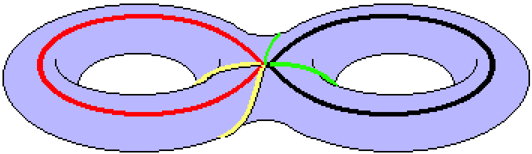
by the Pesin entropy formula.

Periodic orbit growth

Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Suppose the original geodesic flow X_0 comes from a surface of constant curvature.





$$\overbrace{h_{\mathrm{top}}(X_0) = h_{\mathrm{Liouville}}(X_0)}^{\text{constant curvature}} \le \overbrace{h_{\mathrm{Liouville}}(X_h) < h_{\mathrm{top}}(X_h)}^{\text{Foulon Entropy Rigidity}}$$

$$\underbrace{h_{\mathrm{Liouville}}(X_0) \le h_{\mathrm{Liouville}}(X_h)}_{\text{“no less Lyapunov exponent”}}$$

$\Rightarrow$ More periodic orbits: $\quad h_{\mathrm{top}}(X_h) > h_{\mathrm{top}}(X_0)$

NB: $\displaystyle\lim_{t\to\infty} t h_{\mathrm{top}}(\varphi) P_t(\varphi) e^{-t h_{\mathrm{top}}(\varphi)} = 1$, *i.e.*, $\displaystyle\underbrace{P_t(\varphi)}_{:=\#\text{closed orbits of length up to } t} \sim \frac{e^{t h_{\mathrm{top}}(\varphi)}}{t h_{\mathrm{top}}(\varphi)}$.

Increased complexity by surgery

Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples

Summary

Foulon surgery
Definitions
Statement of results

Proofs
Surgery
Local computations

Periodic orbit growth

Complexity of contact *structure*

Many flows

Mind the gap

Given a contact structure $\xi = \ker\alpha$, consider another contact form $\alpha' = f\alpha$ where $f \in \mathscr{C}^\infty(M, \mathbb{R} \smallsetminus \{0\})$. Then $\ker\alpha' = \ker\alpha = \xi$.

But $\boxed{d\alpha' = df \wedge \alpha + f\,d\alpha}$—so $\ker d\alpha'$ is quite different.
The condition $\iota_{R_{\alpha'}} d\alpha' = 0$ implies that R_α and $R_{\alpha'}$ are not collinear unless f is constant.

A Reeb field on a contact manifold (M, ξ) is the Reeb field of any contact form α with $\xi = \ker\alpha$. *"A contact form on (M, ξ)"*

By Lieberman's Theorem, these are exactly the nowhere-vanishing vector fields transverse to ξ whose flows preserve ξ.

This contact *structure* forces orbit growth

(Not new: if a Reeb flow is Anosov then others have positive entropy)

Increased complexity by surgery

Foulon Hasselblatt Vaugon

Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Theorem (Periodic orbit growth in homotopy classes)

(M_F, A_h) = *contact manifold from surgery (not assumed hyperbolic), X_h Anosov, ρ a primitive free homotopy class containing an X_h-periodic orbit, λ hypertight*[6] *contact form.*

$$\exists a > 0, b \in \mathbb{R} \text{ with } N_T^\rho(\lambda) \geq a\ln(T) + b \quad \text{for all} \quad T > 0.$$ [7]

[We use estimates by Barthelmé–Fenley.]

Theorem (Entropy without hyperbolicity—after Alves) ("separating")

(M_F, A_h) = *contact manifold from surgery along a simple geodesic (so M_F is not hyperbolic), A_h not assumed Anosov. If λ is a contact form on $(M_F, \ker(A_h))$, then $h_{top}(R_\lambda) > 0$.*

---

[6] :⇔"no contractible periodic Reeb orbit."
If we drop "hypertight" and assume M_F hyperbolic we get "$\ln(\ln T)$."

[7] $N_T^\rho(\lambda)$ = number of closed orbits in ρ of length up to T.

More contact flows

An S^1-family of contact Anosov flows on $S\mathbb{H}$

Increased complexity by surgery

Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples

Summary

Foulon surgery
Definitions
Statement of results

Proofs
Surgery
Local computations

Periodic orbit growth

Complexity of contact *structure*

Many flows

Mind the gap

Structure equations: $[V,X] =: \underbrace{\overset{\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}}{H}}_{\text{horizontal}}, \quad [H,X] = \underbrace{\overset{\begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}}{V}}_{\text{vertical}}, \quad [H,V] = \underbrace{\overset{\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}}{X}}_{\text{geodesic}}$

Contact flows: $B := dA(V,\cdot)$ is a contact form with $R_B = H$;
so is $E := E_\theta := \cos\theta A + \sin\theta B$,[8]
and $P := R_E = \cos\theta X + \sin\theta H$ is Anosov:
$Q := \cos\theta H - \sin\theta X$ and $\zeta^\pm := Q \pm V$ give

$$[P,\zeta^\pm] = \mp\zeta^\pm \qquad \text{so} \qquad \underbrace{0 = [P, f\zeta^\pm]}_{\boxed{f\zeta^\pm \text{ invariant}}} = (\dot f \mp f)\zeta^\pm \Rightarrow \boxed{f = e^{\pm t}}.$$

(S^1-family of Anosov flows)

$\boxed{V = R_C \text{ with } C := dA(H,\cdot) \text{ gives a periodic flow, the fiber rotation.}}$

[8] $E \wedge dE(P,V,Q) \equiv 1$

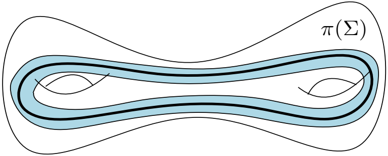
More contact flows

The fiber rotation

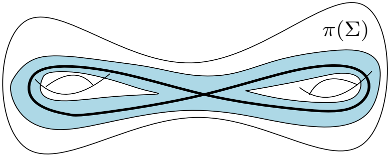
Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

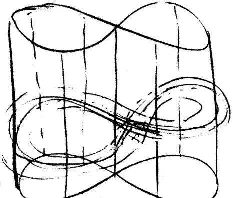
$V \xrightarrow{\text{surgery}}$

- parabolic flow (quadratic growth) *using a simple geodesic*



- (nonuniformly) hyperbolic flow otherwise (Heberle).



[2 transverse shears: $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}=\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$ ]

The entropy gap

Increased complexity by surgery
Foulon
Hasselblatt
Vaugon
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Foulon rigidity: $h_{\text{Liouville}}(\varphi^t) < h_{\text{top}}(\varphi^t)$—but no control of the gap.

Before:

$$\underbrace{\overbrace{h_{\text{top}}(X_0) = h_{\text{Liouville}}(X_0)}^{\text{constant curvature}}}_{=h_{\text{top}}(g^t)\ \ \ \ \text{“no less Lyapunov exponent”}} \le \overbrace{h_{\text{Liouville}}(X_h) < h_{\text{top}}(X_h)}^{\text{Foulon Entropy Rigidity}}$$

$$h_{\text{top}}(X_0) = \underbrace{h_{\text{Liouville}}(X_0) \le h_{\text{Liouville}}(X_h)}_{\text{“no less Lyapunov exponent”}} < \underbrace{h_{\text{top}}(X_h)}_{=h_{\text{top}}(\varphi^t)}$$

$\Rightarrow$More periodic orbits:

$$h_{\text{top}}(\varphi^t) > h_{\text{top}}(g^t).$$

NB: $\lim_{t\to\infty} t h_{\text{top}}(\varphi) P_t(\varphi) e^{-t h_{\text{top}}(\varphi)} = 1$, *i.e.*, $P_t(\varphi) \sim \dfrac{e^{t h_{\text{top}}(\varphi)}}{t h_{\text{top}}(\varphi)}$.

The entropy gap quantified—an open problem

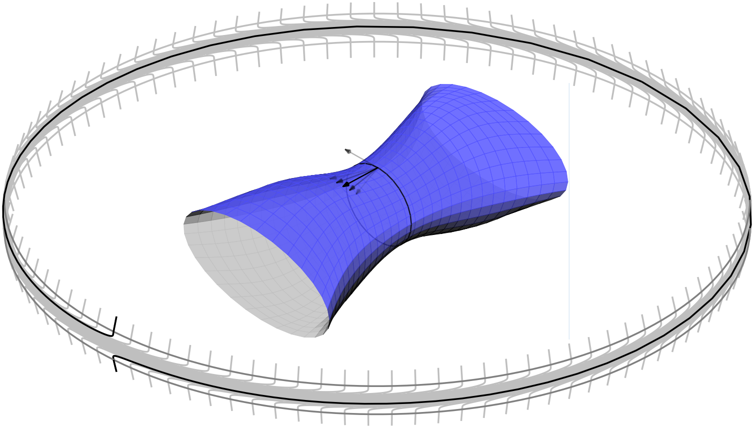
(Idea: Bishop–Hughes–Vinhage–Yang **@MRC2017**)

Increased complexity by surgery

Foulon Hasselblatt Vaugon

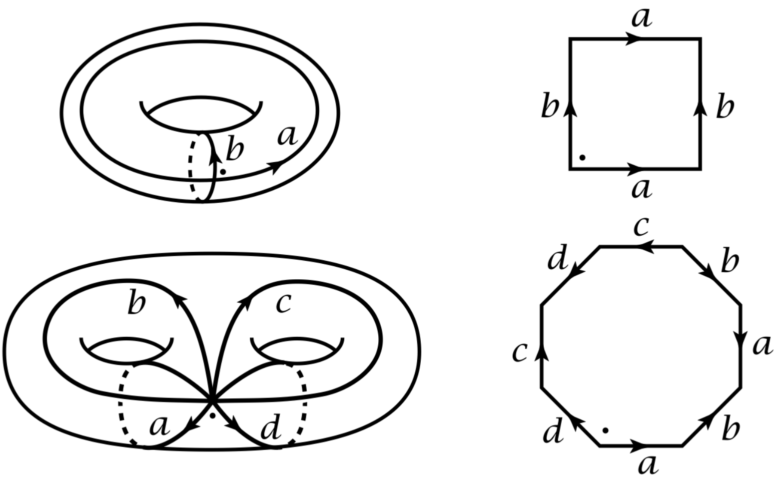
Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

B–H–V–Y idea to quantify the gap in $h_{\mathrm{top}}(\varphi^t) > h_{\mathrm{top}}(g^t)$:



Surgery in an annulus chart $\updownarrow$fiber $\leftrightarrow$base

The ghost of configuration space past



A double torus

Increased complexity by surgery

Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples

Summary

Foulon surgery
Definitions
Statement of results

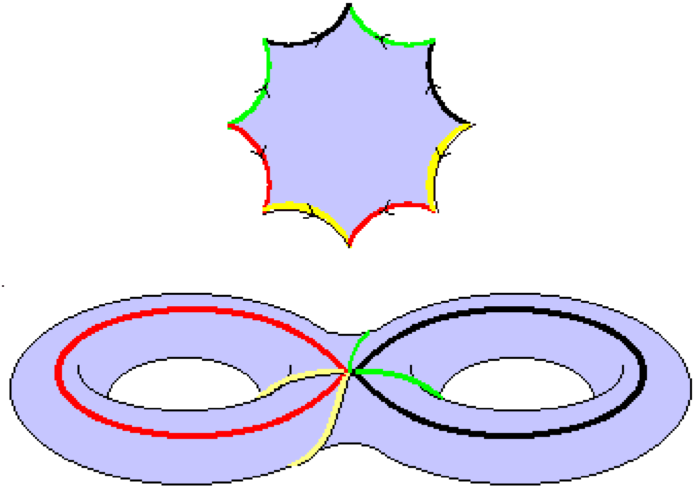
Proofs
Surgery
Local computations

Periodic orbit growth

Complexity of contact *structure*

Many flows

Mind the gap



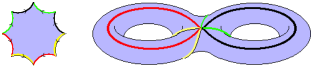
A double torus

The entropy gap quantified—an open problem
(Idea: Bishop–Hughes–Vinhage–Yang **@MRC2017**)

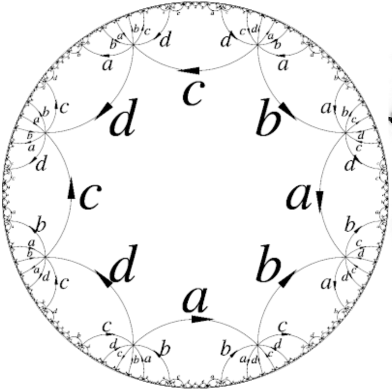
Increased complexity by surgery

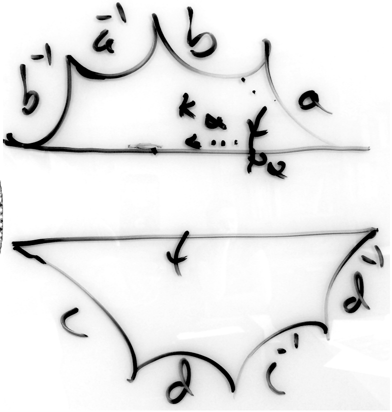
Foulon
Hasselblatt
Vaugon

Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

Bishop–Hughes–Vinhage–Yang: 

Use the configuration space with cutting sequences à la Series:





The Vinhage picture

Thank you for your attention!

Feedback invited on the book draft at `https://tinyurl.com/HypFlows`

Introduction
Algebraic examples
Summary
Foulon surgery
Definitions
Statement of results
Proofs
Surgery
Local computations
Periodic orbit growth
Complexity of contact *structure*
Many flows
Mind the gap

KAVLI IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE