Recent progress of volume conjectures

Qingtao Chen

New York University Abu Dhabi

Joint work with Kefeng Liu and Shengmao Zhu
Joint work with Tian Yang
Joint work with Jun Murakami

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Sketch of my talk

Theme of my talk: Intertwining of Mathematics and Physics

- Mathematics (Volume Conjecture for the colored Jones polynomials and Labastida-Marino-Ooguri-Vafa Conjectures) inspired by physics (Chern-Simons gauge theory, string theory and large N duality)
- Possible new physics interpretation indicated by new discovery in mathematics (Volume Conjectures for the Reshetikhin-Turaev invariants and the Turaev-Viro invariants as well as asymptotics of quantum 6j symbols)

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Part I

History and Basic Setups of Quantum Topology

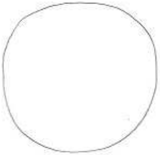
Definition

A knot is a simple closed curve embedded in $\mathbb{R}^3$.

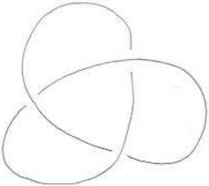
Roughly speaking, a link is several disconnected S^1 embedded in $\mathbb{R}^3$. We focus on the theory of knots in the first part of this talk, because theory of links is similar.

Two mathematical knots are equivalent if one can be transformed into the other via a deformation of $\mathbb{R}^3$ upon itself; these transformations correspond to manipulations of a knotted string that do not involve cutting the string or passing the string through itself.

Examples of knots



Unknot U or 0_1



Trefoil knot 3_1

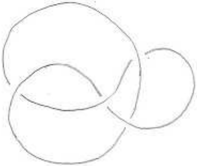
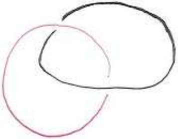


Figure eight knot 4_1

Simplest hyperbolic knot



Hopf link L2a1

Knot Invariants

Question

Is there any **Method** to distinguish different knots?

We have a very good candidate which is called **knot invariant**.

In the mathematical field of knot theory, a knot invariant is a quantity defined for each knot which is the same for equivalent knots.

Research on invariants is not only motivated by the basic problem of distinguishing one knot from another but also to understand fundamental properties of knots and their relations to other branches of mathematics.

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History

Early Development:

- 1928 Alexander polynomial (one variable)
- 1984 Jones polynomial (one variable)
- 1985 HOMFLY-PT polynomial (two variables)
- 1988-90 Kauffman polynomial (one variable and two variables)

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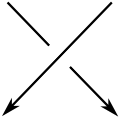
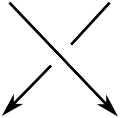
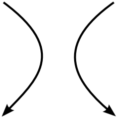
Skein relations

Common Fact: They all can be defined by combinatorial method, i.e. skein relations, as follows
HOMFLY-PT polynomial satisfy:

$$tH(\mathcal{L}_+) - t^{-1}H(\mathcal{L}_-) = (q - q^{-1})H(\mathcal{L}_0)$$

with initial condition imposed on the unknot, i.e. $H(\bigcirc) = 1$.

$\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_0$ are three oriented link diagrams that are identical except in one small region where they differ by the crossing changes or smoothing shown in the figure below:

$\mathcal{L}_+$:  $\mathcal{L}_-$:  $\mathcal{L}_0$: 

Revolution always happens when physics enters into the picture
Modern Development:

- 1988 V. Turaev: recognize HOMFLY-PT polynomial as the invariants associated with the fundamental representation of the quantum group $U_q(sl_N)$ by setting $q^N = t$.
- 1989 E. Witten: interpret the Jones polynomial by introducing Chern-Simons gauge theory and predict the existence of new invariants of knots/links as well as 3-manifolds at roots of unity $q = q(1)$, where $q(s) = e^{\frac{s\pi\sqrt{-1}}{r}}$.
- 1990 Reshetikhin-Turaev: use Quantum Group not only construct new invariants of knots/links and 3-manifolds predicted by Witten at roots of unity $q = q(1)$ but also at roots of unity $q = q(s)$, where s is an odd integer.
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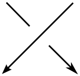
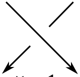
Basic Setups I

Let $\mathfrak{g}$ be a finite dimensional complex semi-simple Lie algebra and $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$. We fix $\mathfrak{g} = sl_N$. For each knot component, we associate it an irreducible representation V of $U_q(sl_N)$. A linear automorphism $\check{R}$ of $V \otimes V$ is said to be an R-matrix, if it is a solution of the Yang-Baxter equation. We use the following figure to represent $\check{R}^{\pm 1}$.



$\check{R}_{V,V}$



$\check{R}^{-1}_{V,V}$

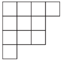
The quantum group invariants $W_V^{sl_N}(\mathcal{K};q)$ of the knot $\mathcal{K}$ can be obtained by taking the quantum trace of endomorphism obtained by those "crossings" of braid representation of a knot $\mathcal{K}$ up to some scaling of q power identity.

Basic Setups II

Definition

A partition of n is a tuple of positive integers $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ such that $|\mu| \triangleq \sum_{i=1}^{k} \mu_i = n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$, where $|\mu|$ is called the degree of μ and k is called the length of μ, denoted by $\ell(\mu)$. A partition can be represented by a Young diagram.

Denote by $\mathcal{P}$ the set of all Young diagrams. Let χ_A be the character of irreducible representation of symmetric group, labeled by partition A . Given a partition μ, define $m_j = \#(\mu_k = j; k \geq 1)$. The order of the conjugate class C_μ of type μ is given by $\mathfrak{z}_\mu = \prod_{j\geq 1} j^{m_j} m_j!$.

For example, if $\mu =$ , then we have $|\mu| = 11$, $\ell(\mu) = 4$ and $\mathfrak{z}_\mu = 4 \cdot 3^2 \cdot 1 \cdot 2! = 72$.

Example

Fact: Irreducible representation of $U_q(sl_N)$ can be identified with Young diagram, especially fundamental representation is identified with a single box $\square$.

$\mathcal{K} = Unknot \bigcirc$

$A = Any\ Young\ diagram$

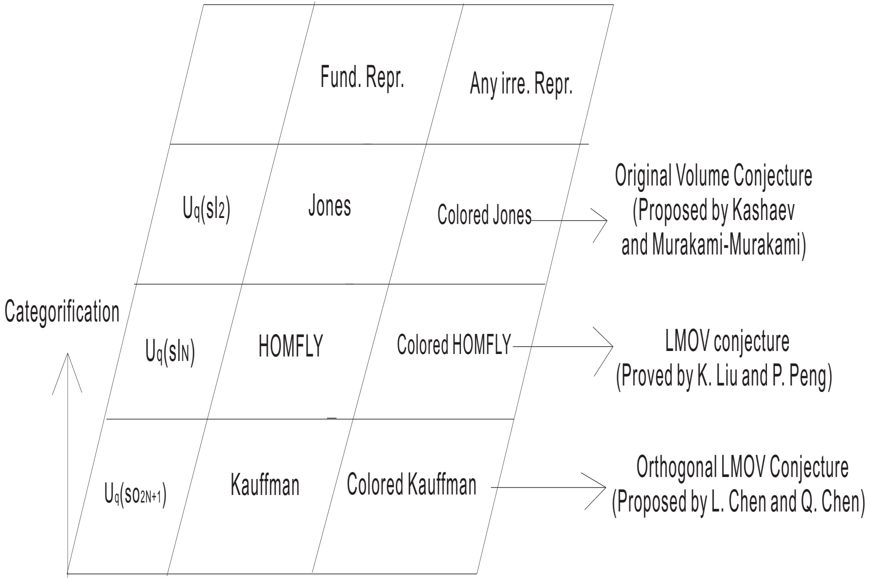
$V_A = Irreducible\ representation\ of\ U_q(sl_N)\ corresponding\ to\ A$

Then we have $W_A(\bigcirc; q) \triangleq W^{sl_N}_{V_A}(\bigcirc; q) = \sum_{|\mu|=|A|} \frac{\chi_A(C_\mu)}{\mathfrak{z}_\mu} \prod_{j=1}^{\ell(\mu)} \frac{q^{N\mu_j} - q^{-N\mu_j}}{q^{\mu_j} - q^{-\mu_j}}$,

where $W_A(\bigcirc; q)$ is called the quantum dimension of the corresponding representation space V_A and it is denoted by $\dim_q(V_A)$.

Fact: For each Young diagram A, there exists $\widetilde{W}^{SL}_A(\mathcal{K}; q, t) \in \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ s.t. $W^{sl_N}_A(\mathcal{K}; q) = \widetilde{W}^{SL}_A(\mathcal{K}; q, t)|_{t=q^N}$. We call $\widetilde{W}^{SL}_A(\mathcal{K}; q, t)$ two variable colored HOMFLY-PT invariant.

Summary


Fund. Repr.
Any irre. Repr.
$U_q(sl_2)$
Jones
Colored Jones
Original Volume Conjecture
(Proposed by Kashaev
and Murakami-Murakami)
Categorification
$U_q(sl_N)$
HOMFLY
Colored HOMFLY
LMOV conjecture
(Proved by K. Liu and P. Peng)
$U_q(so_{2N+1})$
Kauffman
Colored Kauffman
Orthogonal LMOV Conjecture
(Proposed by L. Chen and Q. Chen)

Part II

Hidden relations between quantum invariants

The LMOV Conjecture and generalizations

Mathematical Motivation of the LMOV conjecture I

The original LMOV Conjecture is about colored HOMFLY-PT invariants, so we can still fix $\mathfrak{g} = sl_N$.

It is well known that the classical HOMFLY-PT polynomial

$$H_{\mathcal{L}}(q,t) \in z^{1-L}\mathbb{Z}[z^2, t^{\pm 1}],$$

where $z = q - q^{-1}$.
We know that

$$W_V^{sl_N}(\mathcal{L};q) = \frac{q^N - q^{-N}}{q - q^{-1}} H_{\mathcal{L}}(q,t)|_{t=q^N}$$

holds for $V = \textit{fundamental representation}$ and any link $\mathcal{L}$.
Here $\mathbb{Z}$ means the integrality; $z^2 = (q - q^{-1})^2$ means the symmetry for $W_V^{sl_N}(\mathcal{L};q)$; z^{1-L} means the pole order structure.

If A=partition other than (1) ($V_A = \textit{other irreducible representation}$ of $U_q(sl_N)$), we only have $\widetilde{W}_A^{SL}(\mathcal{L}; q, t) \in \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$

For $\widetilde{W}_A^{SL}(\mathcal{L}; q, t)$, there is

- No obvious integrality
- No obvious symmetry
- No obvious pole order structure

To reveal these **hidden** integrality, symmetry and pole order structure for colored HOMFLY-PT invariants $\widetilde{W}_A^{SL}(\mathcal{K}; q, t)$, we require the deep idea from physics again!

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Physics Motivation of the LMOV conjecture

The LMOV conjecture come from a series of work done by four physicists Labastida, Mariño, Ooguri and Vafa inspired by large N duality which pioneered by a seminal work of 't Hooft in 1974.

The History of the original large N duality:

- 1992 Witten; The Chern-Simons gauge theory on S^3 is equivalent to the open topological string theory on the deformed conifold T^*S^3.
- 1998 Gopakuma-Vafa: The open topological string theory on the deformed conifold T^*S^3 is equivalent to the closed topological string theory on the resolved conifold X_{S^3} (Calabi-Yau 3-folds).
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t'Hooft,74: The duality of large N limit of a U(N) gauge theory and string theory

Chern-Simons, 74: obtain a transgressed form on odd dimensional manifold

Topological closed string theory capture the most information of string theory but rather easy to compute

Jones, 83: Jones polynomial of links

Witten, 89: Chern-Simons interpretation of Jones polynomial and predict more quantum invariants

Gromov-Witten invariants of Calabi-Yau 3-fold give a mathematical rigorous formulation for topological closed string partition function

Combination of Witten, 92 and Goparkumar-Vafa, 98: Chern-Simons of S^3 is equivalent topological closed string on resolved conifold X_S^3

Reshetikhin-Turaev, 90: Using representation theory of quantum group to formulate new invariants predicted by Witten

Katz-Liu, 01:Define the open Gromov-Witten invariants of Resolved conifold with a Lagrangian submanifold determined by unknot

Ooguri-Vafa, 00: Chern-Simons partition function of a link in S^3 (involve colored HOMFLY invariant) is equivalent to the topological open string partition function on resolved conifold X_S^3 with Lagrangian submanifold corresponding to the link

LMOV Conjecture, 01 (proved by K. Liu-P. Peng, 07): Free energy of Chern-Simons partition function of a link in S^3 has an integer coefficient expansion.

Conjecture (Labastida-Marino-Vafa, 01): The total topological open string free energy has an integer coefficient expansion.

Chern-Simons partition function

Let's quickly review the original LMOV conjecture first.

For each knot $\mathcal{K}$, the type$-A$ Chern-Simons partition function of $\mathcal{K}$ is defined by

$$Z_{CS}^{SL}(\mathcal{K};q,t;x)=\sum_{A\in\mathcal{P}}\widetilde{W}_A^{SL}(\mathcal{K};q,t)s_A(x),$$

where $s_A(x)$ are the Schur polynomials.

By using plethystic exponential method (due to Getzler & Kapranov), we can write the free energy $logZ_{CS}^{SL}(\mathcal{K};q,t;x)$ as follows

$$logZ_{CS}^{SL}(\mathcal{K};q,t;x)=\sum_{A\in\mathcal{P}}\sum_{d=1}^{\infty}\frac{f_A(\mathcal{K};q^d,t^d)}{d}s_A(x^d),$$

The original LMOV conjecture (for SL)

Conjecture (LMOV, 2000-2002)

There exists knot invariant $P_B(\mathcal{K};q,t) \in \frac{1}{(q-q^{-1})^2}\mathbb{Z}[(q-q^{-1})^2, t^{\pm 1}]$ s.t.

$$f_A(\mathcal{K};q,t) = \sum_{|B|=|A|} P_B(\mathcal{K};q,t) M_{AB}(q),$$

where $M_{AB}(q) = \sum_{|\mu|=|A|} \frac{\chi_A(C_\mu)\chi_B(C_\mu)}{\mathfrak{z}_\mu} \prod_{j=1}^{\ell(\mu)} (q^{\mu_j} - q^{-\mu_j})$.

Theorem (K. Liu & P. Peng, 2007-2009)

LMOV conjecture is true.

The original LMOV conjecture (for SL)

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LMOV conjecture is true.

Part III

Hidden relations between quantum invariants

Congruence skein relations

Motivation to study congruence skein relation

Not similar to classical HOMFLY polynomials, colored HOMFLY invariants have never been discovered to satisfy any skein relation since the discovery of Quantum invariants. Somehow it is always a dream for mathematician to simplify the calculation after some breakthrough discovery of a new theory. Anyway, searching certain skein relation is one among those dreams.

Along the way we study LMOV type conjecture for framed colored HOMFLY-PT invariants, we discover some cases indicates the existence of certain congruence skein relations. This is the start point from which we obtain a lot of results, conjectures as well as some very interesting phenomenon.

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Framed colored HOMFLY-PT invariants

Definition

Framed colored HOMFLY-PT invariant for a knot $\mathcal{K}$ is given by

$$\widetilde{W}_A^{f,SL}(\mathcal{K};q,t) \triangleq q^{\kappa_A w(\mathcal{K})} t^{|A| w(\mathcal{K})} \widetilde{W}_A^{SL}(\mathcal{K};q,t),$$

where A is a Young diagram, $\kappa_A = \sum_{j=1}^{\ell(A)} A_j(A_j - 2j + 1)$ for partition $A = (A_1, A_2, ..., A_{\ell(A)})$ and $w(\mathcal{K})$ denote the writhe number of knot $\mathcal{K}$. The definition for a link $\mathcal{L}$ with L components is similar.

There is also a framed version LMOV conjecture for framed colored HOMFLY-PT invariants, which seems more deep and hard to prove. But we still obtain a lot of nice property by studying the framed colored HOMFLY-PT invariants.

Reformulated colored HOMFLY-PT invariants I

Definition

Reformulated (framed) colored HOMFLY-PT invariant for a knot $\mathcal{K}$ is given by

$$\breve{\mathcal{Z}}_{\mu}(\mathcal{K};q,t) \triangleq \{\mu\} \sum_{|A|=|\mu|} \chi_A(C_\mu) \widetilde{W}_A^{f,SL}(\mathcal{K};q,t),$$

where A and μ are Young diagrams, $\{\mu\} \triangleq \prod_{j=1}^{\ell(\mu)} \{\mu_j\}$ and $\{n\} \triangleq q^n - q^{-n}$.

The definition for a link $\mathcal{L}$ with L components is similar.

Reformulated colored HOMFLY-PT invariant II

Theorem (C.-Liu-Peng-Zhu, 2014)

$$\breve{\mathcal{Z}}_{\vec{\mu}}(\mathcal{L}; q, t) \in \mathbb{Z}[(q - q^{-1})^2, t^{\pm 1}]$$

Now we introduce the following notation for link colored by same partition (p)

$$\breve{\mathcal{Z}}_p(\mathcal{L}; q, t) \triangleq \breve{\mathcal{Z}}_{((p),(p),\dots,(p))}(\mathcal{L}; q, t).$$

Congruence skein relations for colored HOMFLY-PT

The classical skein relation for framing dependent HOMFLY-PT polynomials can be stated as follows

$\breve{\mathcal{Z}}_1(\mathcal{L}_+) - \breve{\mathcal{Z}}_1(\mathcal{L}_-) = \breve{\mathcal{Z}}_1(\mathcal{L}_0)$ (I: crossing in 1 component of $\mathcal{L}$)

$\breve{\mathcal{Z}}_1(\mathcal{L}_+) - \breve{\mathcal{Z}}_1(\mathcal{L}_-) = \{1\}^2\breve{\mathcal{Z}}_1(\mathcal{L}_0)$ (II: crossing among 2 components of $\mathcal{L}$).

Now we state our conjecture as follows

Conj. (Congruence skein relations, C.-Liu-Peng-Zhu, 2014)

For any prime number p and any link $\mathcal{L}$, we have

$$\breve{\mathcal{Z}}_p(\mathcal{L}_+) - \breve{\mathcal{Z}}_p(\mathcal{L}_-) \equiv (-1)^{p-1}\breve{\mathcal{Z}}_p(\mathcal{L}_0) \mod [p]^2 \text{ (Type I)}$$

$$\breve{\mathcal{Z}}_p(\mathcal{L}_+) - \breve{\mathcal{Z}}_p(\mathcal{L}_-) \equiv (-1)^{p-1}p\{p\}^2\breve{\mathcal{Z}}_p(\mathcal{L}_0) \mod \{p\}^2[p]^2 \text{ (Type II)},$$

where $[n] = \frac{\{p\}}{\{1\}}$ and $A \equiv B \mod C$ means $\frac{A-B}{C} \in \mathbb{Z}[(q-q^{-1})^2, t^{\pm 1}]$.

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Evidence of conjecture of congruence skein relations

Although we can not prove this conjecture at this moment, but we discover a lot of evidence to support our conjecture.

Theorem (C.-Liu-Peng-Zhu, 2014)

The conjecture of congruence skein relations for colored HOMFLY-PT invariants holds for the following link triple

- $(\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0) = (T(2, 2k+1), T(2, 2k-1), T(2, 2k))$ *when* $p = 2, 3$;
- $(\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0) = (T(2, 2k), T(2, 2k-2), T(2, 2k-1))$ *when* $p = 2, 3$;
- $(\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0) = (4_1, \bigcirc, T(2, -2))$ *when* $p = 2, 3$;
- *torus links/knots with larger prime number* p;
- $(\mathcal{L}$ *with a positive kink,* $\mathcal{L}$ *with a negative kink,* $\mathcal{L} \sqcup \bigcirc)$ *with any prime number* p,

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Colored Jones polynomials

It is well-known that colored Jones polynomial $J_N(\mathcal{L};q)$ can be obtained from the colored HOMFLY-PT invariant in the following way.

$$J_N(\mathcal{L};q) \equiv \left(\frac{q^{-2lk(\mathcal{L})\kappa_{(N)}} t^{-2lk(\mathcal{L})N}}{s_{(N)}(q,t)} \widetilde{W}^{SL}_{((N),(N),\ldots,(N))}(\mathcal{L};q,t) \right) |_{t=q^2},$$

where $lk(\mathcal{L})$ is the linking number of a link $\mathcal{L}$ and $s_{(N)}(q,t)$ is the Schur function.

Remark: Our $J_N(\mathcal{L};q)$ denotes $N+1$ dimensional representation in original literature.

Congruence relations of knots for colored Jones polynomial

With the help of Habiro's cyclotomic expansion for colored Jones polynomial, we prove the following congruence relations for colored Jones polynomial of a knot $\mathcal{K}$,

Theorem (Congruence relations for colored Jones polynomial of knots, C.-Liu-Peng-Zhu, 2014)

For any knot $\mathcal{K}$ and any integer N, k and $N \geq k \geq 0$, we have

$$J_N(\mathcal{K}_+;q) - J_N(\mathcal{K}_-;q) \equiv J_k(\mathcal{K}_+;q) - J_k(\mathcal{K}_-;q) \mod \{N-k\}\{N+k+2\},$$

where $A \equiv B \bmod C$ means $\frac{A-B}{C} \in \mathbb{Z}[q^{\pm 1}]$.

Consequence of the Congruence Relations

As an application, we proved the following theorem,

Theorem

For any knot $\mathcal{K}$ and any integer N, k and $N \geq k \geq 0$, we have

$$J_N(\mathcal{K};q) \equiv J_k(\mathcal{K};q) \mod\{N-k\}\{N+k+2\}$$

In particular, we have

$$J_N(\mathcal{K};e^{\frac{\pi\sqrt{-1}}{N+2}}) = 1$$

which recover the following well known result due to V.F. Jones,

$$J_1(\mathcal{K};e^{\frac{\pi\sqrt{-1}}{3}}) = 1.$$

Those results enable us to have a close look at the behavior of colored Jones polynomial at certain roots of unity, which may shed some light on the following Volume conjecture.

Part IV

Asymptotics of quantum invariants of knots

The Original Volume Conjecture and its current situation

Volume Conjecture

Volume Conjecture (Kashaev 97', Murakami-Murakami 01')

The following equality would hold for any knot $\mathcal{K}$ in S^3,

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(\mathcal{K}, e^{\frac{\pi\sqrt{-1}}{N+1}})|}{N+1} = Vol(S^3\backslash\mathcal{K}).$$

Remark

1) This conjecture connects two very profound areas, i.e. quantum invariants founded by Jones, Witten and Reshetikhin-Turaev and modern hyperbolic geometry founded by Thurston.
2) For many years, only proven hyperbolic knot case is the figure-eight knot 4_1 (by Ekholm and full asymptotics by Andersen-Hansen). For other hyperbolic knots, people even don't know the existence of the limit. Recently the full asymptotics of three twists knot 5_2 was proved by Ohtsuki (volume part was proved by Kashaev earlier) by using a powerful analysis method to deal with this Conjecture. See next page.

Recent situation of Volume Conjecture

By a careful analysis of a combination of Poisson summation formula and Saddle point method, Ohtsuki proved not only the volume term of the case 5_2, but also a full asymptotic expansion of the case 5_2.

Ohtsuki's method method is again successfully applied to prove the following cases.

- Ohtsuki-Yokota: 6 crossings
- T. Ohtsuki: 7 crossings
- T. Takata: 8_6, 8_{12}

Remark

A possible physics explanation of the original Volume Conjecture is given by S. Gukov (2003) in the paper "Three-Dimensional Quantum Gravity,¡¡Chern-Simons Theory, and the A-Polynomial" and later by E. Witten (2010) in the paper "Analytic Continuation of Chern-Simons Theory".

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Part V

Hidden relation between congruence relations, cyclotomic expansion and Volume Conjectures

Colored $SU(n)$ invariants

It is well-known that colored $SU(n)$ invariants $J_N^{SU(n)}(\mathcal{L};q)$ can be obtained from the colored HOMFLY-PT invariant in the following way.

$$J_N^{SU(n)}(\mathcal{L};q) \equiv \left(\frac{q^{-2lk(\mathcal{L})\kappa_{(N)}} t^{-2lk(\mathcal{L})N}}{s_{(N)}(q,t)} \widetilde{W}^{SL}_{((N),(N),\dots,(N))}(\mathcal{L};q,t) \right) \Big|_{t=q^n}$$
$$= \frac{q^{-2lk(\mathcal{L})N(N+n-1)}}{s_{(N)}(q,q^n)} \widetilde{W}^{SL}_{((N),(N),\dots,(N))}(\mathcal{L};q,q^n),$$

where $lk(\mathcal{L})$ is the linking number of a link $\mathcal{L}$.

Congruence relations of knots for colored $SU(n)$ invariants

Actually we also formulate the conjecture of congruence relations for colored $SU(n)$ invariants.

Conjecture(Congruence relations of knots for $SU(n)$ invariants, Chen-Liu-Peng-Zhu, 2014)

For any knot $\mathcal{K}$ and any integer N, k and $N \geq k \geq 0$, we have

$$\begin{aligned} & J_N^{SU(n)}(\mathcal{K}_+;q) - J_N^{SU(n)}(\mathcal{K}_-;q) \\ & \equiv J_k^{SU(n)}(\mathcal{K}_+;q) - J_k^{SU(n)}(\mathcal{K}_-;q) \mod\{N-k\}\{N+k+n\} \end{aligned}$$

where $A \equiv B \bmod C$ means $\frac{A-B}{C} \in \mathbb{Z}[q^{\pm 1}]$.

Evidence and Consequence of conjecture of Congruence relations for colored $SU(n)$ invariants

Theorem (C.-Liu-Peng-Zhu, 2014)

The conjecture of congruence relations for colored $SU(n)$ invariants holds for the following knot double

- $(\mathcal{K}_+, \mathcal{K}_-) = (4_1, \bigcirc)$ *for any* $N \geq k \geq 0$, $n \geq 3$.
- *A lot of hyperbolic knots and torus links/knots with small integer pair* $N \geq k \geq 0$*;*

Corollary (C.-Liu-Peng-Zhu)

If the above conjecture holds, then for a knot $\mathcal{K}$, the set of the roots of the equation

$$J_N^{SU(n)}(\mathcal{K}; q) - J_k^{SU(n)}(\mathcal{K}, q) = 0$$

contains $A_{N-k} \cup A_{N+k+n} \cup A_{n-1}$*, where* $A_n \triangleq \{q | q^n = \pm 1\}$.

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Cyclotomic Expansion for colored Jones polynomial and colored $SU(n)$ invariants

Theorem (Harbiro)

The colored Jones polynomial has the following expansion
$J_N(\mathcal{K};q) = 1 + H_1(q)\{N\}\{N+2\} + H_2(q)\{N-1\}\{N\}\{N+2\}\{N+3\} + \cdots + H_N(q)\{1\}\cdots\{N\}\{N+2\}\cdots\{2N+1\}$,
where $H_i(q) \in \mathbb{Z}[q^{\pm 1}]$ *for* $i = 1, ..., N$.

Inspired by congruence relations, we got the following conjecture.

Conjecture (C.-Liu-Zhu, 2015)

The colored $SU(n)$ invariants has the following expansion
$J_N^{SU(n)}(\mathcal{K};q) = 1 + H_1^{(n)}(q)\{N\}\{N+n\} + H_2^{(n)}(q)\{N-1\}\{N\}\{N+n\}\{N+n+1\} + \cdots + H_N^{(n)}(q)\{1\}\cdots\{N\}\{N+n\}\cdots\{2N+n-1\}$,
where $H_i^{(n)}(q) \in \mathbb{Z}[q^{\pm 1}]$ for $i = 1, ..., N$.

Volume Conjecture for colored $SU(n)$ invariants

By solving the "gap equations" in the Habiro type cyclotomic expansion, we could formulate

Volume Conjecture for colored $SU(n)$ invariants(C.-Liu-Zhu, 2015)

The following equality (in our notation) would hold for any hyperbolic knot $\mathcal{K}$ complements in S^3. For any $a = 1, ..., n-1$, we have

$$2s\pi \lim_{N\to\infty} \frac{\log |J_N^{SU(n)}(\mathcal{K}, e^{\frac{s\pi\sqrt{-1}}{N+a}})|}{N+a} = Vol(S^3 \backslash \mathcal{K}).$$

Theorem (C.-Liu-Zhu, 2015)

The above Volume Conjecture is true for figure eight knot.

Cyclotomic Expansion for superpolynomial I

We first write down a special case of Superpolynomial (Dunfield-Gukov-Rasmussen) for the figure eight knot due to Fuji-Gukov-Sulkowski, originally Itoyama-Mironov-Morozov-Morozov.

$$\mathcal{P}_N(4_1; a, q, t) = 1 + \sum_{k=1}^{N} \prod_{i=1}^{k} \left(\frac{\{N+1-i\}}{\{i\}} A_{i-2}(a, q, t) B_{N-1+i}(a, q, t) \right)$$

where $A_i(a, q, t) = aq^i - (-t)^{-1} a^{-1} q^{-i}$ and $B_i(a, q, t) = (-t)^2 aq^i - (-t)^{-1} a^{-1} q^{-i}$.

In particular

$$A_i(q^2, q, -1) = B_i(q^2, q, -1) = \{i+2\}$$

$$\mathcal{P}_N(4_1; q^2, q, -1) = 1 + \sum_{k=1}^{N} \prod_{i=1}^{k} (\{N+1-i\}\{N+1+i\})$$

Inspired by congruence relations, we got the following conjecture. (see next page.)

Cyclotomic Expansion for superpolynomial II

Conjecture (C., 2015)

For each knot $\mathcal{K}$, there exists $\alpha(\mathcal{K}) \in \mathbb{Z}$ s.t. the superpolynomial associated to HOMFLY-PT homology has the following expansion
$(-t)^{N\alpha(\mathcal{K})}\mathcal{P}_N(\mathcal{K};a,q,t) =$
$$1+\sum_{k=1}^{N} H_k(\mathcal{K};a,q,t)\left(A_{-1}(a,q,t)\prod_{i=1}^{k}\frac{\{N+1-i\}}{\{i\}}B_{N+i-1}(a,q,t)\right),$$ where
$H_k(\mathcal{K};a,q,t) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}, t^{\pm 1}]$, $A_i(a,q,t) = aq^i - (-t)^{-1}a^{-1}q^{-i}$ and $B_i(a,q,t) = (-t)^2 aq^i - (-t)^{-1}a^{-1}q^{-i}$.

Remark

If this conjecture is true for $N = 1$, then $\alpha(\mathcal{K})$ is uniquely determined.

Theorem (C., 2016)

The above conjecture is true for torus knots T(m,n) (homologically thick knot) with $N = 1$, and we have $\alpha(T(m,n)) = -(m-1)(n-1)/2$

Relation to smooth 4-ball genus I

Definition

The smooth 4-ball genus $g_4(\mathcal{K})$ of a knot $\mathcal{K}$ is the minimum genus of a surface smoothly embedded in the 4-ball B^4 with boundary the knot. In particular, a knot $\mathcal{K}$ in S^3 is called smoothly slice if $g_4(\mathcal{K}) = 0$.

Remark

The invariant $\alpha(T(m,n)) = -(m-1)(n-1)/2$ suggest a very close relation between the above theorem and the following Milnor Conjecture, which was first proved by P.B. Kronheimer and T.S. Mrowka.

Milnor Conjecture

The smooth 4-ball genus of torus knot $T(m,n)$ is $(m-1)(n-1)/2$.

Relation to smooth 4-ball genus II

Rasmussen introduced a knot invariant $s(\mathcal{K})$ from Khovanov homology, which is a lower bound for the smooth 4-ball genus for knots in the following sense.

Theorem (Rasmussen)

For any knot $\mathcal{K}$ in S^3, we have $|s(\mathcal{K})| \leqslant 2g_4(\mathcal{K})$.

In addition, Rasmussen again proved Milnor Conjecture by a purely combinatorial method.
Based on all the knots we tested, we propose the following conjecture.

Conjecture (C.)

The invariant $\alpha(\mathcal{K})$ (determined by cyclotomic expansion conjecture for $N = 1$) is a lower bound for smooth 4-ball genus $g_4(\mathcal{K})$, i.e.
$|\alpha(\mathcal{K})| \leqslant g_4(\mathcal{K})$.

For all the knots we tested (up to 10 crossings), it is identical to the Ramussen's s invariant and the Ozsvath-Szabo's τ invariant.

Volume Conjecture for Superpolynomials

Again by solving the "gap equations" (2 variables this time) in the Habiro type cyclotomic expansion, we could formulate

Volume Conjecture for specialized Superpolynomials(C., 2016)

The following equality (in our notation) would hold for any hyperbolic knot $\mathcal{K}$ complements in S^3 with $b \geq 1$ and $\frac{n-1-b}{2} \notin \mathbb{Z}_{>0}$, we have

$$\lim_{N\to\infty} \frac{2\pi}{N} \log |\mathcal{P}_N(\mathcal{K}; q^n, q, t = q^{-(N+n-1)})|_{q=\exp(\frac{\pi\sqrt{-1}}{N+b})}| = Vol(S^3\backslash\mathcal{K}).$$

Remark:1) If $b = n-1$, then we have $t = -1$ (original Volume Conjecture). 2) If $n = 2$, then b is chosen from 1(corresponds to the original case), 2, 3,... 3) In a joint work with Joergen Andersen, we fix t to get so called refined Volume Conjectures.

Theorem (C., 2016)

The above Volume Conjecture is true for figure eight knot.

Part VI

Asymptotics of quantum invariants of 3-manifolds

The Volume Conjecture of Reshetikhi-Turaev and (modified) Turaev-Viro invariants

Volume Conjecture for RT invariants at $q(2)$

Set $q(s) = e^{\frac{s\pi\sqrt{-1}}{r}}$ under the quantum integer notation $[N] = \frac{q^N - q^{-N}}{q - q^{-1}}$, where $(r, s) = 1$. So usual Chern-Simons theory evaluated at $e^{\frac{2\pi\sqrt{-1}}{r}}$ under usual notation $[N] = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}$ is actually $q(1)$ under our notations.

Reshtikhin-Turaev not only rigorously defined invariant $RT_r(M)$ of 3-dim closed manifold evaluated at $q(1)$ predicted by Witten but also extend it to $q(odd)$. Blanchet-Habegger-Masbaum-Vogel again extended them to $q(4k+2)$. We propose our Volume conjecture at $q(2)$. ($q(odd)$ later)

Volume Conjecture (3-dim closed orientable manifolds, C.-Yang)

For any 3-dim hyperbolic closed orientable manifold M and let $Vol_{cpx}(M)$ denotes $Vol(M) + \sqrt{-1}CS(M)$, we have

$$4\pi \lim_{r\to\infty,\ r \text{ is odd}} \frac{\log RT_r(M; q(2) = e^{\frac{2\pi\sqrt{-1}}{r}})}{r-2} = Vol_{cpx}(M) (\mathrm{mod}\ \sqrt{-1}\pi^2\mathbb{Z})$$

Let M_p denotes the p-surgery (Dehn) of knot $\mathcal{K}$ in S^3.

The Reshetikhin-Turaev invariants $RT_r(M_p; q(2))$ is given by the following identity

$$\frac{2}{r}e^{(\frac{3+r^2}{r}-\frac{3-r}{4})\pi\sqrt{-1}}\sum_{n=0}^{r-2}\left(\sin\frac{2\pi(n+1)}{r}\right)^2(-e^{\frac{\pi\sqrt{-1}}{r}})^{-p(n^2+2n)}J_n(\mathcal{K}; q(2))$$

In order to verify the Volume Conjecture, it is equivalent to calculate the limit of the following quantity

$$Q_r(M) = 2\pi\log\left(RT_r(M; q(2) = e^{\frac{2\pi\sqrt{-1}}{r}})/RT_{r-2}(M; q(2) = e^{\frac{2\pi\sqrt{-1}}{r-2}})\right).$$

Examples of closed oriented 3-manifolds II

Recall that M_p is hyperbolic when $|p| > 4$, when knot $\mathcal{K}$ is the figure eight knot. For p=6, we have the following according to SnapPy,

$$Vol(M_6) + CS(M_6)\sqrt{-1} = 1.28449 + 1.34092\sqrt{-1} \quad (mod\ \sqrt{-1}\pi^2),$$

We have the following table of $Q_r(M_6)$ modulo $\sqrt{-1}\pi^2\mathbb{Z}$.

r	51	101
$Q_r(M_6)$	$1.22717 + 1.24762\sqrt{-1}$	$1.28425 + 1.30510\sqrt{-1}$
r	151	201
$Q_r(M_6)$	$1.28440 + 1.32496\sqrt{-1}$	$1.28443 + 1.33194\sqrt{-1}$
r	301	501
$Q_r(M_6)$	$1.28446 + 1.33693\sqrt{-1}$	$1.28448 + 1.33948\sqrt{-1}$

Volume Conjecture of RT invariant at $q(odd)$

The Reshetikhin-Turaev invariant of closed 3-manifold M obtained from a $4k+2$-surgery along a knot K, $RT_r(M, q(s))$, vanishes at roots of unity $q(s)$, where r and s are odd numbers (Kirby-Melvin and C.-Liu-Peng-Zhu). Numerical evidence shows that it also goes exponentially large as $r \to \infty$, when s is an odd integer other than 1 but r is an even integer. So it is natural to propose

Volume Conjecture for Reshetikhin-Turaev invariants at $q(odd)$

For any closed 3-manifold M, an odd integer s other than 1 and an integer r s.t. condition $(**)$ $RT_r(M, q(s)) \neq 0$ is satisfied, then we have

$$\lim_{r\to\infty,\ (r,s)=1 \text{ and } r \text{ satisfy } (**)} \frac{2s\pi}{r} \log |RT_r(M, q(s))| = Vol_{cpx}(M) \pmod{\sqrt{-1}\pi^2\mathbb{Z}}$$

Remark

If $RT_r(M, q(s)) \neq 0$ for any even integer r and any 3-manifold closed manifold M, we could change condition "r satisfy $(**)$" to "r is even".

Volume Conjecture for Turaev-Viro invariants

Turaev-Viro invariants was originally defined for closed 3-manifolds and become a TQFT for 3-manifolds with non-empty boundary. Here we just do a small modification. We use the Thurston's ideal triangulation instead of the usual triangulation in the construction. Then we obtain a real valued number instead of a TQFT for 3-manifolds with non-empty boundary.

Volume Conjecture (3-manifolds with boundary, C.-Yang)

For any hyperbolic 3-manifolds M with cusps or with totally geodesic boundary and for r running over all odd integers, we have

$$2\pi \lim_{r\to\infty} \frac{\log TV_r(M; e^{\frac{2\pi\sqrt{-1}}{r}})}{r-2} = Vol(M).$$

Comparison of Volume Conjectures with boundary

tetrahedra #	1	2	3	4	5	6	7	8
$S^3\backslash$ hyperbolic knots/links #	0	1	2	4	22	43	129	299
orientable cusped mfld #	0	2	9	56	234	962	3552	12846
non-orientable cusped mfld #	1	2	7	26	78	258	887	2998

Remark

The original Volume Conjecture was mainly proposed for complements of hyperbolic knots/links in S^3.

Our Volume Conjecture for Turaev-Viro type invariants was proposed for all the cusped manifolds and 3-manifolds with totally geodesic boundary, most of which are NOT knots/links complements.

For example, figure eight knot is a hyperbolic knot whose complement in S^3 consists of two tetrahedra.

5_2 knot is a hyperbolic knot whose complement in S^3 consists of three tetrahedra.

Examples of non-orientable cusped 3-manifolds I

Gieseking manifold N_{1_1} is a non-orientable cusped 3-manifold which consists of only one tetrahedra.
For $r \geqslant 3$, we have

$$TV_r(N_{1_1}) = \sum_{a \in A_r} w_a \begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix},$$

where A_r consist of integers a such that $0 \leqslant a \leqslant (r-2)/3$, $w_i = (-1)^{2i}[2i+1]$ and $\begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix}$ is the quantum $6j$-symbol (Kirillov-Reshetikhin).

Examples of non-orientable cusped 3-manifolds II

For each odd integer $r \geqslant 3$ let Quantum Volume

$$QV_r(M) = \frac{2\pi}{r-2} \log \left(TV_r(M; e^{\frac{2\pi\sqrt{-1}}{r}}) \right).$$

According to SnapPy and Regina, the Gieseking manifold has volume $Vol(N_{1_1}) \approx 1.014942$. We have the following table of values of $QV_r(N_{1_1})$.

r	11	31	51	101
$QV_r(N_{1_1})$	1.62276	1.33012	1.23174	1.14319
r	201	301	401	501
$QV_r(N_{1_1})$	1.08943	1.06872	1.05748	1.05035

Question

Why we call the invariant **Quantum Volume**?

See next page.

Examples of non-orientable cusped 3-manifolds II

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See next page.

Why "Quantum Volume" ?

Remark

We witness very drastic/huge cancellations for the figure eight knot complements. Some summands of Turaev-Viro invariant even reach 10^{500}, while the final Turaev-Viro invariant is just around 10^{140} for $r = 1001$.

In contrast, the convergence behavior of colored Jones polynomial in original Volume Conjecture of figure eight knot is quite simple, which is just the summation of positive numbers. (The original Volume Conjecture of hyperbolic knot 5_2 is more complicated but it still does not involve that dramatic cancellations)Thus it is rather easy to prove the figure eight case for original Volume Conjecture, which only involve some simple estimations.

Volume Conjecture of TV invariant at $q(odd)$

The Turaev-Viro invariant of non-orientable 3-manifold N_{2_1} (Callahan-Hildebrand-Weeks census) vanishes at roots of unity $q(s)$, where r and s are odd integers. Numerical evidence shows that it also goes exponentially large as $r \to \infty$, where s is an odd integer other than 1 but r is an even integer. So it is natural to propose

Volume Conjecture for Turaev-Viro invariants at $q(odd)$

For any 3-manifold M with boundary, an odd integer s other than 1 and an integer r s.t. condition $(*)$ $TV_r(M, q(s)) \neq 0$ is satisfied, then we have

$$\lim_{\substack{r \to \infty,\ (r,s)=1 \\ \text{and } r \text{ satisfy } (*)}} \frac{s\pi}{r} \log |TV_r(M, q(s))| = Vol(M)$$

Remark

If $TV_r(M, q(s)) \neq 0$ for any even integer r and any 3-manifold M with boundary, we could change condition "r satisfy $(*)$" to "r is even".

Recent Development of these Volume Conjectures I

Ohtsuki refined our Volume Conjecture of Reshetikhin-Turaev invariants at $q(2)$ to full asymptotic expansion (physics flavour conjecture by D. Gang-M.Romo-M. Yamazaki) and proved the case of closed hyperbolic 3-manifold obtained by integral surgery along the figure-eight knot 4_1 in S^3.

Ohtsuki-Takata recognized the secondary term in asymptotic expansion of RT invariant as Reidemester torsion for the above example.

Detcherry-Kalfagianni-Yang proved the Volume Conjecture of (modified) Turaev-Viro invariant of complements of 4_1 and Borromean ring in S^3 by establishing a relation involving Turaev-Viro invariants of link complements in S^3 and certain sum of colored Jones polynomials of that link.

Recent Development of these Volume Conjectures II

Detcherry-Kalfagianni first established a relation between the asymptotics of the (modified) Turaev-Viro invariants at $q(2)$ and the Gromov norm of 3-manifolds. Then they obtained a lower bound for the Gromov norm of any compact, oriented 3-manifold with empty or toroidal boundary. They also proved C.-Yang Volume Conjecture for (modified) Turaev-Viro invariants of Gromov norm zero links complements.

Detcherry-Kalfagianni first related the Andersen-Masbaum-Ueno conjecture to the growth of the Turaev-Viro invariants of hyperbolic 3-manifolds at $q(2)$. Then they proved that C.-Yang Volume Conjecture implies the AMU conjecture. They also answer an integrality conjecture of C.-Yang for the (modified) Turaev-Viro invariants of torus links.

Belletti-Detcherry-Kalfagianni-Yang proved fundamental shadow link.

K.H. Wong-T. K.-K. Au obtained asymptotic expansion formula for the Turaev-Viro invariant of figure eight knot evaluated at $q(2)$.

C.-Murakami proved that the primary/secondary terms of the asymptotic expansion of a Kirillov-Reshetikhin quantum $6j$ symbol are dominated by the Volume/Gram matrix of a single tetrahedra respectively for majority cases. Then we also proposed a conjecture for a symmetric property of asymptotics of quantum $6j$ symbol at $q(2)$. Highly nontrivial cases are checked. This conjecture also explains the very mysterious big cancellations when one compute (modified) Turaev-Viro invariants at $q(2)$. This conjecture uncovers an extraordinary hidden nature of the quantum $6j$ symbols.

Z. Wang related Anyon system, 3d quantum gravity to our volume conjecture for closed hyperbolic 3-manifolds and he pointed out that it is puzzling how the non-unitarity arises from the unitary 3d quantum gravity.

Reshetikhin pointed out that Conformal Field Theory corresponding to the quantum $6j$ symbols evaluated at non-conventional root of unity such as $q(2)$ is non-unitary.

Volume Conjecture for quantum 6j symbol I

Since the Turaev-Viro invariant is constructed by using Krillov-Reshetikhin quantum $6j$ symbols, the Volume conjecture for Turaev-Viro invariants suggests that the asymptotics of the quantum $6j$ symbol is expressed by certain geometric data of the corresponding tetrahedron. The volume conjecture can be reformulated for the quantum $6j$ symbols as follows. Let

$$\begin{vmatrix} a & b & e \\ d & c & f \end{vmatrix}_q$$

be the quantum $6j$ symbol which is defined for six non-negative half integers $a, b, \cdots, f$ with the quantum parameter q. For a odd integer $r \geq 3$, a triplet (a, b, c) is called r-*admissible* if a, b, $c \in \{0, 1/2, 1, \cdots, (r-2)/2\}$, (a, b, c) satisfies the Clebsch-Gordan condition and $a + b + c \leq r - 2$. The Clebsch-Gordan condition means that $|a - b| \leq c \leq a + b$ and $a + b + c \in \mathbb{Z}$.

Volume Conjecture for quantum 6j symbol II

Conjecture(C.-Murakami)

Let T be a tetrahedron in hyperbolic, Euclidean or spherical 3-space with dihedral angles θ_a, θ_b, θ_c, θ_d, θ_e, θ_f at edges $a, \cdots, f$. Let a_r, b_r, $\cdots$, f_r be sequences of non-negative half integers satisfying

$$\lim_{r\to\infty} \frac{2\pi}{r}(2a_r+1) = \pi - \theta_a, \quad \cdots, \quad \lim_{r\to\infty} \frac{4\pi}{r}(2f_r+1) = \pi - \theta_f$$

and the triplets (a_r, b_r, e_r), (a_r, d_r, f_r), (b_r, d_r, f_r) and (c_r, d_r, e_r) are all r-admissible for odd $r \geq 3$. Then

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \begin{vmatrix} a_r & b_r & e_r \\ d_r & c_r & f_r \end{vmatrix}_{q(2)} \right| = \mathrm{Vol}(T),$$

where $\mathrm{Vol}(T)$ is the hyperbolic volume of T if T is hyperbolic and $\mathrm{Vol}(T) = 0$ if T is Euclidean or spherical.

Basic Hyperbolic Geometry Setup

Here we show that the above conjecture is true if T is hyperbolic and has at least one ideal or ultra-ideal vertex. Vertices of our hyperbolic tetrahedron are classified into three cases. Let v be a vertex of a tetrahedron T, F_1, F_2, F_3 be the three planes of T to specify v, and θ_{ij} be the dihedral angle of F_i and F_j at the intersection of $F_i \cap F_j$.

- Normal vertex: The first case is the usual vertex, which is the intersection $F_1 \cap F_2 \cap F_3$. In this case, $\theta_{12} + \theta_{13} + \theta_{23} > \pi$.
- Ideal vertex: The second case is the ideal vertex, which is the vertex at ∞, which means that the three edges around the vertex do not intersect in the hyperbolic space, but the infimum of their distances are zero. In this case, $\theta_{12} + \theta_{13} + \theta_{23} = \pi$.
- Ultra-ideal vertex: The last one is the ultra-ideal vertex. In this case, $F_1 \cap F_2 \cap F_3 = \phi$, but there is a plane perpendicular to F_1, F_2, F_3, and adding this plane to F_1, F_2, F_3, we get a truncated vertex as the second tetrahedron. In this case, $\theta_{12} + \theta_{13} + \theta_{23} < \pi$.

Asymptotic Expansion of the Quantum 6j Symbol

Theorem(C.-Murakami,16-17')

Let T be a hyperbolic tetrahedron with one vertex ideal or ultra-ideal, then

$$\left| \begin{vmatrix} a_r & b_r & e_r \\ d_r & c_r & f_r \end{vmatrix}_{q(2)} \right| \underset{r\to\infty}{\sim} \frac{\sqrt{2}\,\pi}{r^{3/2}\sqrt[4]{\det G}} e^{\frac{r}{2\pi}\mathrm{Vol}(T)}, \qquad (1)$$

where G is the Gram matrix of T given by

$$G = \begin{pmatrix} 1 & -\cos\theta_a & -\cos\theta_b & -\cos\theta_f \\ -\cos\theta_a & 1 & -\cos\theta_e & -\cos\theta_c \\ -\cos\theta_b & -\cos\theta_e & 1 & -\cos\theta_d \\ -\cos\theta_f & -\cos\theta_c & -\cos\theta_d & 1 \end{pmatrix}.$$

Remark

Quantum 6j symbol evaluated at $q(1)$ after certain "evaluation" can also be related to the hyperbolic Volume of a tetrahedra which was systematically studied by F. Costantino.

Summary of Volume Conjectures

	Colored Jones (colored SU(n), Superpolynomial of HOMFLY-PT homology) for links	Reshetikhin-Turaev invariants for closed oriented 3-manifolds	Turaev-Viro invariants for 3-manifolds with boundary (C.-Yang)	Quantum 6j symbols (Krillov-Reshetikhin)
q(1)	Volume Conjecture (Kashaev-Murakami-Murakami, SU(n) by C.-Liu-Zhu, Super. by C.)	Witten's Asymptotic Expansion Conjecture (WAE), which asserts RT invariants polynomial growth in terms of r	No Volume Conjecture, should be something similar to WAE Conjecture	Woodward's Asymptotic Expansion Conjecture, which asserts the quantum 6j symbol polynomial growth in terms of r
q(2)	Almost the same as above	Volume Conjecture (proposed by C.-Yang)	Volume Conjecture (proposed by C.-Yang)	Volume Conjecture (proposed by C.-Murakami and majority cases proved by C.-Murakami)

Remark

Formerly, the Reshetikhin-Turaev invariants evaluated at roots of unity other than $q(1)$ was considered to be related to $q(1)$ via certain Galois transformations and thus not significant at all. Now the huge difference between roots $q(1)$ and $q(2)$ has been revealed.

Potential impact of these Volume Conjectures I

We hope this Volume Conjecture of Reshetikhin-Turaev and Turaev-Viro type invariants of 3-manifolds (closed oriented or with non-empty boundary) may uncover certain new geometric/physical interpretation other than the usual $SU(2)$ Chern-Simons gauge theory.

Quantum integer $[N] = \frac{q^N - q^{-N}}{q - q^{-1}} = \frac{sin(\frac{2\pi N}{r})}{sin(\frac{2\pi}{r})}$ for $q = q(2) = e^{\frac{2\pi\sqrt{-1}}{r}}$, thus quantum integer $[N] < 0$ become possible during the computation. Unlike the original Witten's Chern-Simons theory, this indicate a non-unitary Physics theory, which seems quite wild at this moment. But it is confirmed by many top mathematical physicists such as Giovanni Felder, Rinat Kashaev, Nicolai Reshetikhin and Edward Witten.

Potential impact of these Volume Conjectures I

There are a lot of Non-Unitary physics theory, but none of them have any experimental support nor have mathematical significance! Our discovery for the Reshetikhin-Turaev and the Turaev-Viro invariants probably is the first time that the mathematics of a potential non-unitary physics theory looks so extraordinary.

	Physics		Mathematics
$q(1)$	Witten′s CS gauge theory	$\Rightarrow$	Polynomially growth Reshetikhin-Turaev invariants (Witten's Asym. Expan. Conj.)
$q(2) = q(1)^2$	**A new non-unitary physics theory?**	$\Leftarrow$	Exponentially growth Reshetikhin-Turaev invariants (C.-Yang's new Volume Conjectures)

Future

The Turaev-Viro theory is very close to quantum gravity. Although TV theory is only a 2+1D theory, it may shed some new light on the Black Holes Information Paradox, which may require a non-unitary theory to explain.

- 19th Century: from Linear to Non-linear
- 20th Century: from Commutative to Non-commutative (one variable)
- 21th Century: from Unitary to Non-unitary??

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Thank You!