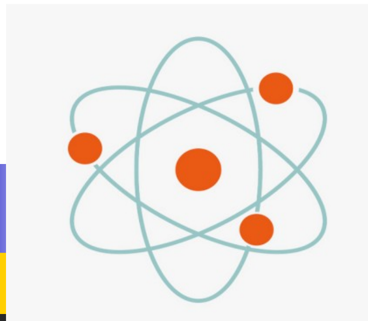
Schrödinger, Klein-Gordon and Dirac equations, atomic wave functions and Operator Product Expansion



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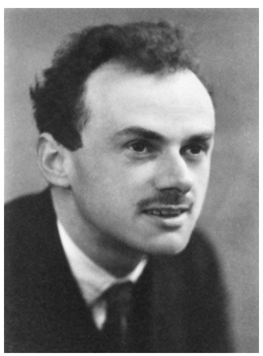

In collaboration with Yingsheng Huang and Rui Yu

Based on ***arXiv: 1809.09023, 1812.11957, 1901.04971***

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Outline of the talk

- Review of universal electron-nucleus coalescence behavior of atomic wave function
- Brief introduction to EFT and OPE – modern tools of QFT
- Non-relativistic Coulomb-Schrödinger EFT – Standard QFT for atomic physics
- Rigorous proof of an OPE relation to all orders in perturbation theory
- 90-year puzzle about wave function at the origin for KG and Dirac hydrogen
- Insight from Schrödinger perturbation theory in QM – UV div. and Renormalization
- OPE and RGE in nonrelativistic EFT implementing relativistic corrections
- Summary

梦回唐朝—穿越回物理学的黄金年代—英雄辈出的 1920 年代

- Golden age of physics, many young heroes in deve ping quantum mechanics

- Non-relativistic wave mechanics, relativistic wave echanics were invented in almost same time – fina y give way to more fundamental framework: Quan m Field Theory

  QFT = Special Relativity + Quantum Mechanics

Figure: Erwin Schrödinger (left), Paul Dirac (right) and Oskar Klein (down)

- Application of single-particle wave mechanics (**Sch ödinger, Klein-Gordon and Dirac equations**) to hy drogen spectroscopy plays a vital role in shaping the modern physics

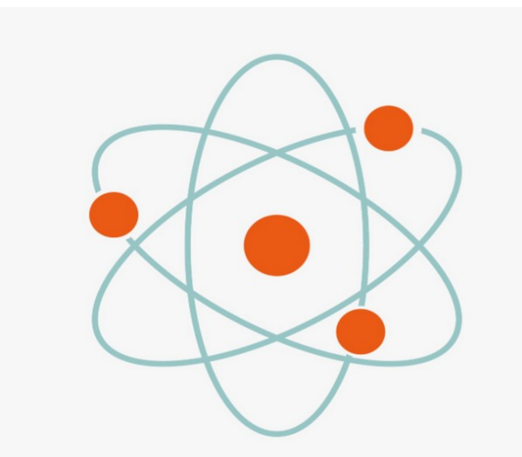
Triumph of QM in early days

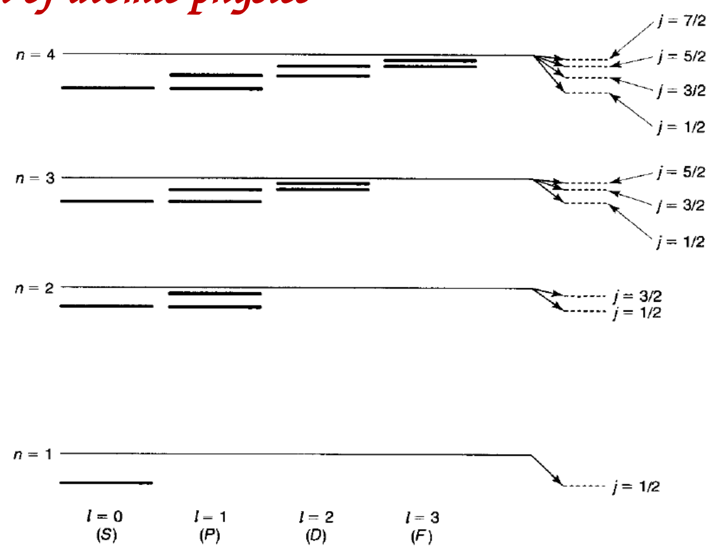
Schrödinger equation with Coulomb potential: the standard theory for atomic physics and quantum chemistry

$$H_{\mathrm{Coul}} = -\sum_{i=1}^{N} \frac{\nabla_i^2}{2m} - \sum_{i=1}^{N} \frac{Z\alpha}{r_i} + \sum_{j>i=1}^{N} \frac{\alpha}{r_{ij}}$$

$$H_{\mathrm{Coul}}\Psi = E\Psi;$$

Standard Model of atomic physics





Orientation: electron-nucleus coaelesence behavior [KG equation with a Coulomb potential] (Schrödinger, 1926, unpublished note)

$$\left[\left(E+\frac{Ze^2}{4\pi r}\right)^2+\hbar^2c^2\nabla^2-m^2c^4\right]\Psi(\mathbf{r})=0.$$

$$\Psi_{nlm}(\mathbf{r})=R_{nl}(r)Y_{lm}(\hat{\mathbf{r}}),$$

$$E_{nl} = mc^2\left\{1+\frac{Z^2\alpha^2}{\left(n-l-\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-Z^2\alpha^2}\right)^2}\right\}^{-\frac{1}{2}}$$

$$= mc^2\left\{1-\frac{Z^2\alpha^2}{2n^2}-\frac{Z^4\alpha^4}{2n^4}\left(\frac{n}{l+\frac{1}{2}}-\frac{3}{4}\right)+\cdots\right\},$$

Wrong fine structure, spin-0 electron

$\rho = (2/n)\,(r/a0)$, a0 is Bohr radius

For S-wave hydrogen atom, KG wave function at short-distance scales as

$$R_{n0}^{\rm KG}(r)\approx R_{n0}^{\rm Sch}(0)\left(\frac{\rho}{2}\right)^{\sqrt{\frac{1}{4}-Z^2\alpha^2}-\frac{1}{2}}\approx R_{n0}^{\rm Sch}(0)\left(\frac{\rho}{2}\right)^{-Z^2\alpha^2}=R_{n0}^{\rm Sch}(0)\left(1-Z^2\alpha^2\ln r+\cdots\right),$$

Long-standing puzzle: why KG wave function at the origin diverges? And so weakly (logarithmically)?

Orientation: electron-nucleus coaelesence behavior: Dirac equation with a Coulomb potential (Darwin and Gordon 1928)

$$(-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2 - \frac{Z\alpha c}{r})\Psi = E\Psi$$

$$\Psi_{njm}(\mathbf{r}) = \begin{bmatrix} f_{n_r\kappa}(\mathbf{r})\Omega_{jlm} \\ (-1)^{\frac{1}{2}(1+l+l')}g_{n_r\kappa}(\mathbf{r})\Omega_{jl'm} \end{bmatrix},$$

$$\kappa = \begin{cases} -(l+1), & j = l + \frac{1}{2} \\ l+1, & j = l - \frac{1}{2} \end{cases},$$

$$\begin{aligned} E_{nj} &= mc^2 \left\{ 1 + \frac{Z^2\alpha^2}{\left(n - j - \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 - Z^2\alpha^2}\right)^2} \right\}^{-\frac{1}{2}} \\ &= mc^2 \left\{ 1 - \frac{Z^2\alpha^2}{2n^2} - \frac{Z^4\alpha^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right) + \cdots \right\}, \end{aligned}$$

For $nS_{1/2}$ hydrogen taom, Dirac wave function at short-distance scales as

$$f_{n0}^{\rm Dirac}(r) \approx R_{n0}^{\rm Sch}(0)\left(\frac{\rho}{2}\right)^{\sqrt{1-Z^2\alpha^2}-1} \approx R_{n0}^{\rm Sch}(0)\left(\frac{\rho}{2}\right)^{-Z^2\alpha^2 over 2} = R_{n0}^{\rm Sch}(0)\left(1 - \frac{Z^2\alpha^2}{2}\ln r + \cdots\right),$$

Long-standing puzzle since 1928:
why Dirac wave function at the origin diverges? and so weakly (logarithmically)? What is the physics behind?

Universal behavior of wave function near the origin in Schrödinger hydrogen (S-wave)

Expand the radial wave function near the origin ($r \ll a_0$)

$$R_{n0}^{Schr}(r) \propto \begin{cases} 1 - \alpha m r Z + \frac{1}{2}\alpha^2 m^2 r^2 Z^2 + \cdots & (n=1) \\ 1 - \alpha m r Z + \frac{3}{8}\alpha^2 m^2 r^2 Z^2 + \cdots & (n=2) \\ 1 - \alpha m r Z + \frac{19}{54}\alpha^2 m^2 r^2 Z^2 + \cdots & (n=3) \\ 1 - \alpha m r Z + \frac{11}{32}\alpha^2 m^2 r^2 Z^2 + \cdots & (n=4) \end{cases}$$

Non-universal

Universal behavior of wave function near the origin in Klein-Gordon hydrogen (S-wave)

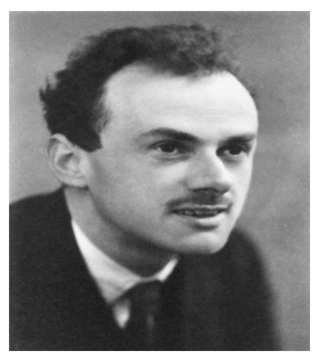
Non-universal

$$R_{n0}^{KG}(r) \propto \begin{cases} 1 - \alpha mrZ + \cancel{\frac{1}{2}\alpha^2 m^2 r^2 Z^2} - Z^2\alpha^2 \log(Z\alpha mr) + Z^3\alpha^3 mr \log(Z\alpha mr) + \cdots & (n=1) \\ 1 - \alpha mrZ + \cancel{\frac{3}{8}\alpha^2 m^2 r^2 Z^2} - Z^2\alpha^2 \log(Z\alpha mr) + Z^3\alpha^3 mr \log(Z\alpha mr) + \cdots & (n=2) \\ 1 - \alpha mrZ + \cancel{\frac{19}{54}\alpha^2 m^2 r^2 Z^2} - Z^2\alpha^2 \log(Z\alpha mr) + Z^3\alpha^3 mr \log(Z\alpha mr) + \cdots & (n=3) \\ 1 - \alpha mrZ + \cancel{\frac{11}{32}\alpha^2 m^2 r^2 Z^2} - Z^2\alpha^2 \log(Z\alpha mr) + Z^3\alpha^3 mr \log(Z\alpha mr) + \cdots & (n=4) \end{cases}$$

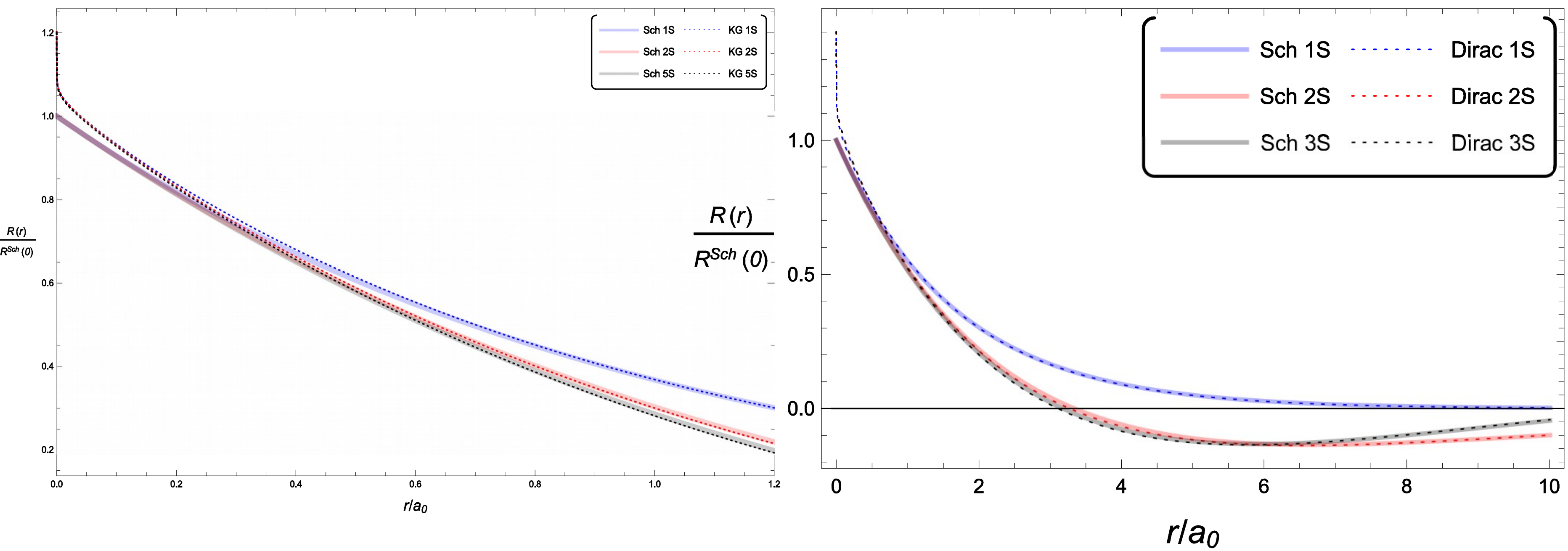
Universal behavior of wave function near the origin in Dirac hydrogen (The $nS_{1/2}$ state)

Non-universal

$$R_{n0}^{Dirac}(r) \propto \begin{cases} 1 - \alpha mrZ + \cancel{\frac{1}{2}\alpha^2 m^2 r^2 Z^2} - \frac{Z^2\alpha^2 \log(\alpha mrZ)}{2} + \frac{Z^3\alpha^3 mr \log(\alpha mrZ)}{2} + \cdots & (n=1) \\ 1 - \alpha mrZ + \cancel{\frac{3}{8}\alpha^2 m^2 r^2 Z^2} - \frac{Z^2\alpha^2 \log(\alpha mrZ)}{2} + \frac{Z^3\alpha^3 mr \log(\alpha mrZ)}{2} + \cdots & (n=2) \\ 1 - \alpha mrZ + \cancel{\frac{19}{54}\alpha^2 m^2 r^2 Z^2} - \frac{Z^2\alpha^2 \log(\alpha mrZ)}{2} + \frac{Z^3\alpha^3 mr \log(\alpha mrZ)}{2} + \cdots & (n=3) \\ 1 - \alpha mrZ + \cancel{\frac{11}{32}\alpha^2 m^2 r^2 Z^2} - \frac{Z^2\alpha^2 \log(\alpha mrZ)}{2} + \frac{Z^3\alpha^3 mr \log(\alpha mrZ)}{2} + \cdots & (n=4) \end{cases}$$



Explicit forms of Schrodinger, KG and Dirac radial wave functions for S-wave hydrogen



Why various w.f. exhibit universal short-distance behavior for a given orbital angular momentum l? What is the physics behind the divergence, and the universality?

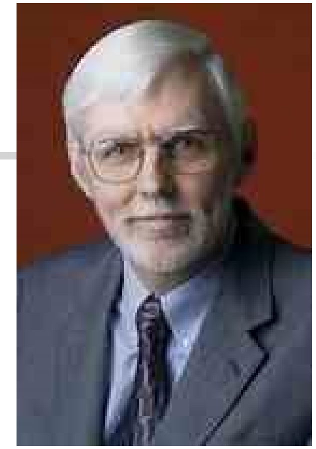
Thanks to X. D. Ji for very inspiring discussions that lead to this study

One afternoon in fall of 2011 at Maryland, Xiangdong informed me that the wave function near the origin in Dirac equation logarithmically diverges… And likely to be related with renormalization effect…

It takes me for seven years to finally figure out how to solve this puzzle

Peter Lepage's pedagogical 1997 Summer School lecture: **How to renormalize Schrödinger equation**

-- Lots of inspiration from Lepage's 1997 Summer School lecture



HOW TO RENORMALIZE THE SCHRÖDINGER EQUATION

Lectures at the VIII Jorge André Swieca Summer School (Brazil, Feb. 1997)

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These lectures illustrate the key ideas of modern renormalization theory and effective field theories in the context of simple nonrelativistic quantum mechanics and the Schrödinger equation. They also discuss problems in QED, QCD and nuclear physics for which rigorous potential models can be derived using renormalization techniques. They end with an analysis of nucleon-nucleon scattering based effective theory.

1 Renormalization Revisited

These lectures are about effective field theories — low-energy approximations to arbitrary high-energy physics — and therefore they are about modern renormalization theory.[1]

Despite the complexity of most textbook accounts, renormalization is based upon a very familiar and simple idea: a probe of wavelength λ is insensitive to details of structure at distances much smaller than λ. This means that we can mimic the *real* short-distance structure of the target and probe by *simple* short-distance structure. For example, a complicated current source $\mathbf{J}(\mathbf{r},t)$ of size d that generates radiation with wavelengths $\lambda \gg d$ is accurately mimicked by a sum of point-like multipole currents ($E1$, $M1$, etc). In thinking about the long-wavelength radiation it is generally much easier to treat the source as a sum of multipoles than to deal with the true current directly. This is particularly true since usually only one or two multipoles are needed for sufficient accuracy. The multipole expansion is a simple example of a renormalization analysis.

In a quantum field theory, QED for example, the quantum fluctuations probe arbitrarily short distances. This is evident when one computes radiative corrections in perturbation theory. Ultraviolet divergences, coming from loop momenta $k \to \infty$ (or wavelengths $\lambda \to 0$), result in infinite contributions — radiative corrections seem infinitely sensitive to short distance behavior. Even ignoring the infinities, this poses a serious conceptual problem since we don't really know what happens as $k \to \infty$. For example, there might be new supersymmetric interactions, or superstring properties might become important, or electrons and muons might have internal structure. The situation is saved

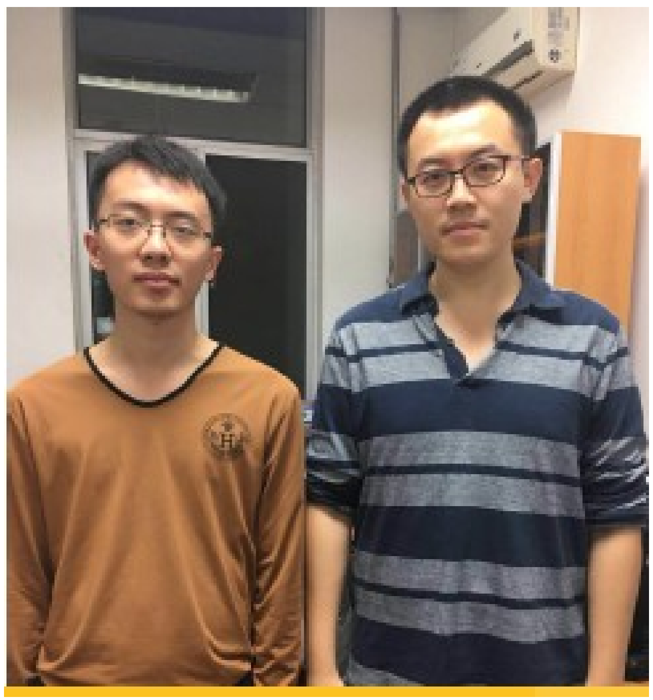
2.6 *Operators and the Operator Product Expansion*

So far we have used our effective theory to compute binding energies and phase shifts. We now examine quantities that depend in detail on the wavefunctions. Consider, for example, the matrix element $\langle n|\mathbf{p}^4|n\rangle$, which might be important if we wished to include relativistic corrections in our potential model. In Table 3 I list values of this matrix element for several S-states both for the true theory, and for our corrected theory (with $a=1$). The values disagree by more than a factor of two, even for very low-energy states, despite the fact that the two theories agree on the corresponding binding energies to several digits.

The problem is that the operator in the effective theory that corresponds to $\mathbf{p}^4$ in the true theory is not $\mathbf{p}^4$. As is true of the hamiltonian, there are local corrections to $\mathbf{p}^4$ in the effective theory. Thus, for any S state, we expect

$$\langle \mathbf{p}^4\rangle_{\rm true} = Z\,\langle \mathbf{p}^4\rangle_{\rm eff} + \frac{\gamma}{a}\,\langle \delta^3_a(\mathbf{r})\rangle_{\rm eff} + \eta\, a\,\langle \nabla^2\delta^3_a(\mathbf{r})\rangle_{\rm eff} + \mathcal{O}(a^3) \qquad (15)$$

Lots of efforts by two of my students in the past three years



山东大学

学 士 学 位 论 文

论文题目：薛定谔方程的有效理论和重整化研究

作 者 姓 名 **黄应生**

专 业 **物理学**

指 导 教 师 **贾宇，李世渊**

2016 Y.-S. Huang Bachelor thesis

Part 1: Schrödinger wave function

EFT and OPE

Coalescence behavior of atomic wave function

- electron-electron coalescence (Kato, 1957; Hofmann, et al., 2013)
- electron-nucleus coalescence (Löwdin, 1954; Kato, 1957; Hofmann, et al., 2013)
- two-electron and nucleus coalescence (Fournais, et al., 2005)
- molecule coalescence (Kolos, 1960; Pack, 1966, . . .)
- more. . .
- a general N-fermion coalescence analysis (Hoffmann-Ostenhof, et al., 1992)

Electron-electron coalescence

- Hamiltonian:

$$H_{\text{Coul}} = -\sum_{i=1}^{N} \frac{\nabla_i^2}{2m} - \sum_{i=1}^{N} \frac{Z\alpha}{r_i} + \sum_{j>i=1}^{N} \frac{\alpha}{r_{ij}}$$

- Kato's Cusp condition (S-wave) (Kato, 1957):

$$\left.\frac{\partial \Psi}{\partial r_{12}}\right|_{r_{12}=0} = \gamma\, \Psi(r_{12}=0)$$

- Leads to:

$$\psi = 1 - \sum_{l=1}^{N} \sum_{i=1}^{n} mZ\alpha r_{il} + \sum_{i>j=1}^{n} \frac{m\alpha r_{ij}}{2} + \mathcal{O}(r^2)$$

Beyond Kato's cusp condition

- **Arbitrary orbital angular momentum** (Löwdin, 1954):

$$R_{nl}(x) = \frac{x^l}{l!}\frac{d^l R_{nl}}{dx^l}(0)\left[1 - \frac{1}{l+1}\frac{x}{a_0} + \mathcal{O}\left(x/a_0\right)^2\right].$$

- **Three-particle coalescence**: (Fournais et al. 2005):

$$F = -\sum_{l=1}^{N}\sum_{i=1}^{n} mZ\alpha r_{il} + \sum_{i>j=1}^{n}\frac{m\alpha r_{ij}}{2} + \frac{2-\pi}{6\pi}\sum_{l=1}^{N}\sum_{i>j=1}^{n} m^2 Z\alpha^2 \mathbf{r}_{il}\cdot\mathbf{r}_{jl}\log\left(m^2(r_{il}^2 + r_{jl}^2)\right)$$

$$\psi = e^F\phi$$

Two electrons approach the nucleus for an arbitrary atom

How to understand this coalescence behavior of wave function from QFT?

In the field-theoretical context, the Bethe-Salpeter wave function can be viewed as the vacuum-to-atom matrix elements of two nonlocal field operator!

$$\Psi_{nlm}(\mathbf{x}) = \langle 0 \,|\psi(\mathbf{x}) N(\mathbf{0})|\, nlm\rangle \,,$$

This suggests that the coalescence behavior can be inferred from OPE: Wilson coefficients are universal!

OPE is operator relation, does not depend on external states, thus applies to an arbitrary atom!

Principle of EFT

- Identifying relevant degree of freedom
- Symmetry as building guidance
- Power counting
- Long-range effects is insensitive about short range physics;
- Short-range effects encoded by Wilson coefficients
- Nonrenormalizable theory is renormalizable

有效理论的基本 idea:

IR　　　　insensitive to UV

- 低能(长程)物理不依赖高能(短程)物理的细节;

　　　　　　　　　　　　(UV)

- 聚焦于感兴趣的能标的物理自由度，短程物理用一些参数 + 刻划。

　　local operators　　underlying physics: uncertainty principle

EFT 的 essential ingredients:

1. Scale separation: 确定系统的UV能标和IR能标

2. Identify active (effective) degrees of freedom:

确定理论需保留的重要的、有效的自由度

3. Impose a UV cutoff; Λ

所有有效理论都有一个适用范围.(不再追求 Theory of Everyth

4. Specify the symmetries of EFT.

利用对称性构造有效理论所有可能的相互作用

5. Dimensional Analysis : Power counting.

如何组织计算,使之满足小参数展开的指定精度.

$(p/\Lambda)^n$

Don't fool yourself!

EFT 成为了物理学研究的基本范式 (paradigm).

EFT 提供了对 renormalization 最全面的理解.

- No theory can work at all scales, even string theory!
- Standard Model must break down long before hitting Planck scale.
- 即使我们知道了 underlying theory, 但 EFT (可能是唯一的) 提供了更加有效的描述手段.

例子1: QCD是所有量子化学和condensed matter物理的underlying theory.

但基于库仑势的薛定谔方程对quantum chemistry and atomic physics更加有效!

phonon

基于电子-声子相互作用的多体理论对理解superconductivity更有效!

例子2: QCD是描述强相互作用的基本理论

(ChPT)

但Chiral perturbation theory提供了理解low-energy hadronic physics更加有效的理论框架 [以π, N作为有效自由度]

Nonrenormalizability is no longer an issue!

Nonrenormalizable theories are renormalizable/predictive. ③

在EFT的框架下，renormalization意味着，即使我们不理解UV ($k>\Lambda$) 的物理，我们依然可以做出model-independent预言，因为the effects of short-distance physics can be localizable in terms of Wilson coefficients in EFTs.

Wilson系数可以 $\begin{cases} \text{从实验值fit.} & \text{ChPT/SMEFT} \\ \text{可从}\underset{(\text{已知})}{\underline{\text{UV理论}}}\text{推导} & \text{HQET/NRQED/NRQCD/SCET} \end{cases}$

H. Georgi (1989): In this picture, the presence of infinities in quantum field theory is neither a disaster, nor an asset. It is simply a reminder of practical limitation — we do not know what happens at distances much smaller than those we can look at directly.

总结: All QFTs are EFTs!

Related reasoning：理解氢原子能级，不需知道质子的夸克胶子结构，只需知道质子的一些 bulk properties：

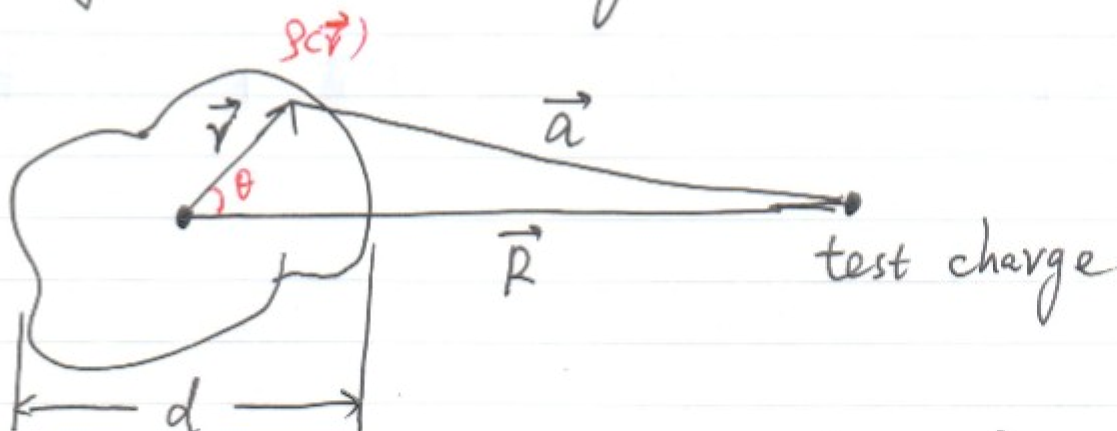
mass, charge, spin-$\frac{1}{2}$, $\mu_p = 2.793\,\mu_B$ (反常磁矩), … ④

EFT example 1. [最好的学习是通过 examples]

静电势的多极矩展开 (multipole expansion)

(even not a QFT)

localized electric charge distribution, with density $\rho(\vec{r})$



源之 size 为 d

scale hierarchy:

$d \ll R$

$$V=\int d^3\vec{r}\ \frac{\rho(\vec{r})}{|\vec{a}|}=\int d^3r\ \frac{\rho(\vec{r})}{\sqrt{R^2+2Rr\cos\theta+r^2}}$$

非常复杂!

但利用 $|\vec{r}|\sim d\ll R$，做 Taylor expansion in $\frac{r}{R}$.

$$V\approx\sum_{n=0}^{\infty}\frac{1}{R^{n+1}}\int d^3r\ r^n\rho(\vec{r})\,P_n(\cos\theta)$$

$$=\frac{q}{R}+\frac{p}{R^2}+\frac{Q}{R^3}+\cdots$$

mono pole/charge　dipole　quadrupole

- 级数收敛很快
- Long-distance physics only sensitive to bulk properties: $q, p, Q, \cdots$

EFT example 2: 氢原子—量子力学预言

Underlying theory: Dirac equation (in Coulomb potential)

4分量

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\overset{=1}{\cancel{Z}}\alpha}{r}\right)\psi_{Dirac} = -i\hbar c\,\nabla\cdot\vec{\alpha}\,\psi_{Dirac} + mc^2\beta\,\psi_{Dirac}$$

量子化的束缚态能级为: Darwin, Gordon (1928)

$$E_{n,j} = \frac{mc^2}{\sqrt{1 + \frac{\alpha^2}{\left(n - j - \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 - \alpha^2}\right)^2}}}$$

$$\approx mc^2\left[1-\frac{\alpha^2}{2n^2}-\frac{\alpha^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right)+\cdots\right]$$

Bohr 能级 fine-structure

cutoff: $\Lambda \simeq m_e$

有效理论： Schrödinger equation： valid at $p < m_e$

$$\boxed{\begin{aligned} & i\hbar\frac{\partial}{\partial t}\psi_{sch} = H\psi_{sch}, \\ & \psi_{sch} = \psi_{space}\otimes\underbrace{\psi_{spin}}_{=\text{分量}} \end{aligned}}$$

$$H = H_0 + \Delta H$$ → taken as perturbation, recover

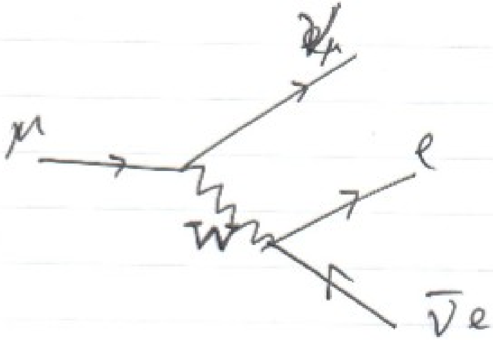
where $H_0 = \frac{\vec{p}^2}{2m_e} - \frac{\alpha}{r}$ Coulomb hamiltonian

$$\Delta H = \frac{\vec{p}^4}{8m_e^3c^2} - \frac{e\hbar}{4m_e^2c^2}\vec{\sigma}\cdot\vec{E}\times\vec{p} - \frac{e\hbar^2}{8m_e^2c^2}\nabla\cdot\vec{E}$$

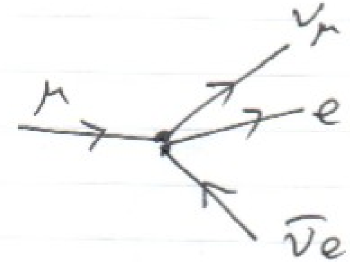
vel. corr. spin-orbital Darwin

$$\Delta H = -\frac{\vec{p}^4}{8m_e^3c^2} - \frac{\alpha}{2r^3m_e^2}\vec{L}\cdot\vec{S} + \frac{\pi\alpha}{2m_e^2c^2}\delta^3(\vec{r})$$

EFT Example 3: Weak interaction: 4-fermion interaction.



$\xrightarrow{\quad}$ contract W propagator to a point



$P_L = \frac{1-\gamma_5}{2}$

$$\mathcal{M}(\mu\to e\,\nu_\mu\,\bar\nu_e) = \left(\frac{g_2}{\sqrt{2}}\right)^2 \left[\bar u(p_{\nu_\mu})\,\gamma^\mu P_L\, u(p_\mu)\right]\left[\bar u(p_e)\,\gamma_\mu P_L\, v(p_{\bar\nu_e})\right]$$

$$\times \frac{1}{(p_\mu - p_{\nu_\mu})^2 - M_W^2}.$$

因为 $\boxed{p_\mu^2 = m_\mu^2 \ll M_W^2,}$ W boson is highly virtual, can not propagate far,

$$\frac{1}{(p_\mu - p_{\nu_\mu})^2 - M_W^2} \approx -\frac{1}{M_W^2}.$$

$$\mathcal{M}(\mu \to e\, \nu_\mu \bar{\nu}_e) \simeq -\frac{4G_F}{\sqrt{2}} \left[\bar{u}(p_{\nu_\mu}) \gamma^\mu P_L u(p_\mu)\right] \left[\bar{u}(p_e) \gamma_\mu P_L v(p_{\bar{\nu}_e})\right]$$

Fermi constant $\dfrac{G_F}{\sqrt{2}} = \dfrac{g_2^2}{8M_W^2}$.

Effective weak Hamiltonian :

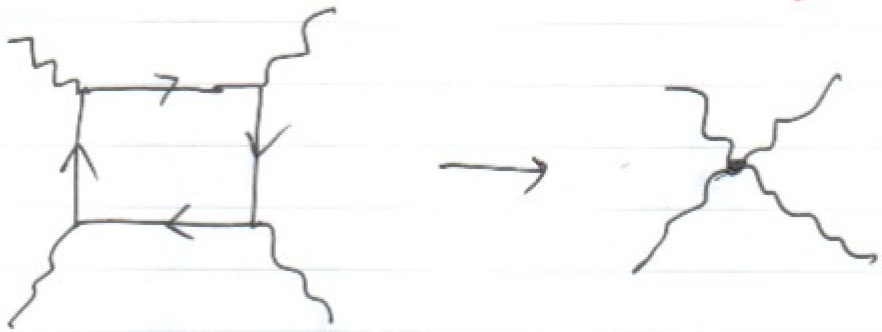
$$\mathcal{H}_W = -\mathcal{L}_W = \frac{4G_F}{\sqrt{2}} \left[\bar{\nu}_\mu \gamma^\mu P_L \mu\right] \left[\bar{e}\, \gamma_\mu P_L \nu_e\right].$$

EFT example 4: Light-by-light scattering

$\gamma\gamma \to \gamma\gamma$

光子能量为 E_γ

假设 $\boxed{E_\gamma \ll m_e}$



underlying theory

我们可以从QED作用量中积掉电子场.

(Euler, Heisenberg, Kockel 1936)

$$\mathcal{L}_{QED}(\psi, \bar{\psi}, A_\mu) \longrightarrow \mathcal{L}_{EH}(A_\mu)$$

单圈图 matching 得到

$$\mathcal{L}_{EH} = \frac{1}{2}(\vec{E}^2-\vec{B}^2) + \frac{e^4\hbar}{360\pi^2 m_e^4 c^7}\left[(\vec{E}^2-\vec{B}^2)^2 + 7(\vec{E}\cdot\vec{B})^2\right] + \cdots$$

Gauge inv. explicit: $F_{\mu\nu}F^{\mu\nu} \sim \vec{E}^2-\vec{B}^2$, $F_{\mu\nu}\tilde{F}^{\mu\nu} \sim \vec{E}\cdot\vec{B}$

expansion parameter: $\left(\frac{E_\gamma}{m_e}\right)^{2n}$

Cross section $\sigma[\gamma\gamma\to\gamma\gamma]\Big|_{E_\gamma \ll m_e} = \frac{1}{16\pi^2}\frac{973}{10125\pi}\frac{e^8}{m_e^2}\left(\frac{E_\gamma}{m_e}\right)^6$.

$$\propto \frac{\alpha^4 E_\gamma^6}{m_e^8}.$$

Seen in heavy-ion collisions ATLAS, Nature Phys 13 (2017) 852

EFT example 5: GR as low-energy EFT of Quantum Gravity

[Donoghue 1995]

广义相对论可以从 Einstein-Hilbert action 推出

Ricci 曲率标量

$$S_{EH} = \int d^4x \sqrt{-g}\left(-\frac{2}{\kappa^2}R\right) \qquad \kappa = \sqrt{32\pi G_N}$$

牛顿万有引力常数

$$S = S_{EH} + S_{matter}$$

$$\delta S = 0 \Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$$

目前我们并不知道自洽的量子引力 (Nobody knows what happens at Planck scale)

但可以构造 low energy EFT of quantum gravity:

宇宙学常数

$$S_{eff} = \int d^4x \sqrt{-g}\left[-\Lambda - \frac{2}{\kappa^2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + \dots\right]$$

Allowed by general coordinate transformation symmetry.

Similarly, write down the most general interaction terms allowed by symmetry for matter sector

One loop corrections to Newton's law:

 代表 graviton

post-Newtonian

quantum correction

$$V(r) = -\frac{G_N m_1 m_2}{r}\left[1 + \frac{3G_N(m_1+m_2)}{rc^2} + \frac{41}{10\pi}\frac{G_N \hbar}{r^2c^3}\right].$$

$$+\ C_1 G_N \delta^{(3)}(\vec{r}).$$

non-analytic

EFT example 6: (量纲分析)

Rayleigh scattering: Why the sky is blue ?

可见光和大气中的原子散射，涉及多个length scales.

1) 可见光波长 $\lambda \sim 5000\,\mathring{A} \gg$ 原子的size ~ 几个 $\mathring{A}$

因此原子可以作为点粒子处理；

2) 可见光子能量 << 原子的激发能，因此只需考虑光子和中性原子的弹性散射.

3) 原子几乎是静止的，可以用非相论性的场 $\psi(x)$ 来描述

more details later!

$$\mathcal{L}_{eff} = \psi^{+}\left(i\partial_t + \cancel{\frac{\nabla^2}{2M}}\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{int}$$

原子不带电荷，协变导数 → 普通导数

U(1) Gauge invariance demands $F_{\mu\nu} = (\vec{E}, \vec{B})$

$$\mathcal{L}_{int} = a_0^3\, \psi^\dagger \psi \left(c_1 \vec{E}^2 + c_2 \vec{B}^2 \right) + \cdots$$

Note $[\psi_{NR}] = \frac{3}{2}$, $[\mathcal{L}] = 4$ $\quad a_0$: size of atom.

$(\vec{E} \sim \dot{\vec{A}},\ \vec{B} \sim \nabla \times \vec{A})$

$$\sigma_{\text{Rayleigh}} \propto a_0^6\, \underline{E_\gamma^4}$$

Simple dimensional analysis reproduces the famous E_γ^4 dependence of the Rayleigh scattering X section, thus explains why sky is blue! Shows strength of EFT

EFT for atoms (analogue of heavy-quark bound states)

- Effective Lagrangian: NRQED (Caswell & Lepage, 1986)+HNET (similar to HQET, E. Eichten, Hill & H. Georgi, 1990)

$$\mathcal{L} = \mathcal{L}_{\mathrm{Max}} + \mathcal{L}_{\mathrm{NRQED}} + \mathcal{L}_{\mathrm{HNET}} + \delta\mathcal{L}_{\mathrm{contact}}$$

$$\mathcal{L}_{\mathrm{Max}} = -\frac{1}{4} d_\gamma F_{\mu\nu} F^{\mu\nu} + \cdots ,$$

$$\mathcal{L}_{\mathrm{NRQED}} = \psi^\dagger \left\{ iD_0 + \frac{\mathbf{D}^2}{2m} + \frac{\mathbf{D}^4}{8m^3} + c_D e \frac{[\boldsymbol{\nabla} \cdot \mathbf{E}]}{8m^2} + c_F e \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{2m} + i c_S e \frac{(\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D}) \cdot \boldsymbol{\sigma}}{8m^2} + \cdots \right\} \psi ,$$

$$\mathcal{L}_{\mathrm{HNET}} = N^\dagger i D_0 N + \cdots ,$$

$$\delta\mathcal{L}_{\mathrm{contact}} = \frac{c_4}{m^2} \psi^\dagger \psi N^\dagger N + \cdots$$

where $D^\mu = \partial^\mu + ieA^\mu .$

EFT for Schrödinger atoms

- **Coulomb-Schrödinger EFT** for atoms:

$$\mathcal{L}_{\mathrm{Coul-Schr}} = \psi^\dagger \left\{ iD_0 + \frac{\nabla^2}{2m} \right\} \psi + N^\dagger iD_0 N + \frac{1}{2} \left(\nabla A^0 \right)^2 .$$

Field theoretical realization of Schrodinger eq.

$$H_{\mathrm{Coul}} = -\sum_{i=1}^{N} \frac{\nabla_i^2}{2m} - \sum_{i=1}^{N} \frac{Z\alpha}{r_i} + \sum_{j>i=1}^{N} \frac{\alpha}{r_{ij}} \qquad H_{\mathrm{Coul}} \Psi = E\Psi ,$$

- Coulomb gauge (only retain instantaneous Coulomb potential)
- No dynamic photons (set **A**=0): so will not see Lamb shift
- No relativistic corrections included

Feynman rules in NREFT

The nucleus propagator is defined as

$$D_N(k) \equiv \frac{i}{k^0 + i\epsilon}.$$

Nucleus HNET propagator

NRQED electron propagator

$$D_e(k) \equiv \frac{i}{k^0 - \frac{\mathbf{k}^2}{2m} + i\epsilon}.$$

The photon propagator in Coulomb gauge is defined as

$$D_{00}(k) \equiv \frac{i}{\mathbf{k}^2},$$

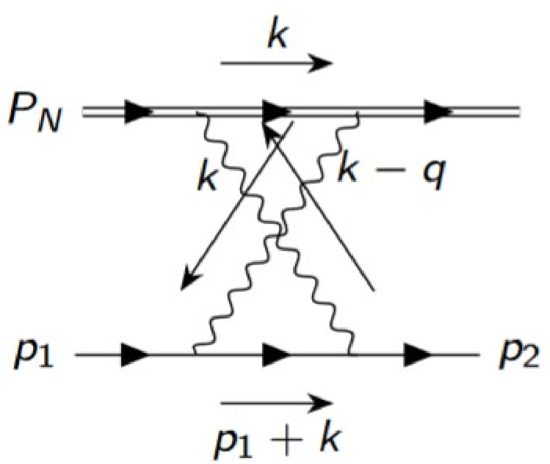
Instantaneous Coulomb photon

$$D_{ij}(k) \equiv \frac{i}{k^2 + i\epsilon}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right), \qquad i \neq 0, j \neq 0 \longrightarrow$$

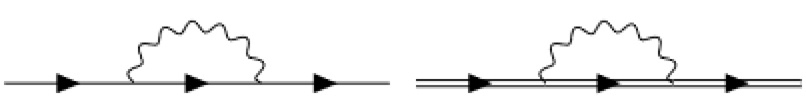
Not needed in this work!

Only ladder diagrams survives in NREFT calculation

- All crossed ladder diagrams are zero due to the co ntour integral.



$$\propto \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{\mathbf{k}^2} \frac{i}{|\mathbf{k}-\mathbf{q}|^2} \frac{i}{p_1^0 + k^0 - \frac{|\mathbf{k}+\mathbf{p}_1|^2}{2m} + i\epsilon} \frac{i}{k^0 + i\epsilon}$$



Vanish with single pole!

Operator Product Expansion (Wilson, 1969; Zimmermann, 1971)

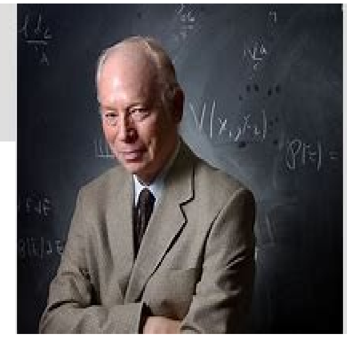
OPE assumes the following form:

$$T\phi(x)\phi(0) \sim \sum_{\mathcal{O}} C_{\mathcal{O}}(x^{\mu})[\mathcal{O}(0)]_R$$

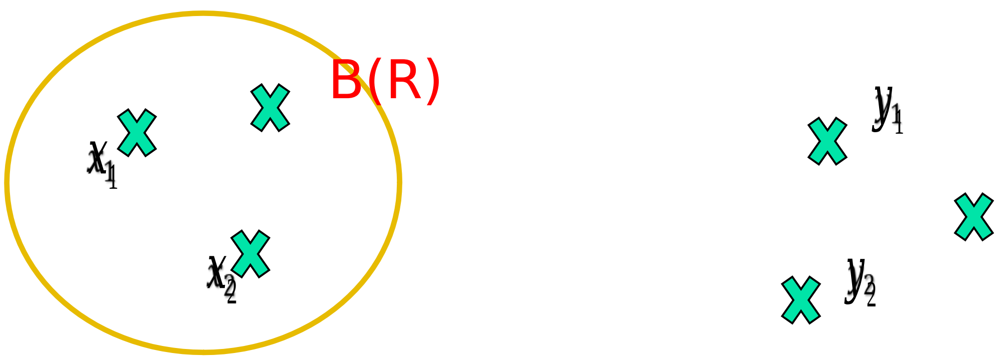
- Applied in various areas of particle physics (light-cone expansion, Minkowski spacetime, and Euclidean OPE)

Can be used to defined the renormalized composite local operators.

Proof of OPE using path integrals (see Weinberg, QFT Vol. 2)



$$\langle T\{A_1(x_1),\, A_2(x_2),\, \ldots B_1(y_1),\, B_2(y_2)\, \ldots\}\rangle_0$$
$$= \int \left[\prod_{\ell,\, z} d\phi_\ell(z)\right]\; a_1(x_1)\, a_2(x_2) \cdots b_1(y_1)\, b_2(y_2) \cdots \exp(iI\,[\phi])\,,$$



$$I = \int_{z\in B(R)} d^4z\; \mathscr{L}(z) + \int_{z\notin B(R)} d^4z\; \mathscr{L}(z)\,.$$

Locality of action is crucial for existence of OPE

Proof of OPE (QFT, Weinberg)

$$\langle T\{A_1(x_1),\, A_2(x_2),\, \ldots B_1(y_1),\, B_2(y_2)\, \ldots\}\rangle_0$$

$$= \int \left[\prod_{z\notin B(R),\,\ell} d\phi_\ell(z)\right] \; b_1(y_1)\, b_2(y_2) \ldots \exp\left(i\int_{z\notin B(R)} \mathscr{L}(z)\right)$$

$$\times \int \left[\prod_{z\in B(R),\,\ell} d\phi_\ell(z)\right] \; a_1(x_1)\, a_2(x_2) \ldots \exp\left(i\int_{z\in B(R)} \mathscr{L}(z)\right)$$

$$\langle T\{A_1(x_1),\, A_2(x_2),\, \ldots B_1(y_1),\, B_2(y_2)\, \ldots\}\rangle_0 \to \int \left[\prod_{\ell,\,z} d\phi_\ell(z)\right]$$

$$\times\, b_1(y_1)\, b_2(y_2) \ldots \; \exp\left(i\int \mathscr{L}(z)\right)$$

$$\times \sum_O U_O^{A_1,A_2,\ldots}(x_1-x,\, x_2-x,\, \ldots)\;\; o(x)$$

$$= \sum_O U_O^{A_1,A_2,\ldots}(x_1-x,\, x_2-x,\, \ldots)\, \langle T\{O(x),\, B_1(y_1),\, B_2(y_2)\, \ldots\}\rangle_0$$

OPE: an important tool in high energy physics

DIS:

$$T[J^\mu(x)J^\nu(0)] = \sum \mathcal{C}_i\left(x^2,\mu^2\right) x_{\mu_1}\dots x_{\mu_n} \mathcal{O}_i^{\mu\nu\mu_1\dots\mu_n}(\mu),$$

Twist expansion

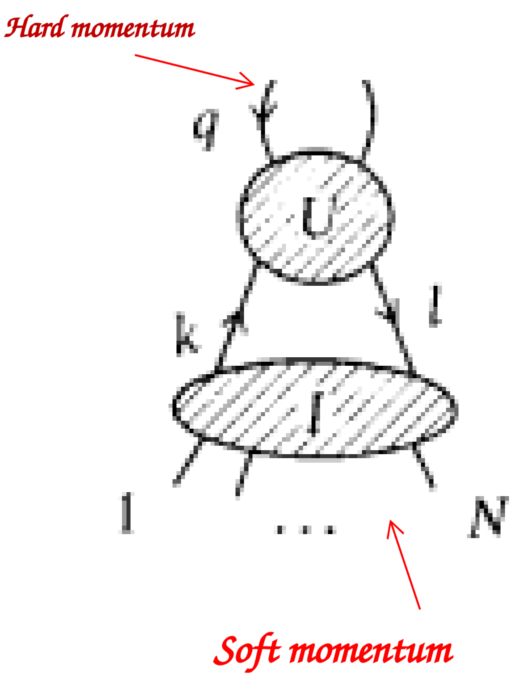
$$O_{q,V}^{\mu_1\cdots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} \mathcal{S}\left\{\bar{q}\,\gamma^{\mu_1}\,\overset{\leftrightarrow}{D}^{\mu_2}\cdots\overset{\leftrightarrow}{D}^{\mu_n}\,q\right\},$$

$$O_{q,A}^{\mu_1\cdots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} \mathcal{S}\left\{\bar{q}\,\gamma^{\mu_1}\,\overset{\leftrightarrow}{D}^{\mu_2}\cdots\overset{\leftrightarrow}{D}^{\mu_n}\,\gamma_5 q\right\},$$

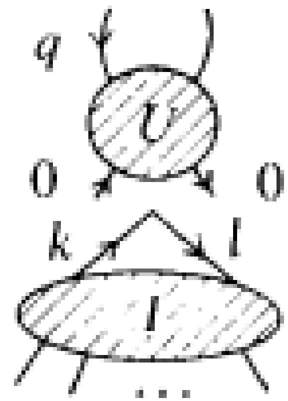
QCD (SVZ) Sum rule Shifman et al., 1978

OPE + dispersion relation: useful phenomenological model to predict some hadronic nonperturbative quantities.

The essential idea of OPE is factorization: momentum flow of Green function: separate hard and soft (lucidly explained by John Collins, text on Renormalization)



Factorization property of Green function



Consider a ϕ^4 theory

$$\mathscr{L} = \partial\phi^2/2 - m^2\phi^2/2 - g\phi^4/24 + \text{counterterms.}$$

we may have OPE

A toy example!

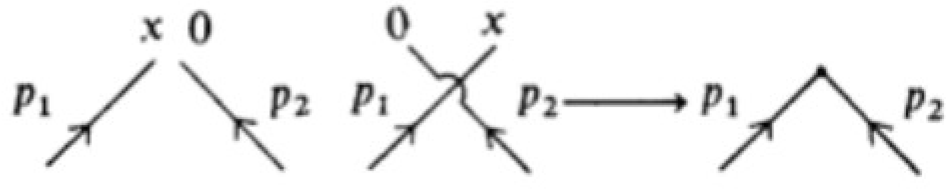
$$T\phi(x)\phi(0) \sim \sum_{\mathcal{O}} C_{\mathcal{O}}(x^\mu)[\mathcal{O}(0)]$$

To derive the exact Wilson coefficient $C_{\mathcal{O}}$, we consider Green function

$$\langle 0|T\phi(x)\phi(0)\tilde{\phi}(p_1)\tilde{\phi}(p_2)|0\rangle.$$

At leading order we have disconnected diagrams

OPE can be obtained from asymptotic behavior of Green function when large momentum injected ✉ factorized form



(a) (b)

which gives

$$\frac{i}{p_1^2 - m^2}\frac{i}{p_2^2 - m^2}[\exp(-ip_1\cdot x) + \exp(-ip_2\cdot x)].$$

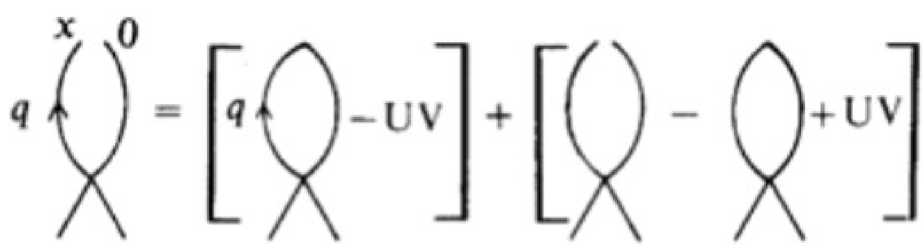
Expanding it at $x = 0$

$$\frac{i}{p_1^2 - m^2}\frac{i}{p_2^2 - m^2}[2 - i(p_1 + p_2)\cdot x - (p_1\cdot x^2 + p_2\cdot x^2)/2 + \cdots].$$

which is equivalent to

$$T\phi(x)\phi(0) = \phi^2(0) + \tfrac{1}{2}x^\mu\partial_\mu\phi^2 + \tfrac{1}{2}x^\mu x^\nu\phi\partial_\mu\partial_\nu\phi + \cdots$$

However this is only the leading contribution. For next-to-leading contribution, we have the following 1-loop diagram



(a) (b) (c)

q soft; local

q hard; Wilson coef.

where the diagram l.h.s is

$$\frac{i^2}{(p_1^2 - m^2)(p_2^2 - m^2)} ig \int \frac{d^4 q}{(2\pi)^4} \frac{e^{-iq\cdot x}}{(q^2 - m^2)[(q - p_1 - p_2)^2 - m^2]}.$$

The Wilson coefficient we actually care for is

$$\left.\begin{aligned} T\phi(x)\phi(0) &\sim C_{\phi^2}(x)[\phi^2], \\ C_{\phi^2} &= 1 + (g/16\pi^2)c_1(x^2). \end{aligned}\right\}$$

which is extracted via 1-loop diagram (c)

$$\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)} \frac{-ig}{(2\pi)^4} \times$$
$$\times \left\{ \int d^4 q \frac{e^{iq\cdot x} - 1}{(q^2 - m^2)[(q - p_1 - p_2)^2 - m^2]} - \text{UV divergence} \right\}.$$

The contribution of order 1 from large q is

$$c_1(x) = \frac{1}{2\pi^2} \left\{ \frac{i}{(2\pi\mu)^{d-4}} \int d^d q \frac{(e^{iq\cdot x} - 1)}{(q^2)^2} + \frac{2}{d-4} \right\}.$$

while the rest are of order $|x|$.

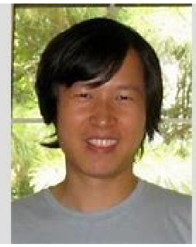
Using Schwinger parametrization

$$1/(q^2)^2 = \int_0^\infty \mathrm{d}z\, z\, \mathrm{e}^{-z(-q^2)}$$

one could determine c_1 to be

$$c_1(x) = \tfrac{1}{2}[\gamma + \ln(-\pi^2\mu^2x^2)].$$

Previous application of OPE to atomic physics



Braaten & Platter, PRL (2008)

To reproduce the famous Tan relation in cold atoms

PRL **100,** 205301 (2008) PHYSICAL REVIEW LETTERS week ending 23 MAY 2008

Exact Relations for a Strongly Interacting Fermi Gas from the Operator Product Expansion

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(Received 12 March 2008; published 21 May 2008)

The momentum distribution in a Fermi gas with two spin states and a large scattering length has a tail that falls off like $1/k^4$ at large momentum k, as pointed out by Tan. He used novel methods to derive exact relations between the coefficient of the tail in the momentum distribution and various other properties of the system. We present simple derivations of these relations using the operator product expansion for quantum fields. We identify the coefficient as the integral over space of the expectation value of a local operator that measures the density of pairs.

DOI: 10.1103/PhysRevLett.100.205301 PACS numbers: 67.85.Lm, 03.75.Nt, 31.15.−p, 34.50.−s

OPE for electron Coulomb gas

Hofmann, Barth and Zwerger in 2013

PHYSICAL REVIEW B **87**, 235125 (2013)

Short-distance properties of Coulomb systems

Johannes Hofmann,[1,*] Marcus Barth,[2,†] and Wilhelm Zwerger[2]
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(Received 15 April 2013; revised manuscript received 27 May 2013; published 20 June 2013)

We use the operator product expansion to derive exact results for the momentum distribution and the static structure factor at high momentum for a jellium model of electrons in both two and three dimensions. It is shown that independent of the precise state of the Coulomb system and for arbitrary temperatures, the asymptotic behavior is a power law in the momentum, whose strength is determined by the contact value of the pair distribution function $g(0)$. The power-law tails are quantum effects which vanish in the classical limit $\hbar \to 0$. A leading-order virial expansion shows that the classical and the high-temperature limit do not agree.

DOI: 10.1103/PhysRevB.87.235125 PACS number(s): 71.10.Ca, 05.30.Fk, 31.15.−p

Why starting from HNET+NRQED?

- Necessary for manifesting the OPE operator relation!!!

- With external Coulomb potential, it is impo ssible to write down the OPE.

- Nucleus infinitely heavy. Electron moves sl owly. Still local QFT

OPE relation for coalescence

- Naïve Taylor expansion:

$$\psi(\mathbf{x})N(\mathbf{0}) = [\psi N](\mathbf{0}) + \mathbf{x}\cdot[\nabla\psi N](\mathbf{0}) + \cdots$$

This is incomplete!!!

- Correct expansion in coordinate space:

$$\psi(\mathbf{x})N(\mathbf{0}) = (1 - mZ\alpha|\mathbf{x}|)\,[\psi N](\mathbf{0}) + (1 - mZ\alpha|\mathbf{x}|/2)\mathbf{x}\cdot[\nabla\psi N](\mathbf{0}) + \cdots$$

- And momentum space:

$$\widetilde{\psi}(\mathbf{q})N(\mathbf{0}) \equiv \int d^3\mathbf{x}e^{-i\mathbf{q}\cdot\mathbf{x}}\psi(\mathbf{x})N(\mathbf{0})$$
$$= \frac{8\pi Z\alpha m}{\mathbf{q}^4}[\psi N](\mathbf{0}) - \frac{16i\pi Z\alpha m}{\mathbf{q}^6}\boldsymbol{q}\cdot[\nabla\psi N](\mathbf{0}) + \cdots\ .$$

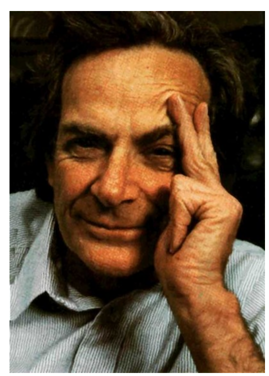
Determine Wilson coefficients

- First define 4-point connected Green functions

$$\Gamma\left(\mathbf{q};\, \mathbf{p}, E \equiv p^0 + k^0\right) \equiv \int d^4y\, d^4z\; e^{-ip\cdot y - ik\cdot z} \langle 0 | T\{\widetilde{\psi}(\mathbf{q}) N(\mathbf{0}) \psi^\dagger(y) N^\dagger(z)\} | 0 \rangle_{\mathrm{amp}},$$

$$\Gamma_S(\mathbf{p}, E) \equiv \int d^4y d^4z\; e^{-ip\cdot y - ik\cdot z} \langle 0 | T\{[\psi N](\mathbf{0}) \psi^\dagger(y) N^\dagger(z)\} | 0 \rangle_{\mathrm{amp}}$$

- There're only ladder diagrams defined as r.h.s.



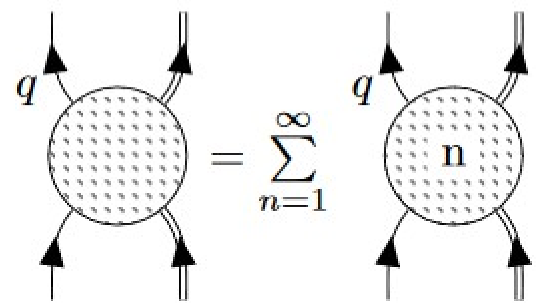


Figure: $\Gamma = \sum_{n=1}^{\infty} \Gamma^{(n)}$.

Momentum space: Asymptotic behavior of 4-point Green function as the injected momentum q->m gets hard

- Leading diagram is th e tree diagram.
- Integrand is expanded given q~m.

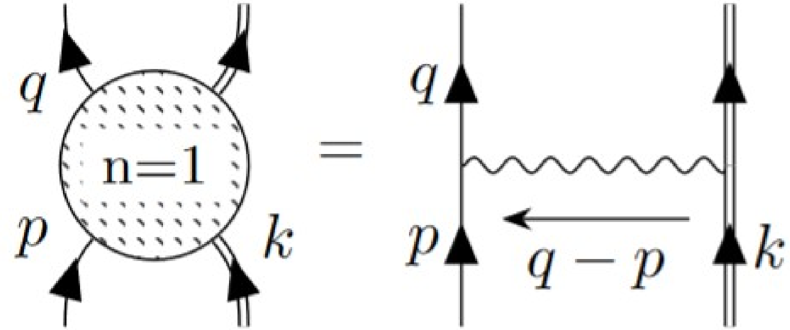


Figure: The tree-level amputated Green function $\Gamma^{(1)}$.

$$\Gamma^{(1)} = Ze^2 \int \frac{dq^0}{2\pi} \frac{i}{q^0 - \frac{\boldsymbol{q}^2}{2m} + i\epsilon} \frac{i}{E - q^0 + i\epsilon} \frac{i}{|\boldsymbol{q} - \boldsymbol{p}|^2}$$

$$= Ze^2(-2m) \frac{i}{\boldsymbol{q}^2 - 2mE - i\epsilon} \frac{i}{|\boldsymbol{q} - \boldsymbol{p}|^2}$$

$$\xrightarrow{\mathbf{q} \to m} \frac{8\pi m Z\alpha}{\mathbf{q}^4} + \cdots , \qquad (20)$$

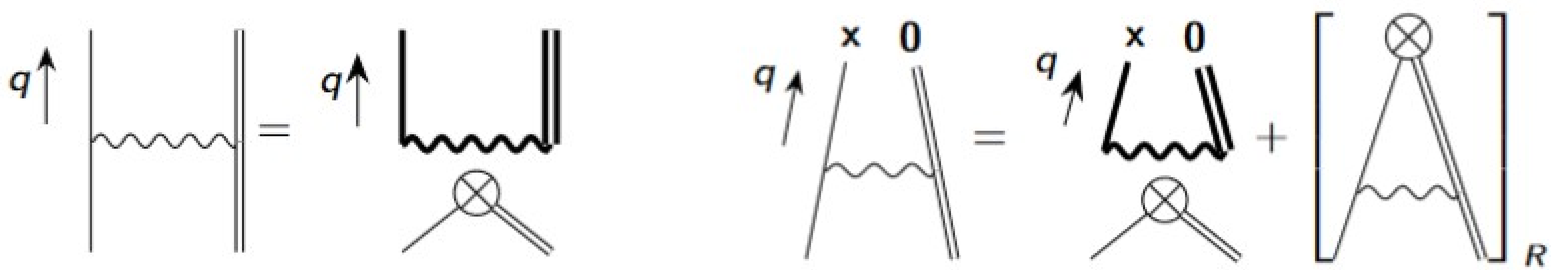
Factorized form

- Define a n-ladder diagram

$$\Gamma^{(n)}(\mathbf{q};\mathbf{p},E)=\sum_{i=1}^{n}\tilde{C}^{(i)}(\mathbf{q})\Gamma_S^{R(n-i)}(\mathbf{p},E)$$

The factorized form is seen.

- Diagrammatically, the 1st order



Proof to all orders by method of induction

- Analyse the loop momentum l_n:
- Hard: q^{-5}; Soft: q^{-4}
- Keep leading region: Soft

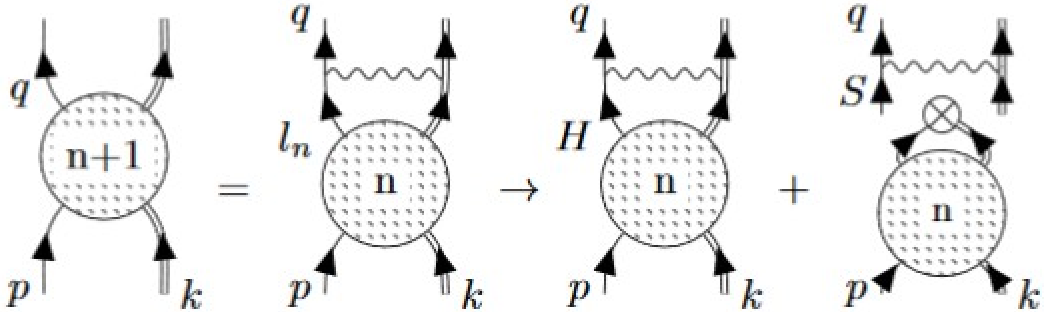


Figure: Reexpressing the four-point Green function $\Gamma^{(n+1)}$ with $n+1$ Coulomb ladders a one loop integral involving the Green function containing n-ladder. H and S indicate whether the loop momentum is hard or soft. The crossed dot marks composite operator $[\psi N]$.

OPE in coordinate space

- Coordinate space Green function (additional diagram abandoned in momentum space as disconnected gives 1)

$$\Gamma_X(\mathbf{x};\mathbf{p},E) \equiv \int d^4y\, d^4z\, e^{-ip\cdot y-ik\cdot z}\langle 0|T\{\psi(\mathbf{x})N(\mathbf{0})\psi^\dagger(y)N^\dagger(z)\}|0\rangle_{\mathrm{amp}}$$

- Given Fourier integral: $\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot\mathbf{x}}-1}{\mathbf{q}^4} = -\frac{1}{8\pi}|\mathbf{x}|$
- Coordinate Wilson coefficient:

Use $\Gamma^{(n)}(\mathbf{q};\mathbf{p},E) = \sum_{i=1}^{n}\tilde{C}^{(i)}(\mathbf{q})\Gamma_S^{R(n-i)}(\mathbf{p},E)$ as n=1, gives:

$$-mZ\alpha|\mathbf{x}|.$$

P-wave hydrogen-like atom

- Similar to S-wave
- The only difference is the local operator $[\nabla\psi N](\mathbf{0})$
- Subtraction to get coordinate Wilson coefficient is different:

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{x}} - i\mathbf{q}\cdot\mathbf{x}}{\mathbf{q}^6}\mathbf{q} = -\frac{i}{32\pi}|\mathbf{x}|\mathbf{x}$$

gives $-\frac{mZ\alpha|\mathbf{x}|\mathbf{x}}{2}$.

Application to hydrogen-like atoms

- Matrix element definition of wave function

$$\Psi_{nlm}(\mathbf{x}) = \langle 0 \,|\psi(\mathbf{x}) N(\mathbf{0})|\, nlm\rangle$$

gives ($a_0 \equiv 1/mZ\alpha.$)

$$R_{n0}(x) = R_{n0}(0)\left[1 - \frac{x}{a_0} + \mathcal{O}\left(x/a_0\right)^2\right]$$

$$\tilde{R}_{n0}(q)(2\pi)^{-3/2} = 2^2 R_{n0}(0)\frac{(2\pi)}{a_0 q^4} + \mathcal{O}\left(\frac{1}{a_0 q}\right)^2$$

$$R_{n1}(x) = x\frac{dR_{n1}}{dx}(0)\left[1 - \frac{1}{2}\frac{x}{a_0} + \mathcal{O}\left(x/a_0\right)^2\right]$$

$$\tilde{R}_{n1}(q)(2\pi)^{-3/2} = 2^3 \frac{dR_{n1}(0)}{dx}\frac{(2\pi)}{a_0 q^5} + \mathcal{O}\left(\frac{1}{a_0 q}\right)^2$$

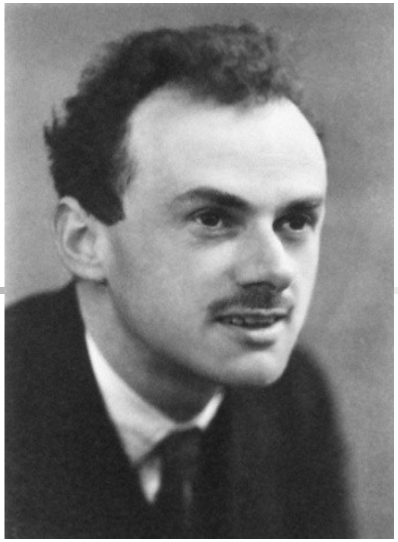
Compare with the old knowledge

- Our results agree with the old QM results:

$$R_{nl}(x) = \frac{x^l}{l!}\frac{d^l R_{nl}}{dx^l}(0)\left[1 - \frac{1}{l+1}\frac{x}{a_0} + \mathcal{O}\left(x/a_0\right)^2\right]$$ (Löwdin, 1954)

$$\tilde{R}_{nl}(q) = 2^{l+2}\frac{dR_{nl}^l(0)}{d^l x}\frac{(2\pi)^{\frac{5}{2}}}{a_0 q^{l+4}} + \mathcal{O}\left(\frac{1}{a_0 q}\right)^2$$ (Bethe & Salpeter, 1957)





Part 2: Klein-Gordon and Dirac

EFT and OPE

Klein-Gordon equation


$$\left[\left(E+\frac{Ze^2}{4\pi r}\right)^2+\hbar^2c^2\nabla^2-m^2c^4\right]\Psi(\mathbf{r})=0. \qquad \Psi_{nlm}(\mathbf{r})=R_{nl}(r)Y_{lm}(\hat{\mathbf{r}}),$$

$$\left[\frac{d^2}{dr^2}+\frac{Z^2\alpha^2-l(l+1)}{r^2}+\frac{2Z\alpha\,E}{\hbar cr}+\frac{E^2-m^2c^4}{\hbar^2c^2}\right]rR_{nl}(r)=0,$$

$$R_{nl}(r)=N_{nl}\rho^{l'}\,e^{-\frac{\rho}{2}}\,{}_1F_1(l'+1-\lambda,2l'+2;\rho),$$

$$l'\equiv-\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-Z^2\alpha^2}, \quad \beta\equiv\frac{2}{\hbar c}\sqrt{m^2c^4-E_{nl}^2}, \quad \rho\equiv\beta r, \quad \lambda=\frac{2Z\alpha E_{nl}}{\hbar c\beta}.$$

Near-the origin behavior of S-wave hydrogen KG wave function

$$R''_{nl} + \frac{2}{r} R'_{nl} + \frac{Z^2\alpha^2 - l(l+1)}{r^2} R_{nl} = 0.$$

Substituting the ansatz $R_{nl}(r) \propto r^{l'}$, one can solve l' from the following quadratic equation:

$$l'(l'+1) = l(l+1) - Z^2\alpha^2,$$

$$l' \equiv -\frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - Z^2\alpha^2},$$

Expanding the wave function near origin

- Expanding $(\rho)^{s_l}$ gives

$$R_{n0}^{\mathrm{KG}}(r) \approx R_{n0}^{\mathrm{Sch}}(0)\rho^{\sqrt{\frac{1}{4}-Z^2\alpha^2}-\frac{1}{2}} \approx R_{n0}^{\mathrm{Sch}}(0)\rho^{-Z^2\alpha^2} = R_{n0}^{\mathrm{Sch}}(0)\left(1 - Z^2\alpha^2 \ln\frac{2r}{na_0} + \cdots\right)$$

- Expanding the rest gives Schrödinger results

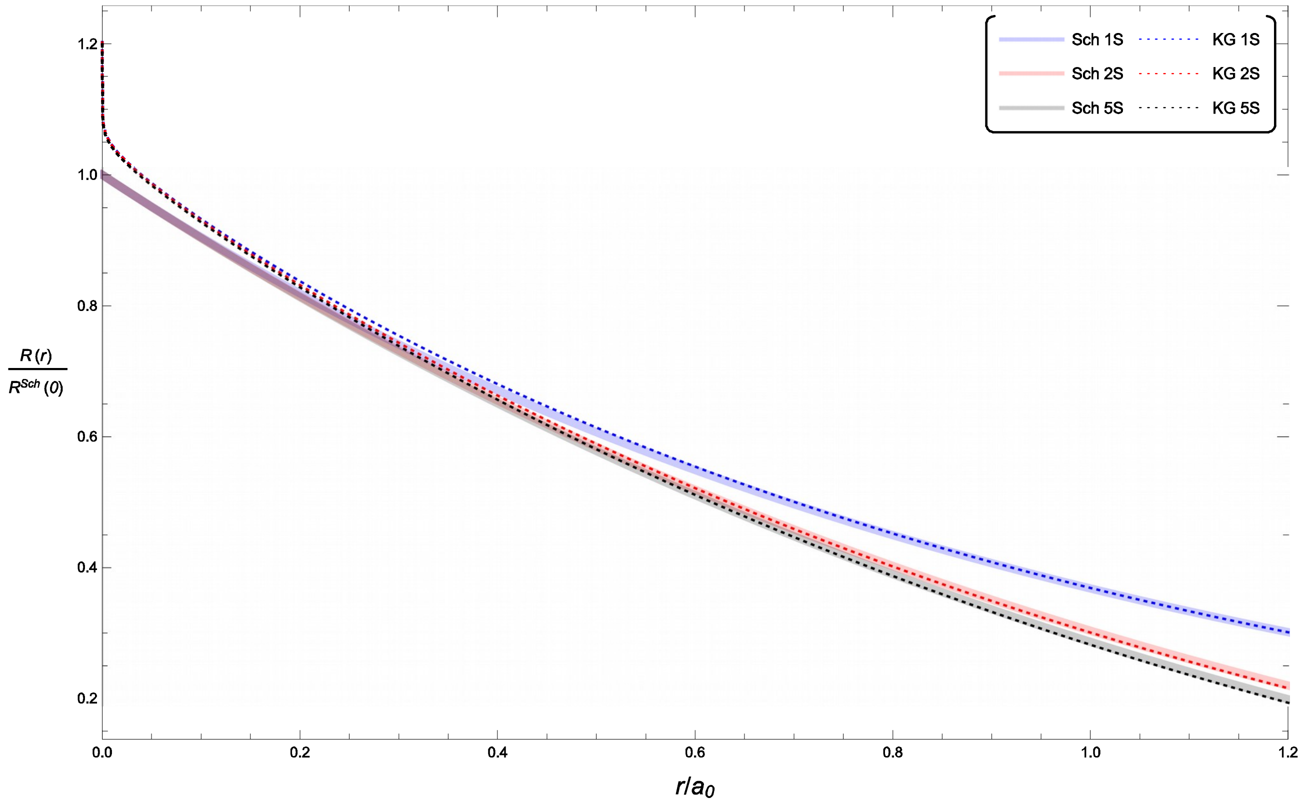
$$1 - \alpha m r Z$$

- Sum up to

$$R_{n0}^{\mathrm{KG}}(r) = R_{n0}^{\mathrm{Sch}}(0)(1 - mZ\alpha r)\left(1 - Z^2\alpha^2 \ln\frac{2r}{na_0} + \cdots\right)$$

S-wave Klein-Gordon wave functions near the origin

- Wave function near the origin:



Perturbation theory in QM

- Hamiltonian:

$$H_{\text{eff}} = H_0 + \Delta H = H_0 + H_{\text{kin}} + H_{\text{Darwin}},$$

$$H_0 = -\frac{\boldsymbol{\nabla}^2}{2m} - \frac{Ze^2}{4\pi r},$$

$$H_{\text{kin}} = -\frac{\boldsymbol{\nabla}^4}{8m^3c^2}, \qquad H_{\text{Darwin}} = \frac{1}{32m^4c^4}\left[\boldsymbol{\nabla}^2, \left[\frac{Ze^2}{4\pi r}, \boldsymbol{\nabla}^2\right]\right]$$

- Energy

$$\Delta E_{nl}^{(1)}\Big|_{\text{kin}} = \langle nl|H_{\text{kin}}|nl\rangle = -mc^2\frac{Z^4\alpha^4}{2n^4}\left(\frac{n}{l+\frac{1}{2}} - \frac{3}{4}\right)$$

$$\Delta E_{nl}^{(1)}\Big|_{\text{Darwin}} = \langle nl|H_{\text{Darwin}}|nl\rangle = 0.$$

Wave function correction

- First-order QM perturbation

$$\Delta R_{10}^{(1)}(0) = \sum_{n>1} R_{n0}^{\mathrm{Sch}}(0) \frac{\langle n0|\Delta H|10\rangle}{E_{10} - E_{n0}} + \int_0^\infty \frac{dk}{2\pi} R_{k0}^{\mathrm{Sch}}(0) \frac{\langle k0|\Delta H|10\rangle}{E_{10} - E_{k0}},$$

- Coutinuum spectrum gives UV divergence
- Coutinuum Coulomb wave function

$$R_{k0}^{\mathrm{Sch}}(r) = \sqrt{\frac{8\pi mcZ\alpha k}{1 - e^{\frac{-2\pi mcZ\alpha}{k}}}} e^{-ikr} {}_1F_1\left(1 + \frac{imcZ\alpha}{k}, 2; 2ikr\right), \qquad E_{k0} = \frac{k^2}{2m},$$

$$\int dr\, r^2 R_{k0}^{\mathrm{Sch}}(r) R_{k'0}^{\mathrm{Sch}}(r) = 2\pi\delta(k - k').$$

Continuum matrix element

$$\langle k0|H_{\rm kin}|10\rangle = -2Z^3\alpha^3\sqrt{\frac{\pi k}{2\left(1-e^{-\frac{2\pi m Z\alpha}{k}}\right)}}\left[1-\frac{2m^2Z^2\alpha^2\cosh\frac{\pi m Z\alpha}{k}\exp\left(\frac{2mZ\alpha}{k}\tan^{-1}\frac{mZ\alpha}{k}\right)}{k^2+m^2Z^2\alpha^2}\right]$$

$$\xrightarrow{k\to\infty} -\sqrt{mZ\alpha}Z^3\alpha^3\left(\frac{k}{mZ\alpha}+\frac{\pi}{2}+\cdots\right), \tag{17a}$$

$$\langle k0|H_{\rm Darwin}|10\rangle = -\frac{Z^3\alpha^3}{4}\sqrt{\frac{\pi k}{2(1-e^{-\frac{2\pi m Z\alpha}{k}})}}\left(\frac{k^2}{m^2}+Z^2\alpha^2\right)$$

$$\xrightarrow{k\to\infty} -\sqrt{mZ\alpha}\frac{Z^3\alpha^3}{8m^2}\left(\frac{k^3}{mZ\alpha}+\frac{\pi k^2}{2}+\frac{(24+\pi^2)k(mZ\alpha)}{24}+\frac{\pi(\pi^2-24)(mZ\alpha)^2}{48}+\cdots\right), \tag{17b}$$

Must impose a UV cutoff, yields divergence

$$\Delta R_{10}^{(1)}(0)\Big|_{\rm kin} = R_{10}^{\rm Sch}(0)\left(\frac{Z\alpha\Lambda}{\pi m}+Z^2\alpha^2\ln\Lambda+\text{finite}\right),$$

$$\Delta R_{10}^{(1)}(0)\Big|_{\rm Darwin} = R_{10}^{\rm Sch}(0)\left(\frac{Z\alpha\Lambda^3}{24\pi m^3}+\frac{Z^2\alpha^2\Lambda^2}{16m^2}+\frac{Z^3\alpha^3\pi\Lambda}{24m}+\text{finite}\right).$$

Scalar QED+HNET and OPE: an unsuccessful attempt!

- Lagrangian:

$$\mathcal{L} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi + N^\dagger iD_0 N - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$D^\mu = \partial^\mu + ieA^\mu ; D^\mu_N = \partial^\mu - iZeA^\mu$$

- OPE relations defined as:

$$\phi_R(\mathbf{r})N_R(\mathbf{0}) = (1 + C(\mathbf{r}))\,[\phi N]_R(\mathbf{0}) + \cdots ,$$

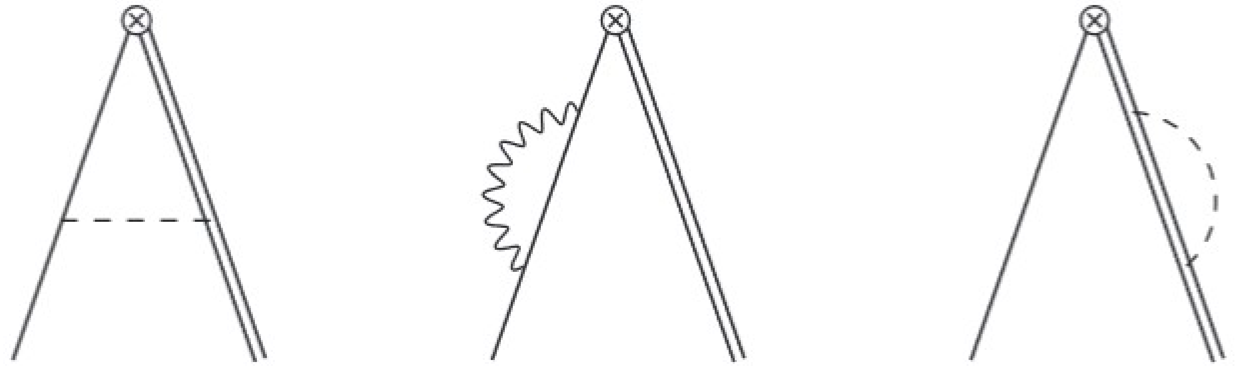
$$\begin{aligned}\tilde{\phi}_R(\mathbf{q})N_R(\mathbf{0}) &\equiv \int d^3\mathbf{r}e^{-i\mathbf{q}\cdot\mathbf{r}}\phi_R(\mathbf{r})N_R(\mathbf{0}) \\ &= \tilde{C}(\mathbf{q})[\phi N]_R(\mathbf{0}) + \cdots ,\end{aligned}$$

Renormalize Local Operators

- Definition:

$$[\phi N]_R(\mathbf{0}) \equiv Z_{\phi N}\phi^0 N^0(\mathbf{0}) = Z_{\phi N}\sqrt{Z_\phi Z_N}\phi_R N_R(\mathbf{0})$$
$$= Z_S \phi_R N_R(\mathbf{0})$$

- Calculate the following diagrams in both Feynman gauge and Coulomb gauge



Renormalize Local Operators

- Relation for counterterms

$$\delta_S = \delta_{\phi N} + \frac{\delta_\phi}{2} + \frac{\delta_N}{2}.$$

	δ_ϕ	δ_N	δ_S	$\delta_{\phi N}$
Feynman Gauge	$\frac{e^2}{4\pi^2\epsilon}$	$\frac{Z^2e^2}{4\pi^2\epsilon}$	$-\frac{Ze^2}{8\pi^2\epsilon}$	$-\frac{Ze^2}{8\pi^2\epsilon}-\frac{e^2}{8\pi^2\epsilon}-\frac{Z^2e^2}{8\pi^2\epsilon}$
Coulomb Gauge	$\frac{e^2}{4\pi^2\epsilon}$	0	$-\frac{Ze^2}{4\pi^2\epsilon}$	$-\frac{Ze^2}{4\pi^2\epsilon}-\frac{e^2}{8\pi^2\epsilon}$

TABLE I: Counterterms in Feynman gauge and Coulomb gauge within MS scheme.

- While $\delta_{\phi N}$ is gauge-invariant with Z=1, what's rela ted to OPE is δ_S.

OPE (similar to point-splitting, smearng a local composite operator)

- Define

$$\tilde{\Gamma}_{\phi}\left(\mathbf{q};\,\mathbf{p},E\equiv k^{0}+p^{0}\right)\equiv\int\frac{d^{3}r}{(2\pi)^{3}}\frac{d^{4}y}{(2\pi)^{4}}\frac{d^{4}z}{(2\pi)^{4}}e^{-i\mathbf{q}\cdot\mathbf{r}}e^{ip\cdot y}e^{ik\cdot z}\langle\Omega|T\left\{\phi(\mathbf{r})N(\mathbf{0})\varphi^{\dagger}(y)N^{\dagger}(z)\right\}|\Omega\rangle_{amp},$$

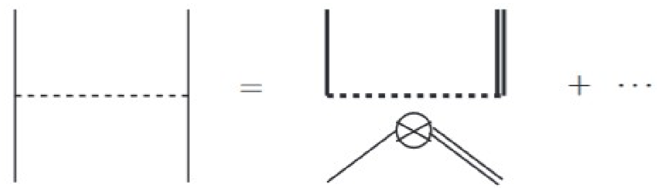
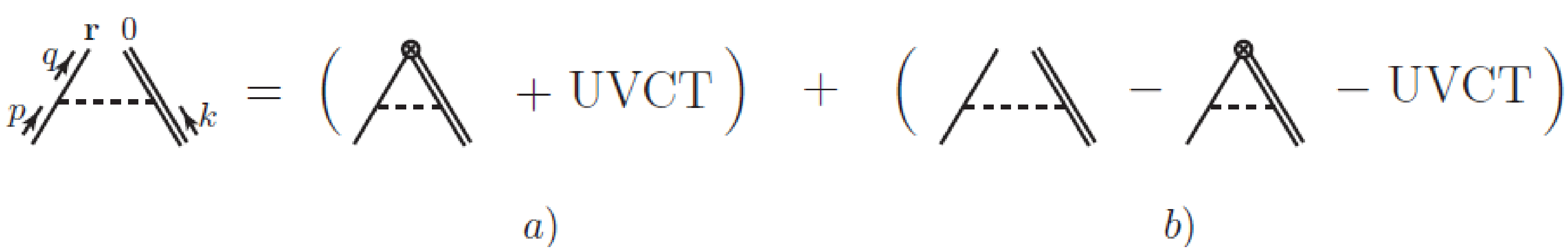


FIG. 4: The OPE relation to order $Z\alpha$ in the momentum space.



OPE in coordinate space (Feynman gauge)

- In Feynman gauge, expanding $\tilde{\Gamma}_{\phi}^{Feyn}(\mathbf{q};\mathbf{p},k^0,p^0)$ gives

$$\tilde{C}^{Feyn}(\mathbf{q}) = \frac{\pi Z\alpha}{|\mathbf{q}|^3} + \cdots$$

- Fourier transform to coordinate space

$$C^{Feyn}(\mathbf{r}) = -\frac{Z\alpha}{2\pi}(\log \mu r + \frac{1}{2}\log \pi e^{\gamma_E}),$$

OPE in coordinate space (Coulomb gauge)

- In Coulomb gauge, expanding $\tilde{\Gamma}_{\phi}^{Coul}(\mathbf{q};\mathbf{p},E\equiv k^0+p^0)$ gives

$$\tilde{C}^{Coul}(\mathbf{q}) = \frac{2\pi Z\alpha}{|\mathbf{q}|^3} + \cdots$$

- Fourier transform to coordinate space

$$C^{Coul}(\mathbf{r}) = -\frac{Z\alpha}{\pi}(\log \mu r + \frac{1}{2}\log \pi e^{\gamma_E}),$$

- Both gauges give logarithm at $Z\alpha$ order

NREFT

- Scalar QED + HNET won't work!
- Drop A^0

$$\mathcal{L} = (D_0\phi)^\dagger D^0\phi - \nabla\phi^\dagger \cdot \nabla\phi - m^2\phi^\dagger\phi + N^\dagger iD_0 N + \frac{1}{2}(\boldsymbol{\nabla} A_0)^2$$

- Non-relativistic approximation

$$\varphi = e^{imc^2 t}\frac{1}{\sqrt{2mc^2}}(iD_0 + mc^2)\phi, \quad \tilde{\varphi} = e^{imc^2 t}\frac{1}{\sqrt{2mc^2}}(-iD_0 + mc^2)\phi$$

- Use with EOM, the following conditions are obtained

$$\tilde{\varphi} = -\frac{iD_0}{2mc^2}(\varphi + \tilde{\varphi}). \qquad iD_0\varphi = \frac{\nabla^2}{2m}(\varphi + \tilde{\varphi}).$$

NREFT: our working basis

- NREFT Lagrangian:

$$\mathcal{L}_{EFT} = \varphi^\dagger \left[iD_0 + \frac{\boldsymbol{\nabla}^2}{2m} + \frac{\boldsymbol{\nabla}^4}{8m^3} + \frac{\boldsymbol{\nabla}^6}{16m^5} + \frac{e}{32m^4}[\boldsymbol{\nabla}^2, [A_0, \boldsymbol{\nabla}^2]] + \cdots \right] \varphi + N^\dagger i D_N^0 N$$
$$+ \frac{1}{2}(\boldsymbol{\nabla} A_0)^2 + c_4 \varphi^\dagger \varphi N^\dagger N + \cdots \, .$$

Matching of contact interaction

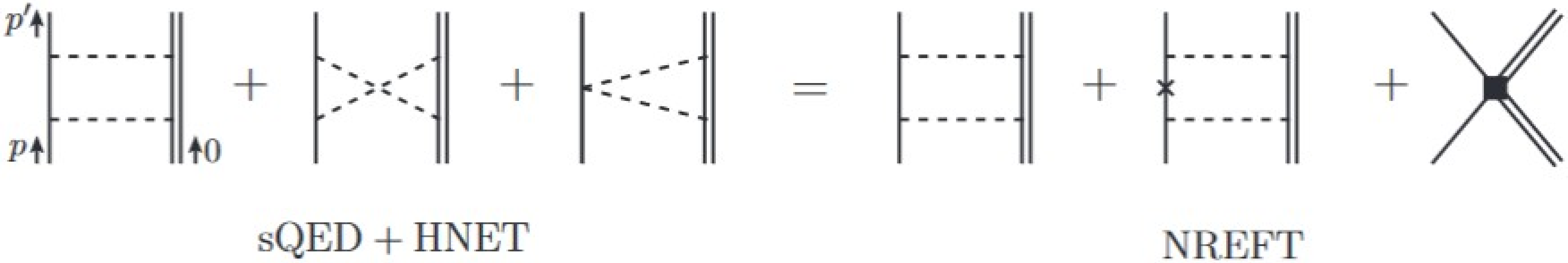


FIG. 7: One-loop diagrams for $eN \to eN$ on-shell amplitude from both sQED+HNET and NREFT. The cross represents the p^4 relativistic correction, and the solid square represents the c_4 electron-nucleus contact interaction.

- Matching gives $c_4 = \mathcal{O}(Z^3\alpha^3)$

OPE result

- Coordinate space

$$\varphi(\mathbf{r})N(\mathbf{0}) = \left(1 - mZ\alpha|\mathbf{r}| - Z^2\alpha^2\left(\log\left(mZ\alpha|\mathbf{r}|\right) + \frac{1}{2}\log 4\pi e^{\gamma_E - 1}\right) + \cdots\right)[\varphi N]_R(\mathbf{0}) + \cdots .$$

- Momentum space

$$\begin{aligned}\tilde{\varphi}(\mathbf{q})N(\mathbf{0}) &\equiv \int d^3\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{x}}\varphi(\mathbf{r})N(\mathbf{0}) \\ &= \left(\frac{8\pi mZ\alpha}{\mathbf{q}^4} + \frac{2\pi^2 Z^2\alpha^2 + \cdots}{\mathbf{q}^3}\right)[\varphi N]_R(\mathbf{0}) + \cdots\end{aligned}$$

- Crossed multiplication

$$\tilde{\Gamma}^{(n)}(\mathbf{q};\mathbf{p},E) = \sum_{i=1}^{n}\tilde{C}^{(i)}(\mathbf{q})F_R^{(n-i)}(\mathbf{p},E)$$

Renormalize local operators

- No need include wave function correction, the ren ormalization only involves vertex correction:

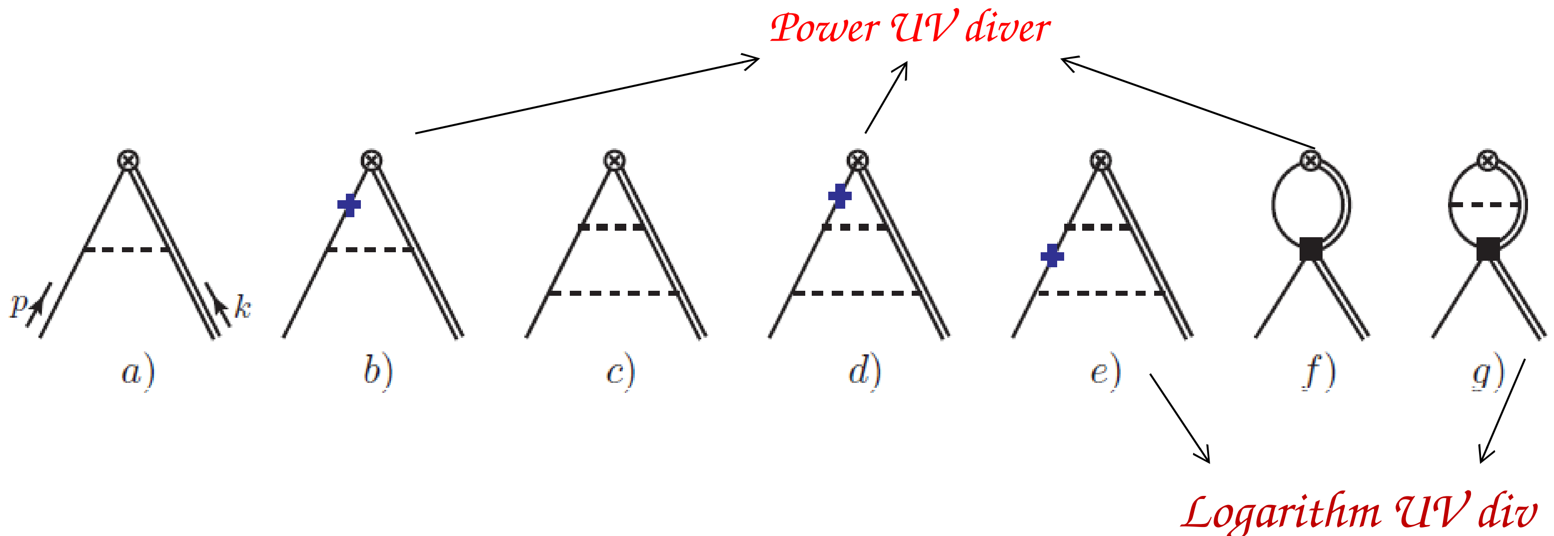
$$[\varphi N]_R(\mathbf{0}) = Z_S \varphi N(\mathbf{0})$$

- Define

$$F(\mathbf{p}, E) \equiv \int \frac{d^4 y}{(2\pi)^4} \int \frac{d^4 z}{(2\pi)^4} e^{ip\cdot y} e^{ik\cdot z} \langle\Omega| T\left\{\varphi N(\mathbf{0}) \tilde{\varphi}^\dagger(p) \tilde{N}^\dagger(k)\right\} |\Omega\rangle_{amp},$$

Renormalize local operators

- Calculate diagram b and d won't give logarithmic divergence.
- Diagram a and c are finite.
- Calculate diagram e gives logarithmic divergence $Z^2\alpha^2\left(\frac{1}{2\epsilon}+\log\mu\right)$.
- Diagram f is higher order contribution.



Renormalize local operators

- With MS scheme in dimensional regularization

$$Z_S = 1 - \frac{Z^2\alpha^2}{2\epsilon} + \mathcal{O}(\alpha^4),$$

- The anomalous dimension

$$\gamma_S \equiv \frac{d\log Z_S}{d\log\mu} = Z^2\alpha^2$$

Momentum space Wilson coefficient

- Define

$$\tilde{\Gamma}\left(\mathbf{q};\mathbf{p},E\equiv k^0+p^0\right)\equiv\int\frac{d^4y}{(2\pi)^4}\int\frac{d^4z}{(2\pi)^4}e^{ip\cdot y}e^{ik\cdot z}\langle\Omega|T\left\{\tilde{\varphi}(\mathbf{q})N(\mathbf{0})\varphi^\dagger(y)N^\dagger(z)\right\}|\Omega\rangle_{amp},$$

- Scales in the problem:

- Hard: $m\,v << q < m$, $v = Z\alpha << 1$
- Soft: $p \sim mv$, $E \sim mv^2$

Leading scaling behavior of diagrams: double-layer form of OPE

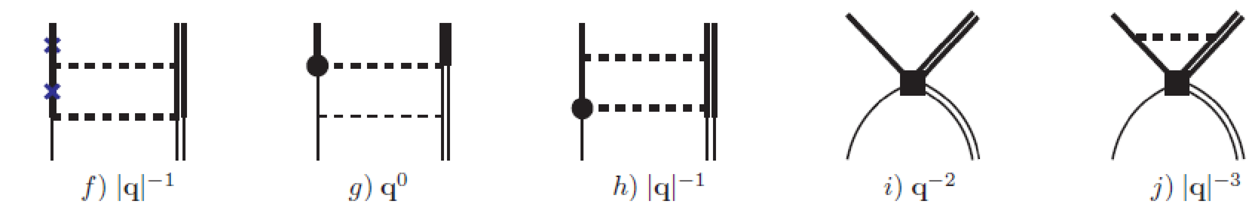


FIG. 8: Leading $\mathbf{q}$-scaling behavior of numerous higher-order diagrams for the momentum-space Green function $\widetilde{\Gamma}$. The thick lines are meant to carry the hard momentum of order $\mathbf{q}$, in order to contribute to the specified leading region. The cross, heavy dot, solid square refer to the $\mathbf{p}^4$ kinetic term, Darwin term and contact interaction, respectively.

$$\widetilde{\Gamma}(\mathbf{q};\mathbf{p},E) \xrightarrow{q\to m} \frac{1}{\mathbf{q}^4} \quad \text{and} \quad \frac{1}{|\mathbf{q}|^3},$$

OPE relation (momentum space)

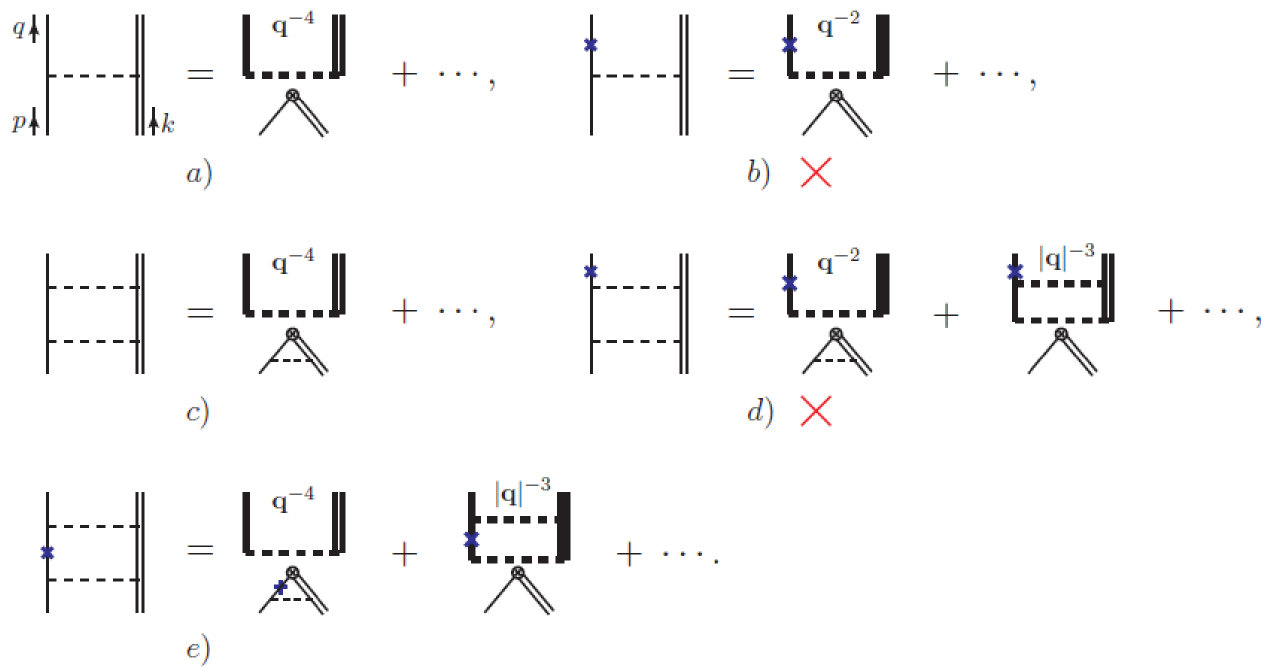
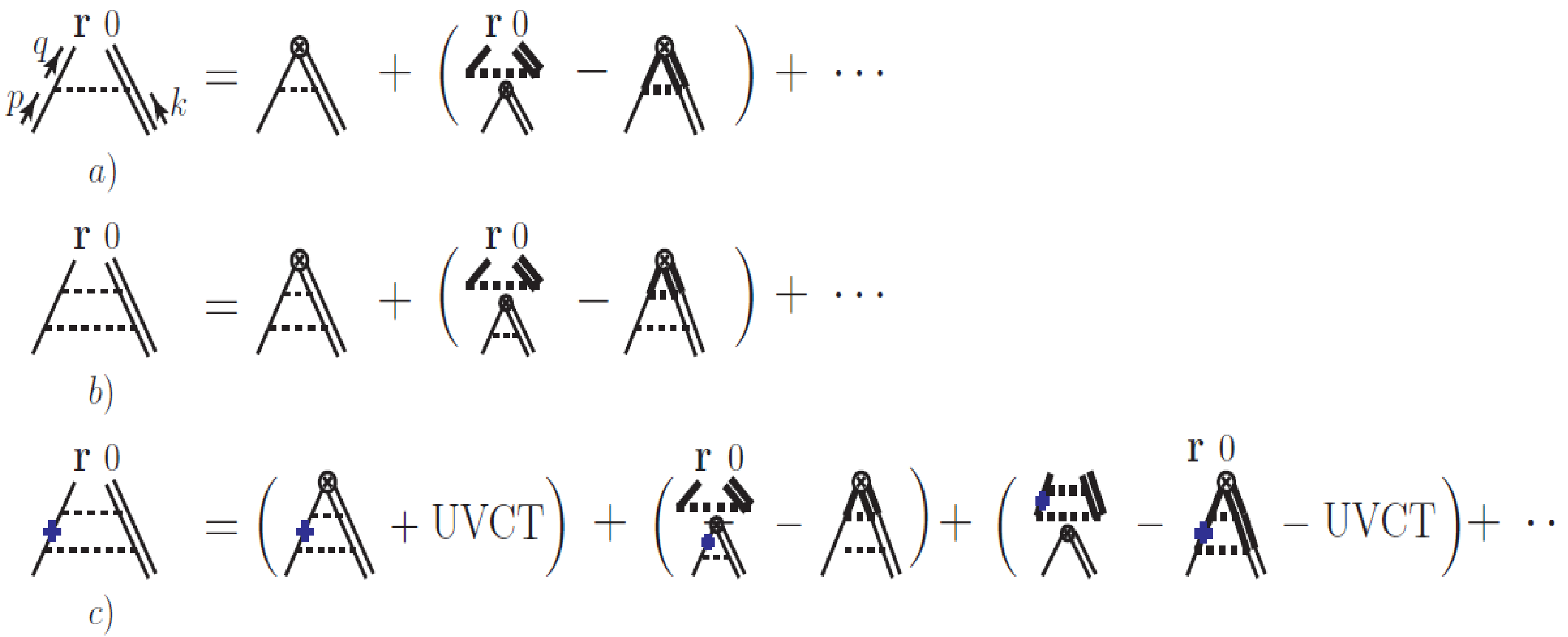


FIG. 9: The factorization of momentum-space four-point function into hard and soft parts through order $Z^2\alpha^2$.

OPE relation (coordinate space)



Wilson coefficient

- Momentum space

$$\tilde{C}^{(2)}(\mathbf{q}) = \frac{2\pi^2 Z^2 \alpha^2}{|\mathbf{q}|^3}.$$

- Coordinate space

$$C^{(2)}(\mathbf{r}) = -Z^2\alpha^2 \left(\log \mu|\mathbf{r}| + \frac{1}{2} \log 4\pi e^{\gamma_E - 1} \right),$$

Renormalization Group Equation: Resum the leading logarithm:

RGE for local composite S-wave operator:

$$\mu\frac{d[\varphi N]_R}{d\mu} = \gamma_S[\varphi N]_R, \qquad \gamma_S = Z^2\alpha^2 + \mathcal{O}(Z^4\alpha^4)$$

$$[\varphi N]_R(\mu) = [\varphi N]_R(\mu_0)\left(\frac{\mu}{\mu_0}\right)^{Z^2\alpha^2}$$

RGE for S-wave Wilson coefficient of OPE:

$$\mu\frac{d\mathcal{C}(r)}{d\mu} + \mathcal{C}(r)\gamma_S = 0. \qquad (\mu\tfrac{d}{d\mu} - r\tfrac{d}{dr})\mathcal{C}(\mu r) = 0.$$

Define $r \equiv r_0\kappa$,

$$\kappa\frac{d\mathcal{C}(r_0\kappa)}{d\kappa} + \mathcal{C}(r_0\kappa)\gamma_S = 0.$$

$$\boxed{\mathcal{C}(r) = \mathcal{C}(r_0)\kappa^{-Z^2\alpha^2} = \mathcal{C}(r_0)\left(\frac{r}{r_0}\right)^{-Z^2\alpha^2}}$$

Choosing $r_0 = a_0$ as the Bohr radius

Fully reproduce the near-the origin anomalous scaling of KG wf. $R^{\mathrm{KG}}_{n0}(r) \approx R^{\mathrm{Sch}}_{n0}(0)\rho^{\sqrt{\frac{1}{4}-Z^2\alpha^2}-\frac{1}{2}} \approx R^{\mathrm{Sch}}_{n0}(0)\rho^{-Z^2\alpha^2}$

Dirac wave function of hydrogen

$$\left(-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta mc^2-\frac{Ze^2}{4\pi r}\right)\Psi=E\Psi,$$

$$E_{nj}=mc^2\left[1-\frac{Z^2\alpha^2}{2n^2}-\frac{Z^4\alpha^4}{2n^4}\left(\frac{n}{j+\frac{1}{2}}-\frac{3}{4}\right)+\cdots\right],$$

We focus on the large component of Dirac wave functions for the j=1/2, positive parity hydrogen

$$\Psi_{n\frac{1}{2}m}(\mathbf{r})=\begin{pmatrix}F_n(r)\sqrt{\frac{1}{4\pi}}\,\xi_m\\ G_n(r)\sqrt{\frac{3}{4\pi}}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\,\xi_m\end{pmatrix},$$

$$F_n(r)\approx R_n^{\rm Sch}(0)\left(\frac{2r}{na_0}\right)^{-\frac{Z^2\alpha^2}{2}},$$

where $a_0=\hbar/(mcZ\alpha)$ is the Bohr radius, and $R_n^{\rm Sch}(0)$ represents the radial Schrödinger wave function at the origin for the nS hydrogen state. We have also taken the nonrelativistic approximation $\sqrt{1-Z^2\alpha^2}\approx 1-Z^2\alpha^2/2$ in the exponent. The singularity has noticeable effect only when $r\lesssim\frac{na_0}{2}\exp(-2/Z^2\alpha^2)\sim\frac{na_0}{2}10^{-16300/Z^2}$ [14], which is even many orders shorter than the length scale related to the QED Landau pole!

Dirac wave function

- Wave function origin behavior of large component

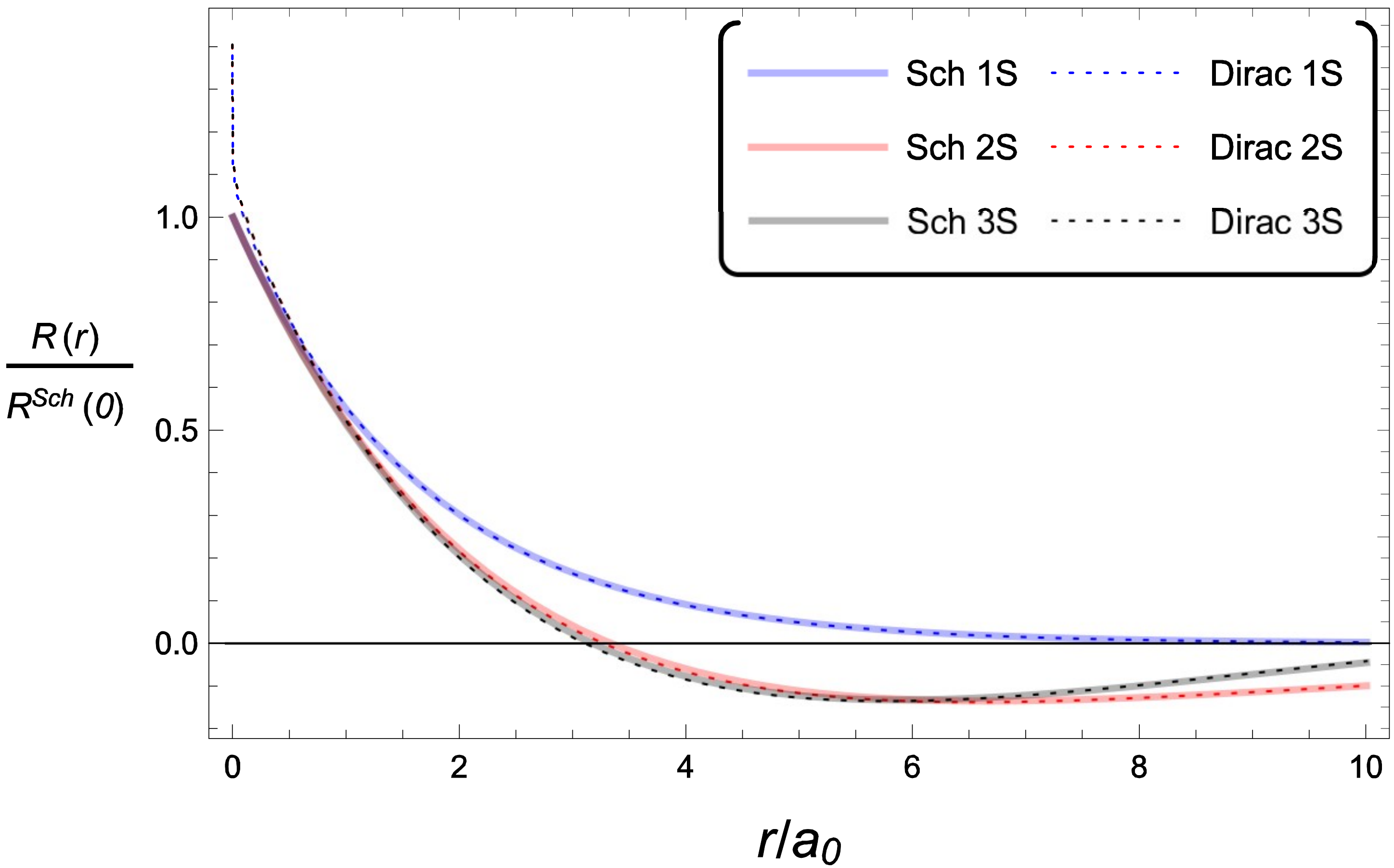
$$F_n(r) \approx R_n^{\mathrm{Sch}}(0)\left(\frac{2r}{na_0}\right)^{-\frac{Z^2\alpha^2}{2}},$$

- Expand the large component spinor:

$$\lim_{r\to 0} F_n(r) = R_n^{\mathrm{Sch}}(0)\,(1 - r/a_0)\left(1 - \frac{Z^2\alpha^2}{2}\ln r + \cdots\right).$$

- Divergence can be reproduced by perturbative QM
- Kinetic correction + Darwin term+ spin orbit

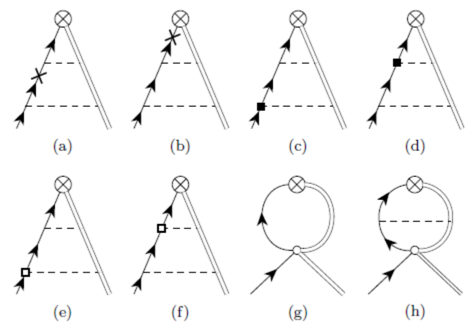
Dirac wave function



Correct OPE formulated in NRQED+HNET

$$\mathcal{L}_{\rm NREFT} = \psi^\dagger\left\{ iD_0 + \frac{\boldsymbol{\nabla}^2}{2m} + \frac{\boldsymbol{\nabla}^4}{8m^3} - c_D e\frac{\boldsymbol{\nabla}^2 A^0}{8m^2} + i c_S e\frac{\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}A^0\times\boldsymbol{\nabla})}{4m^2} + \cdots \right\}\psi + N^\dagger iD^0 N + \frac{c_4}{m^2}\psi^\dagger\psi N^\dagger N + \frac{1}{2}\left(\nabla A^0\right)^2 + \cdots .$$

$$F_n(r)\frac{1}{\sqrt{4\pi}}\,\xi_m = \langle 0|\psi(\mathbf{r})N(0)|nS_{1/2}, m\rangle,$$

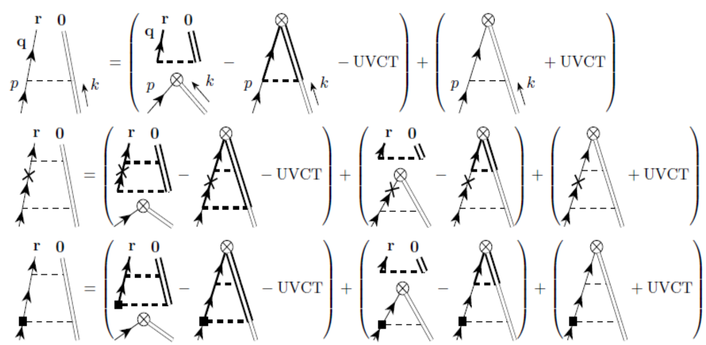


$$\lim_{r\to\frac{1}{m}} \psi(\mathbf{r})N(0) = \mathcal{C}(r)[\psi N]_R(0) + \cdots , \tag{18}$$

$$\mathcal{C}(r) = 1 - mZ\alpha r - \frac{Z^2\alpha^2}{2}(\ln\mu r + \text{const}) + \mathcal{O}(Z^3\alpha^3).$$

$$\widetilde{\psi}(\mathbf{q})N(0) \equiv \int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}}\varphi(\mathbf{r})N(0) \to \widetilde{\mathcal{C}}(q)[\psi N]_R(0) + \cdots , \tag{19}$$

$$\widetilde{\mathcal{C}}(q) = \frac{8\pi m Z\alpha + \mathcal{O}(Z^3\alpha^3)}{\mathbf{q}^4} - \frac{\pi^2 Z^2\alpha^2 + \mathcal{O}(Z^4\alpha^4)}{|\mathbf{q}|^3},$$



Summary

- EFT combined with OPE offers new insights for deeper understanding coalescence behavior of atomic wave functions.
  - It can be extended to study multi-particle coalescence behaviors.
- Applied in relativistic Klein-Gordon and Dirac equations, solving long-standing puzzle about the anomalous scaling behavior of the hydrogen wave functions near the origin
- Leading logarithms are resummed with the aid of RGE
- Lesson: To succeed, one must start from NREFT, not the UV-completed relativistic QED+HNET
- KG and Dirac equation as $r<1/m$ becomes untrustworthy… why?

Thanks for your attention!