Holographic Complexity in the Jackiw-Teitelboim Gravity

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arXiv:1901.00014: Joint work with
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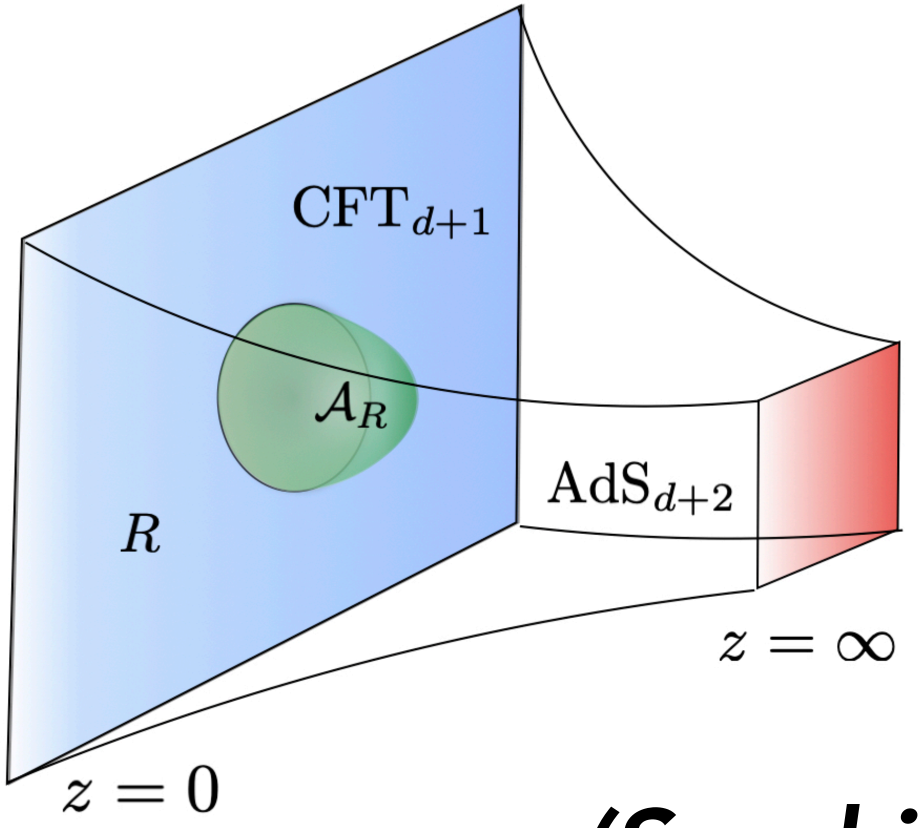
AdS/CFT correspondence *(Maldacena '97)*

Quantum Gravity on AdS = CFT on the boundary of AdS

"bulk" "boundary"

How the bulk geometry encoded in the boundary theory?



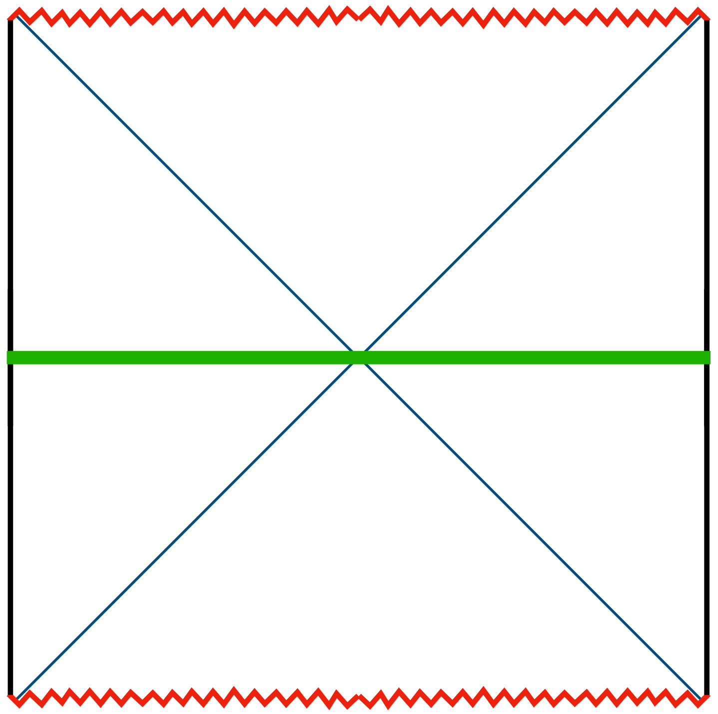
- Entanglement *(Ryu-Takayanagi '06)*

Min area of co-dim 2 surface

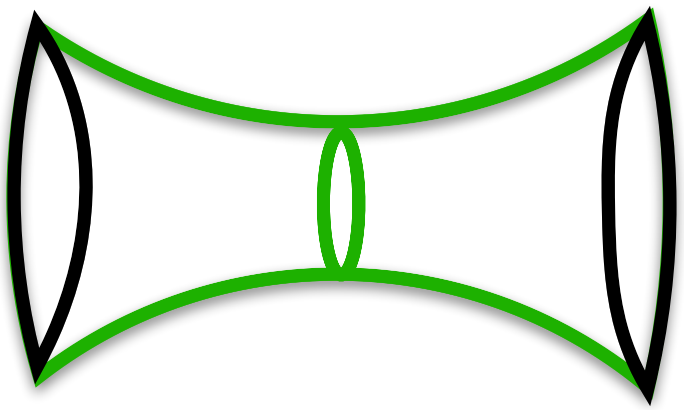
= Entanglement entropy

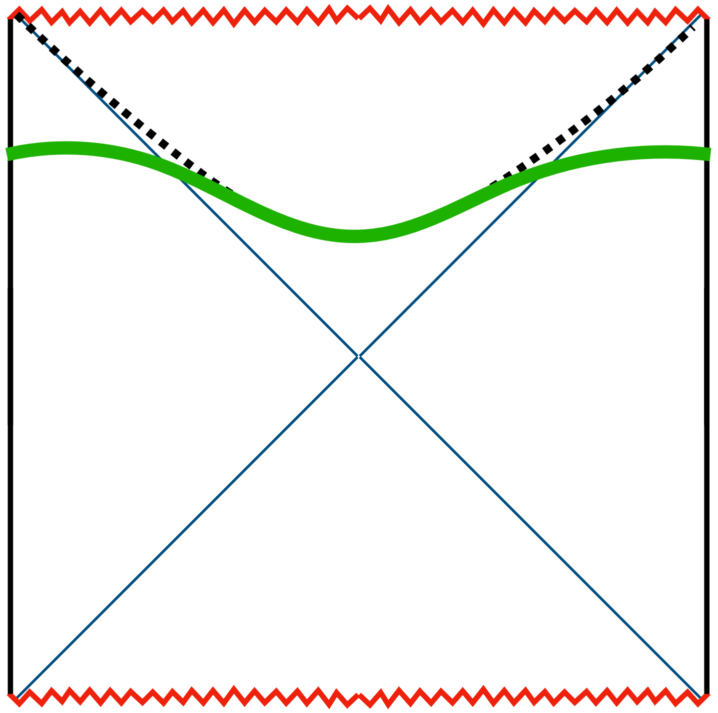
- *(Susskind '14)* Entanglement is enough??

- **The wormhole geometry connecting two boundaries**

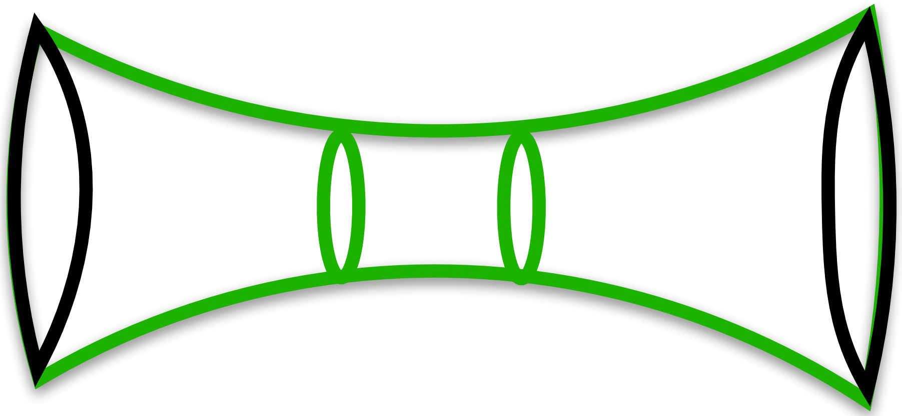


Consider the maximal volume which connects two boundaries



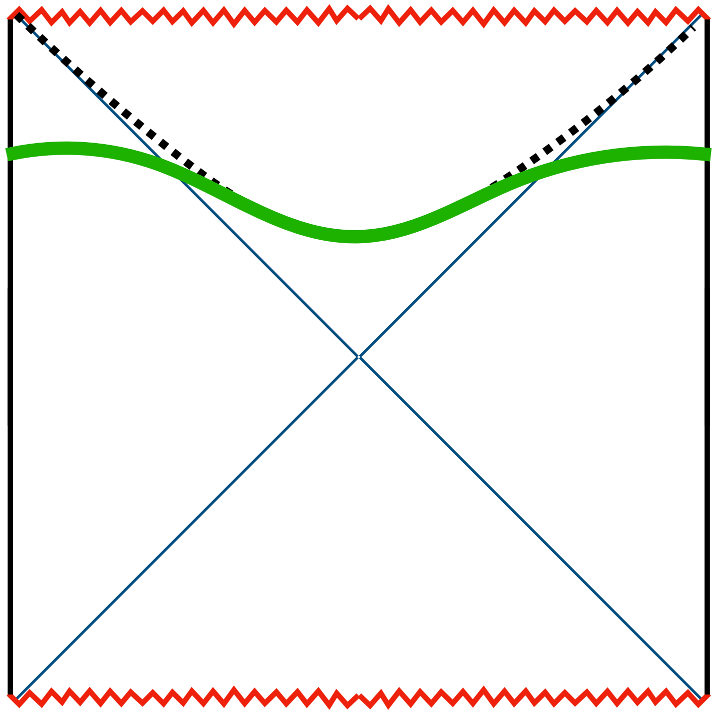


It grows linearly with time (at late times).



Classically, it grows forever.

- **What is there in boundary theory that can capture the continuing growth of the volume?**



Entanglement? → No

(Hartman-Maldacena '13)

The entanglement entropy S_A =the minimal cross-sectional area of the wormhole

When t~ (the size of the region A), the entanglement stops growing.

→**Entanglement is not enough** *(Susskind '14)*

Complexity

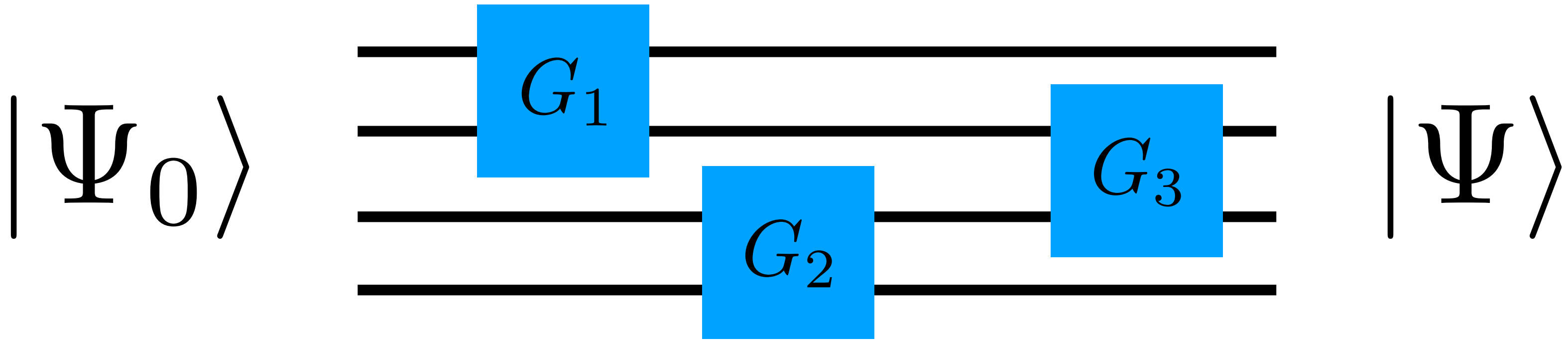
Susskind '14 proposed the continuing growth of the wormhole is related to ``complexity'' in the boundary theory.

Complexity: the measure of the difficulty of transforming a reference state to some other state

simple unitary1 or 2 qubit operators

- reference state $|\Psi_0\rangle$
- elementary gate set (simple operation) $\{G_i\}$

Complexity of a state $|\Psi\rangle$: minimal # of gates necessary for transforming $|\Psi_0\rangle$ to $|\Psi\rangle$



complexity also grows linearly in t ↔ wormhole growth?

A good def. of complexity for generic QFTs has not been known yet.

There are many attempts recently

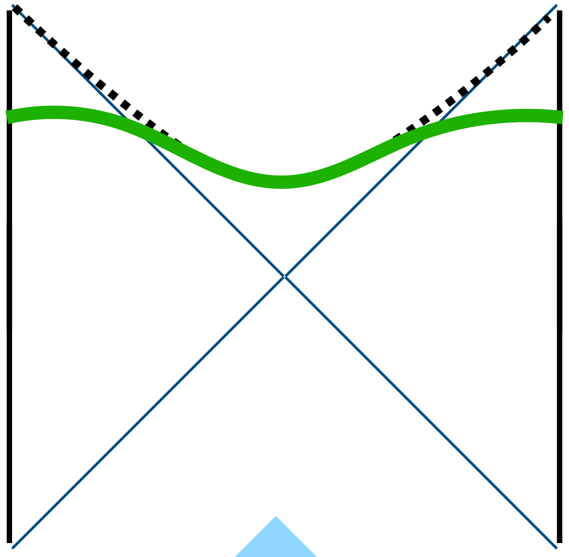
- 2d CFT [Caputa-Kundu-Miyaji-Takayanagi-Watanabe, 1703.00456]
- Free scalars [Jefferson-Myers, 1707.08570, Chapman-Heller-Marrochio- Pastawski,1707.08582],

…

In order to give a good definition of the complexity, we should know much about the corresponding bulk quantity. → Today’s focus

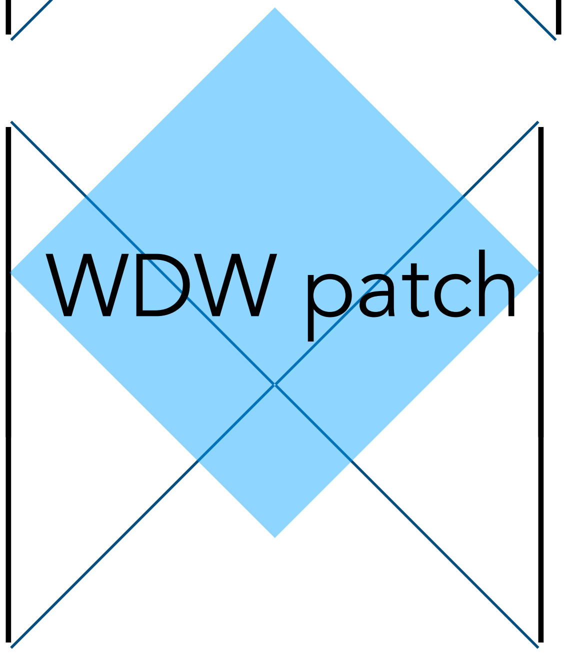
Holographic Complexity

There are two proposals on how to measure ``the size” of the wormhole



Complexity=Volume *(Susskind ‘14)*

$$\mathcal{C}_{\mathcal{V}} \sim \frac{V}{G l_{\mathrm{AdS}}^2}$$



Complexity=Action *(Brown et al.’15)*

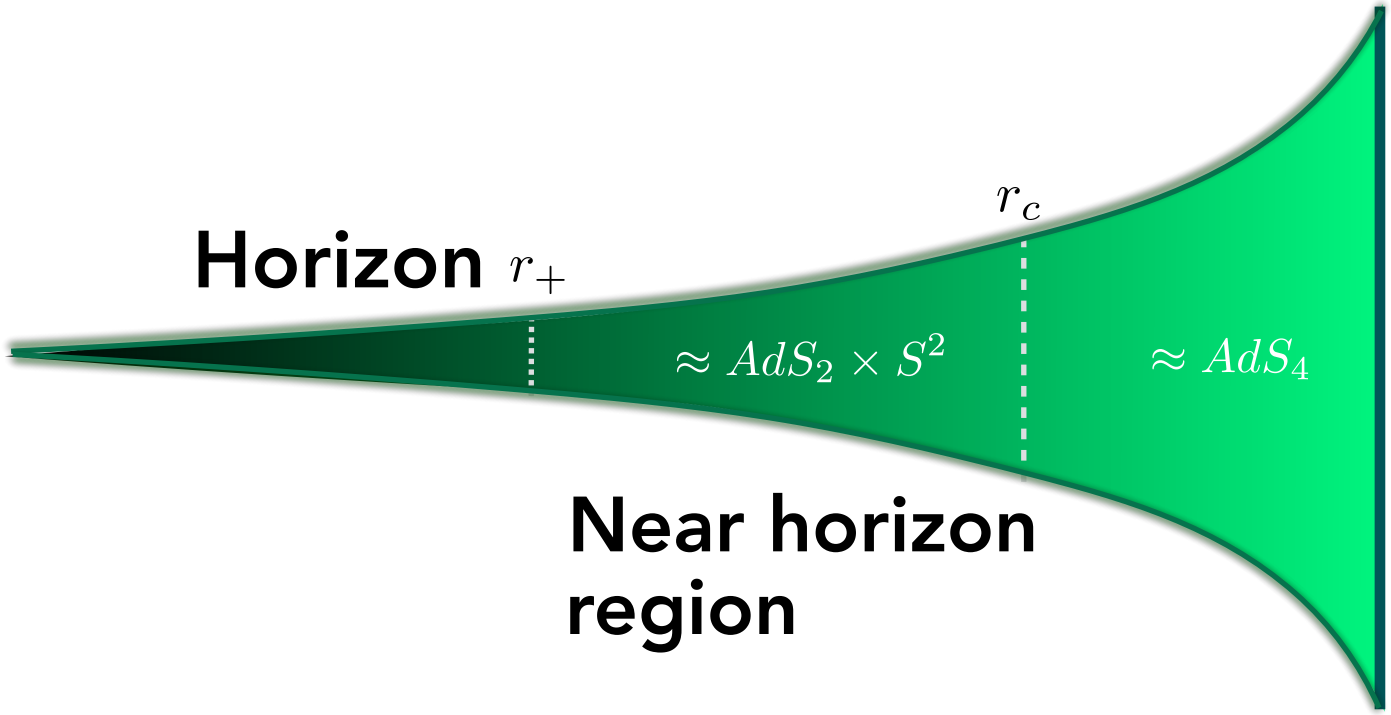
On-shell action evaluated in the Wheeler-DeWitt (WDW) patch

$$\mathcal{C}_{\mathcal{A}} \sim \frac{1}{\hbar} S_{WDW}$$

Both grow linearly with t at late times at least for eternal neutral BHs and the shockwave geometry etc.

Today we mainly consider the holographic complexity in the so-called **Jackiw-Teitelboim model** (**JT model**).

2-dim dilaton gravity describing **AdS_2** spacetime



It can be derived from the higher dim charged black holes by the dimensional reduction in the throat limit

Sachdev-Ye-Kitaev (SYK) model:

quantum mechanical model of Majorana fermion

$$H = i^{q/2} \sum_{1\leq q_1 \leq \cdots \leq i_q \leq N}^{N} J_{i_1 \cdots i_q} \psi_{i_1} \cdots \psi_{i_q}$$

$J_{i_1,\ldots,i_q}$:random coupling

JT model is holographically dual to the **SYK model**:
It can describe IR dynamics of SYK, which is governed by ***Schwarzian action*** *(Maldacena-Stanford-Yang '16)*

Motivation

- We expect it easier to define complexity in the bdy theory←SYK model: QM of Majorana fermions

→Holographic complexity in the JT model gives a hint about complexity in the SYK model ??

- Einstein eq. can be solved *(Almheiri-Polchinski '14)*

→Can calculate complexity in various geometries: Shockwave geometry & Traversable wormhole…

Interesting to see the effect of the perturbation on complexity

What we have done

- We computed the holographic complexity in the JT model both in the **CV** and the **CA** proposal

→ Behavior of complexity in the **CA** proposal disagrees with the **CV** result!

CA stops growing : ❌ linear growth at late times

- The JT model: derived from a 4d **magnetically** charged black holes by the dimensional reduction

→observed the similar behavior in a 4d black hole!

On the other hand…

- We found: complexity of a 4d **electric** black holes shows **linear growth rate** at late times!

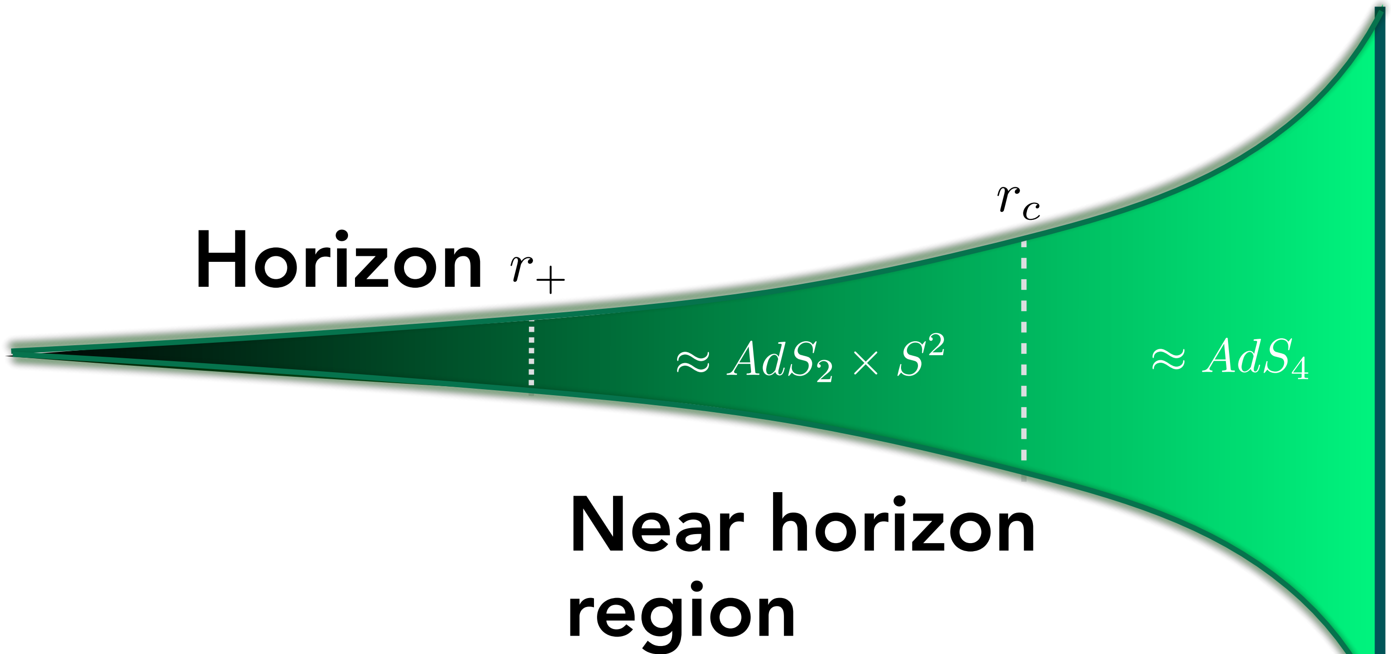
→derived another 2d dilaton gravity theory “**JT-like**” model, which describes the **electric** black holes and found linear growth rate!

- d.o.f of adding Maxwell boundary term to the action →also changes the late time behavior of complexity

Jackiw-Teitelboim model

- **Jackiw-Teitelboim model**: 2d (solvable) dilaton gravity which describes AdS$_2$ geometry
[Jackiw, Teitelboim, Almheiri-Polchinski]

- derived from the near-horizon region of 4d near-extremal magnetic black holes by the dim reduction



- 4d Maxwell-Einstein action

$$I_{EM} = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\sqrt{-\hat{g}}(\hat{R}-2\Lambda) + \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}}\sqrt{-\hat{\gamma}}\hat{K} - \frac{1}{4g^2}\int_{\mathcal{M}} d^4x\sqrt{-\hat{g}}F_{\mu\nu}F^{\mu\nu},$$

ansatz for the metric and the Maxwell field:

$$ds^2 = g_{ab}(x^a)dx^a dx^b + \Psi^2(x^a)(d\theta^2 + \sin^2\theta d\phi^2)$$

keep the off-shell d.o.f in 2d

fix the spherical part and perform the dim reduction

magnetic: $F_{\phi\theta} \neq 0, (\text{other components of } F) = 0$

$$F = \frac{Q}{4\pi}\sin\theta d\phi \wedge d\theta$$

$$I_C^{\text{magnetic}} = \frac{1}{4G_N}\int_{\mathcal{M}} d^2x\sqrt{-g}\left(\Psi^2 R + 2(\nabla\Psi)^2 + 2 - 2\Psi^2\Lambda - \frac{Q^2}{2\pi\Psi^2}\right) + \frac{1}{4G_N}\int_{\partial\mathcal{M}}\sqrt{-\gamma}\Psi^2 K$$

- Near-horizon & near-extremal→expand Ψ^2 around the area of the horizon of the extremal BH

$$\Psi^2 = \frac{1}{4\pi}(\Phi_0 + \Phi)\,,\ \Phi_0 = 4\pi r_h^2\ \ Q = Q_{\text{ext}}$$ with $\Phi/\Phi_0 \ll 1$

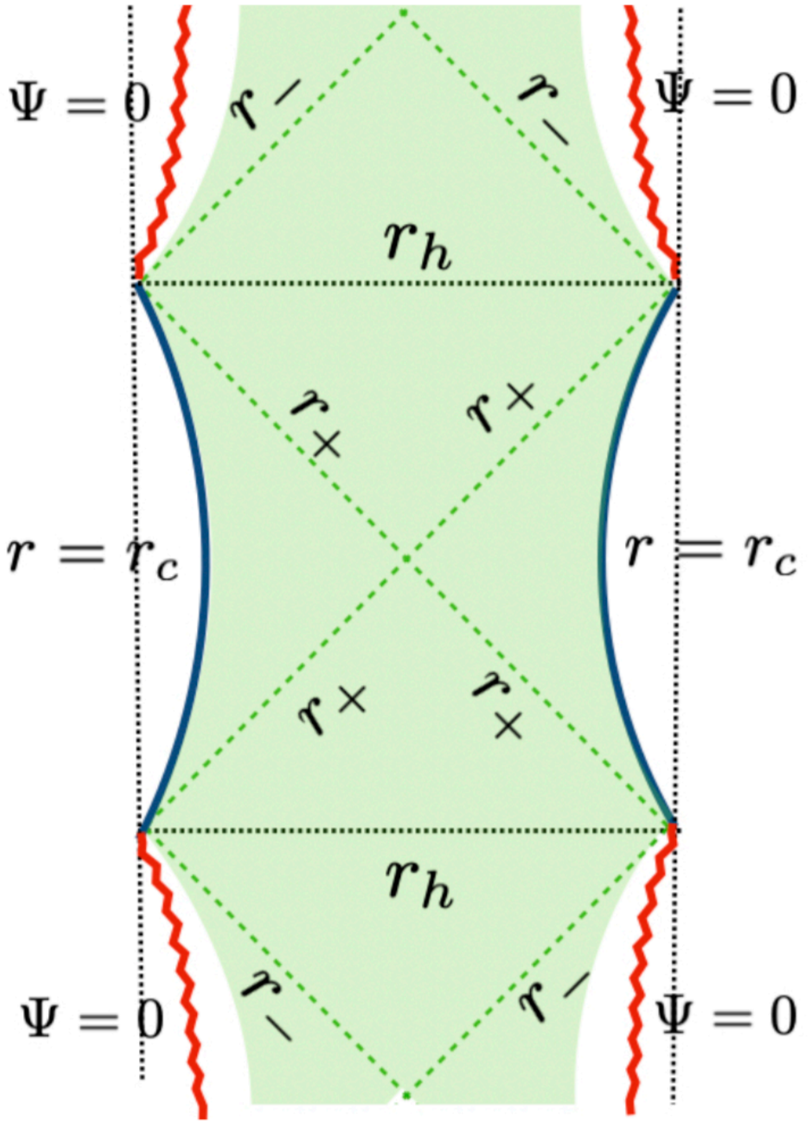
area of the extremal BH **charge of the extremal BH**

The action of the JT model:

$$I_{\text{JT}} = \frac{\Phi_0}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x\sqrt{-g}R + 2\int_{\partial\mathcal{M}} dx\sqrt{-\gamma}K\right] + \frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x\Phi(R - 2\Lambda_2) + 2\int_{\partial\mathcal{M}} dx\sqrt{-\gamma}\Phi_b K\right]$$

dynamical d.o.f : $g_{ab}\,,\Phi$ 2d cosmological const: $\Lambda_2 \equiv \Lambda - \frac{Q_{\text{ext}}}{4\pi r_h^4}$

Solution of the JT model



$$r_\pm = \pm\sqrt{\mu}$$

EOMs:

$$R - 2\Lambda_2 = 0$$

$$\nabla_a \nabla_b \Phi - g_{ab} \nabla^2 \Phi - g_{ab} \Lambda_2 \Phi = 0$$

AdS$_2$ black hole solution:

$$\Phi = \phi_b r$$

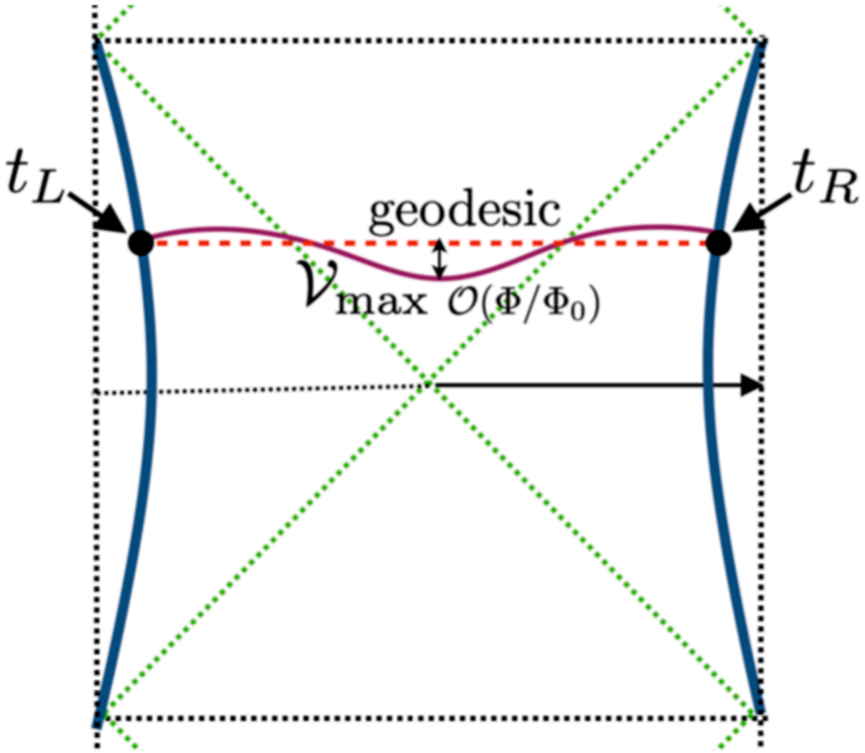
$$ds^2 = -\frac{r^2 - \mu}{L_2^2} dt^2 + \frac{L_2^2}{r^2 - \mu} dr^2$$

the dilaton blows up as $r \to \infty$

Near-horizon & near-extremal

→ cut off the spacetime at $r = r_c$

Complexity=Volume



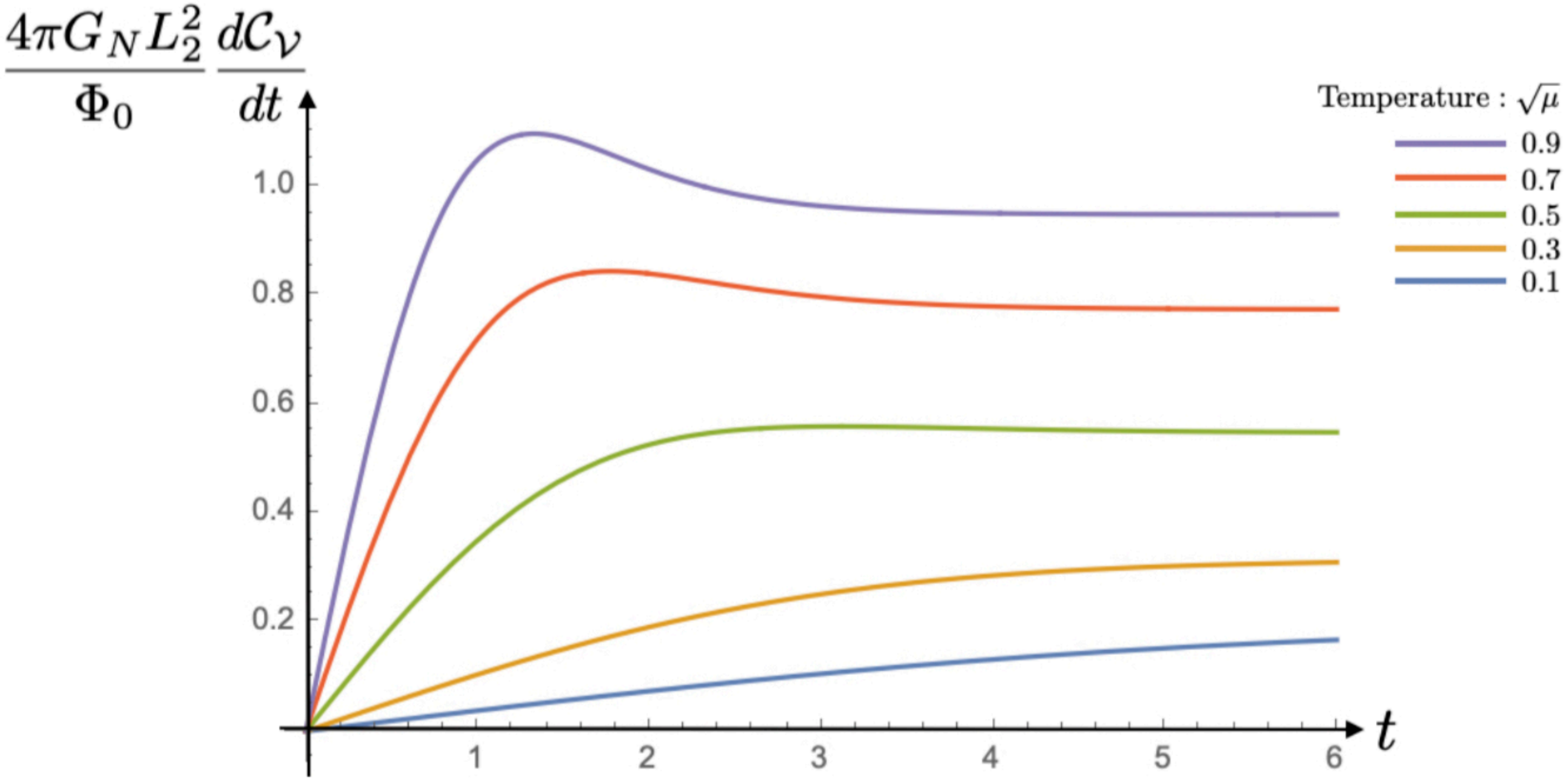
Compute the complexity in the **Complexity**=**Volume** proposal

$$\mathcal{C}_{\mathcal{V}} \equiv \frac{1}{G_N L} \max_{\Sigma=\partial\mathcal{B}} \mathcal{V}(\mathcal{B})$$

From the perspective of the dim reduction, the most natural definition of "volume" is:

$$\mathcal{C}_{\mathcal{V}} = \frac{1}{G_N L_2} \max_{\gamma} \int_{\gamma} d\lambda \sqrt{-h}(\Phi_0 + \Phi)$$

$$\rightarrow \quad \max_{\gamma} \int_{\gamma} d\lambda \sqrt{-h}[\Phi_0 + \Phi] = \int_{\text{geodesic}} d\lambda \sqrt{-h}[\Phi_0 + \Phi] + \mathcal{O}((\Phi/\Phi_0)^2)$$



At late times, the complexity grows linearly with time

$$\lim_{t\to\infty}\frac{d\mathcal{C}_\mathcal{V}}{dt}=2S_0T$$

S_0:extremal entropy T:temperature

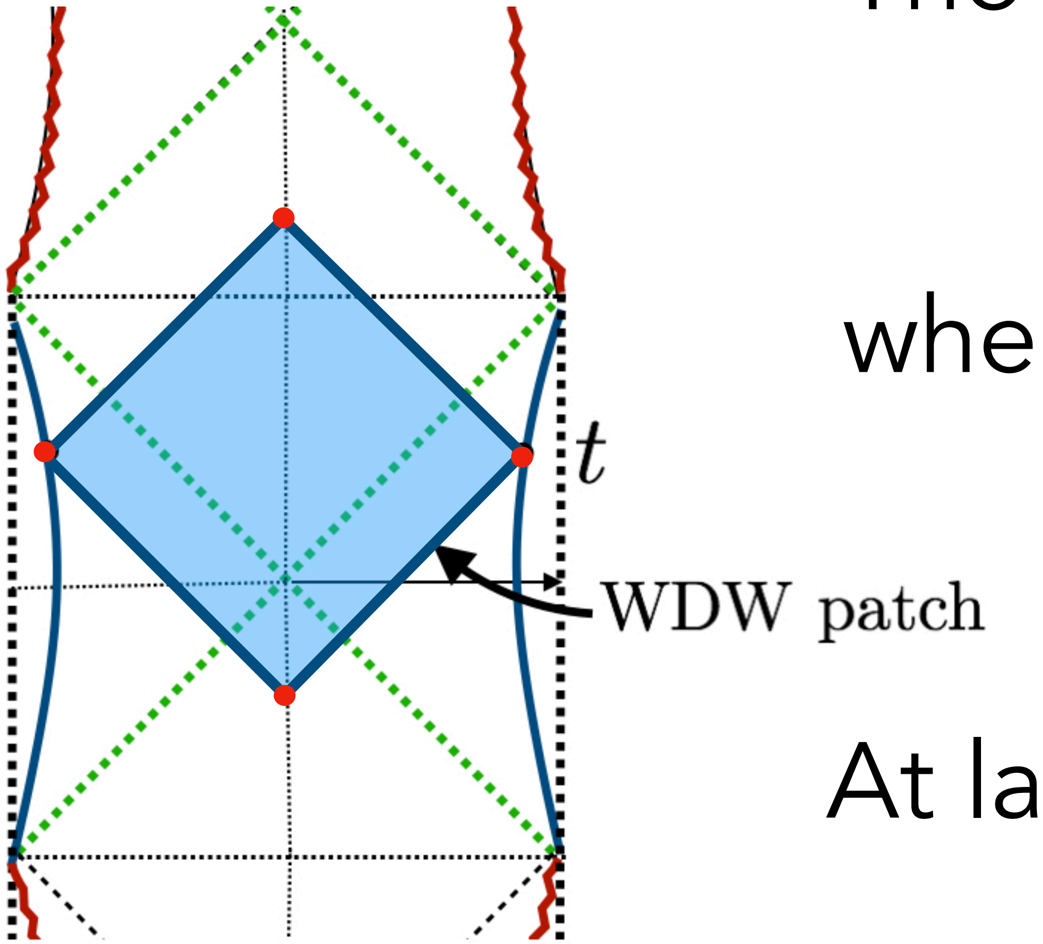
→Lloyd bound (upper bound of the complexity growth)

$$\text{Lloyd bound}=2ST\simeq 2S_0T$$

Complexity=Action

The bulk action of the JT model

$$I^{\mathrm{JT}}_{\mathrm{bulk}} = \frac{\Phi_0}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{-g}R + \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\Phi(R-2\Lambda_2)$$



The action complexity:

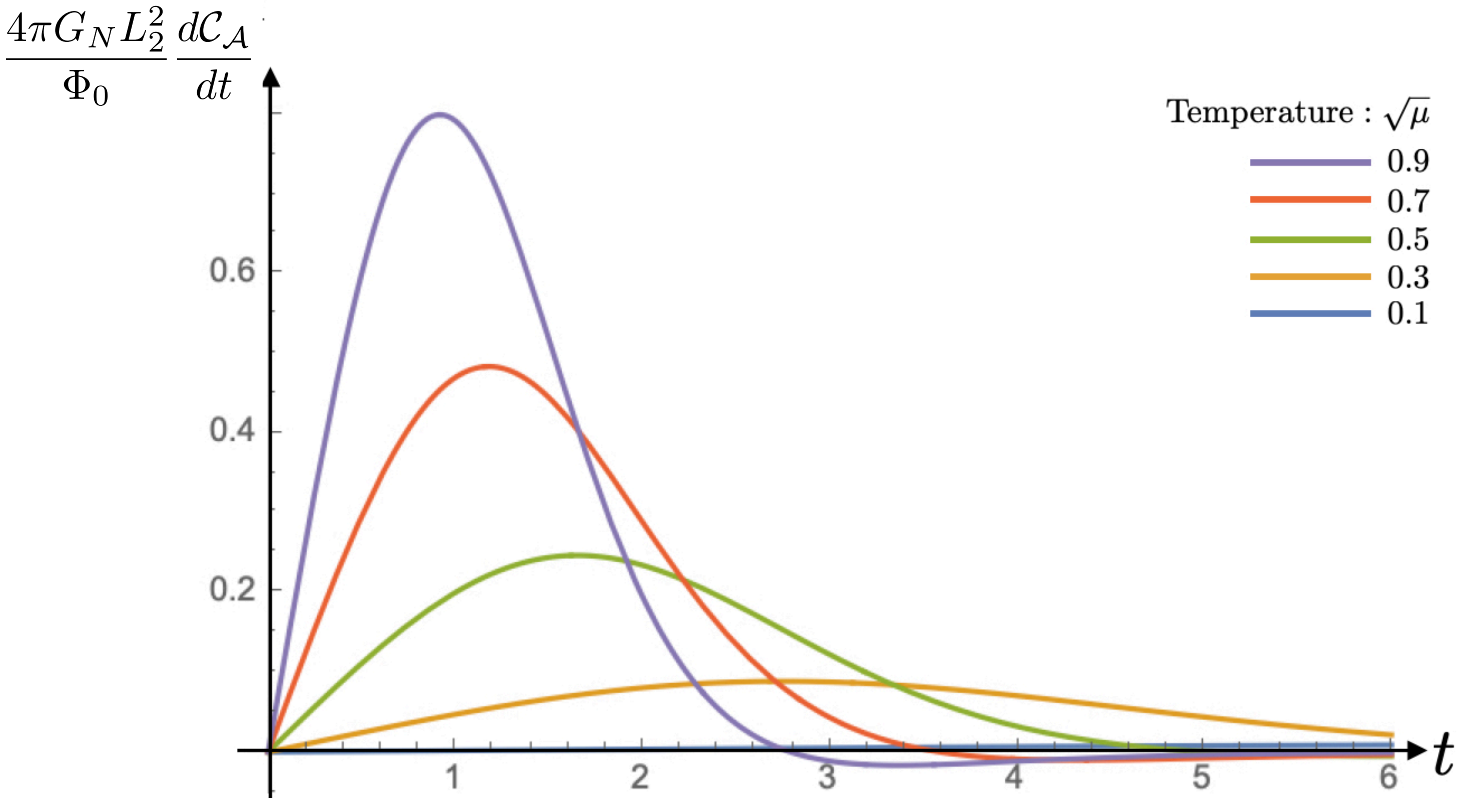
$$\mathcal{C}_{\mathcal{A}} \equiv I^{\mathrm{JT}}_{\mathrm{bulk}} + I^{\mathrm{JT}}_{\mathrm{boundary}}$$

where

$$I^{\mathrm{JT}}_{\mathrm{boundary}} \equiv I^{\mathrm{JT}}_{\mathrm{GHY}} + I^{\mathrm{JT}}_{\mathrm{joint}} + I^{\mathrm{JT}}_{\mathrm{boundary,ct}}$$

At late times,

$$\left.\begin{aligned} I^{\mathrm{JT}}_{\mathrm{bulk}} &\to -2S_0T \\ I^{\mathrm{JT}}_{\mathrm{boundary}} &\to 2S_0T \end{aligned}\right\}$$ cancel out !



Action complexity stops growing at a finite time!

$$\lim_{t\to\infty}\frac{d\mathcal{C}_\mathcal{A}}{dt}=0$$

→The A=C behavior is completely different from V=C.

Can see the same behavior in the 4d near extremal BHs?

The Reissner-Nordstrom black hole

- 4d Einstein-Maxwell action

$$I_{EM} = \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\sqrt{-\hat{g}}(\hat{R}-2\Lambda) + \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}}\sqrt{-\hat{\gamma}}\hat{K}$$
$$-\frac{1}{4g^2}\int_{\mathcal{M}} d^4x\sqrt{-\hat{g}}F_{\mu\nu}F^{\mu\nu}\,,$$

electrically/ magnetically charged BHs in AdS

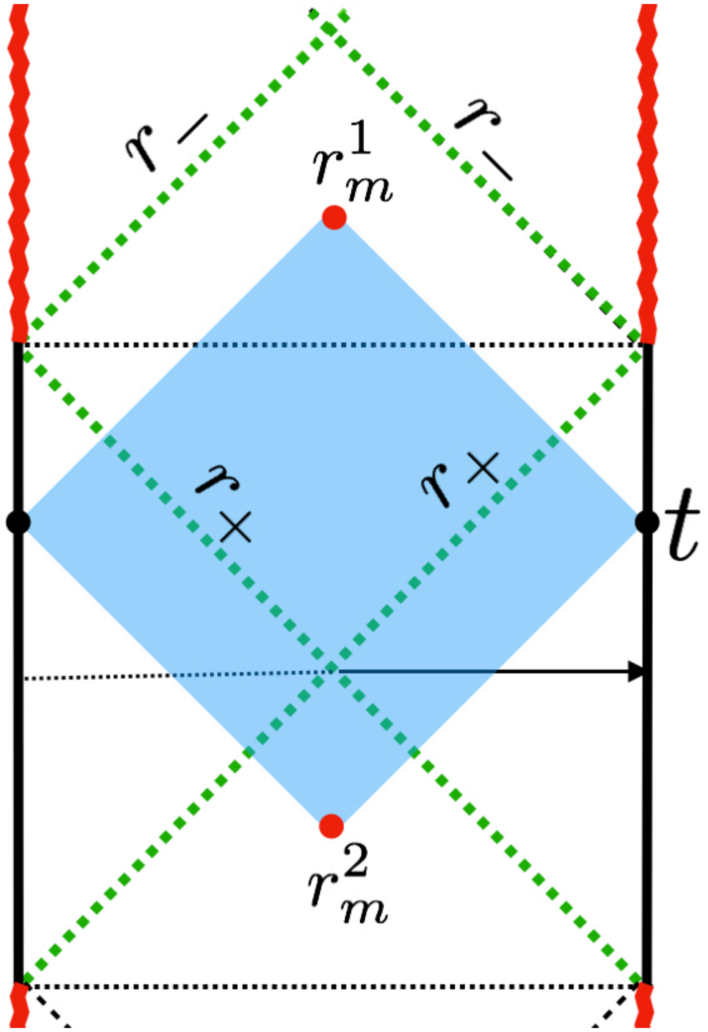
$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2$$

$$f(r) = 1 - \frac{2G_N M}{r} + \frac{4\pi(Q_e^2+Q_m^2)}{r^2} + \frac{r^2}{L^2}$$

magnetic $A_\phi = Q_m(1-\cos\theta)\,,\ \ F_{\theta\phi} = \partial_\theta A_\phi = Q_m\sin\theta$

electric $A_t = \Phi_\infty - \frac{Q_e}{r}\,,\qquad F_{rt} = \partial_r A_t = \frac{Q_e}{r^2}$

Complexity=Action of RN BHs



Evaluate the action complexity in RN BHs

$$\mathcal{C}_\mathcal{A} = I_{\text{bulk}}^{\text{EM}} + I_{\text{boundary}}$$

Bulk contribution:

electric

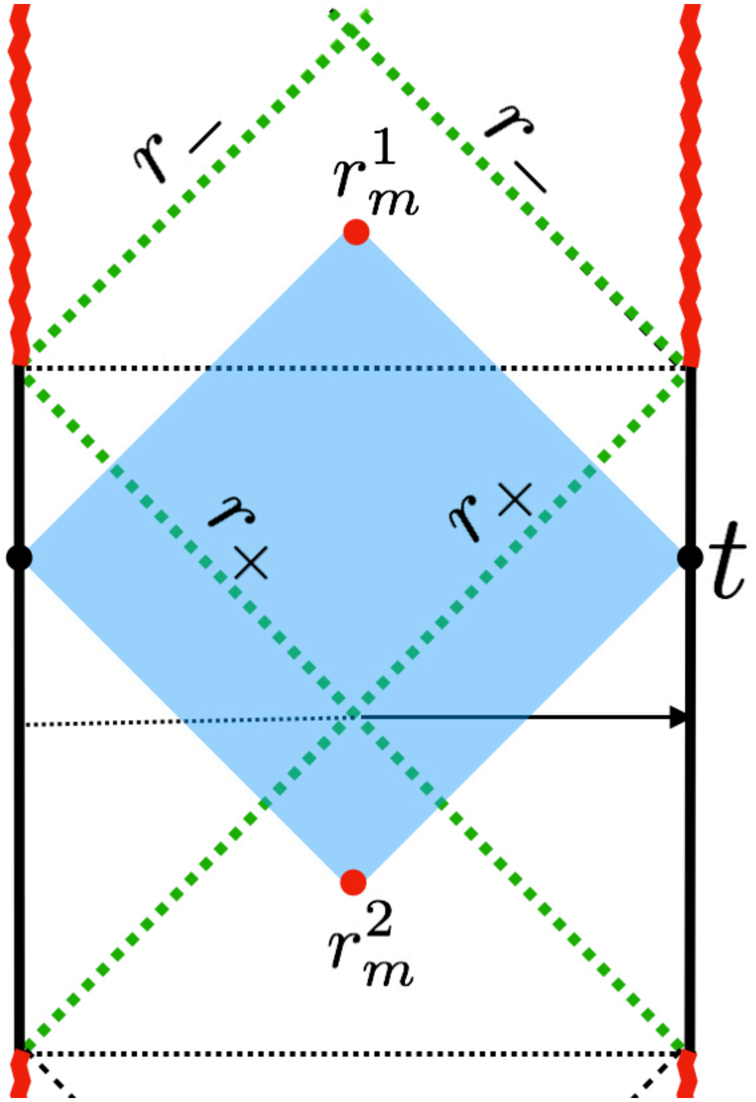
$$\frac{dI_{\text{bulk}}^{\text{EM}}}{dt} = \frac{1}{2G_N}\left[\frac{r^3}{L^2} \pm \frac{4\pi Q^2}{r}\right]_{r_m^2}^{r_m^1}$$

magnetic

$$\because F^2 \sim B^2 - E^2$$

Boundary contribution:

$$\frac{dI_{\text{boundary}}^{\text{EM}}}{dt} = -\frac{1}{4G_N}\left[2rf(r)\log|f(r)| + r^2\partial_r f(r)\right]_{r_m^2}^{r_m^1}$$



Electric:

$$\frac{d\mathcal{C}_{\mathcal{A}}}{dt} = \frac{1}{2G_N}\left[\frac{4\pi Q^2}{r}\right]_{r_m^2}^{r_m^1} - \left[\frac{r f(r)}{2\pi G_N}\log|f(r)|\right]_{r_m^2}^{r_m^1}$$

$$\rightarrow \frac{1}{2G_N}\left[\frac{4\pi Q^2}{r_-} - \frac{4\pi Q^2}{r_+}\right]$$

Magnetic:

$$\frac{d\mathcal{C}_{\mathcal{A}}}{dt} = -\left[\frac{r f(r)}{2\pi G_N}\log|f(r)|\right]_{r_m^2}^{r_m^1} \rightarrow 0$$

Consistent with the fact that JT model is derived from a magnetically charged BHs!

Comments on the Maxwell bdy term

One can add the Maxwell boundary term

$$\mathcal{C}_\mathcal{A} = I^{\text{EM}}_{\text{bulk}} + I_{\text{boundary}} + I_{\text{Max-boundary}}$$

where

$$I_{\text{Max-boundary}} = \frac{\gamma}{G_N} \int_{\partial\mathcal{M}} d^3x \sqrt{-\gamma} F^{\mu\nu} A_\mu n_\nu$$

n_μ :the outward directed normal vector to the boundary

It changes the late time behavior of the complexity:

Electric: $$\lim_{t\to\infty} \frac{d\mathcal{C}_\mathcal{A}}{dt} = (1-\gamma) \left. \frac{2\pi Q_e^2}{r} \right|_{r_+}^{r_-} \xrightarrow{\gamma \to 1} 0$$

Magnetic: $$\lim_{t\to\infty} \frac{d\mathcal{C}_\mathcal{A}}{dt} = \gamma \left. \frac{2\pi Q_m^2}{r} \right|_{r_+}^{r_-} \xrightarrow{\gamma \to 1} \textit{finite}$$

electric charge doesn't contribute to the complexity!

The Max bdy term for a physical boundary:
changes the b.c. imposed on the gauge field A_μ

→changes the thermodynamic ensemble to which gravitational solutions belong from the Euclidean path integral perspective [Hawking-Ross '93]

for example..

electric with Max bdy term
magnetic without Max bdy term

} fixed-charge ensemble

$$\lim_{t\to\infty}\frac{d\mathcal{C}_\mathcal{A}}{dt}=0$$

→Action complexity depends on the ensemble?

We found: action complexity of **electric** BHs described by I_{EM} (without Maxwell boundary term) shows linear growth rate at late times

Q. Can we see the similar behavior of complexity in the dimensionally reduced dilaton gravity theory?

ansatz for the metric and the Maxwell field:

$$ds^2 = \underline{g_{ab}(x^a)}dx^a dx^b + \underline{\Psi^2(x^a)}\underline{(d\theta^2 + \sin^2\theta d\phi^2)}$$

dynamical d.o.f in 2d fix

electric: $\underline{F_{ab} \neq 0}, (\text{other components of } F) = 0$

keep F_{ab} as the dynamical d.o.f in 2d

$$
\begin{aligned}
I_{\mathrm{GC}}^{\mathrm{electric}} &= \frac{1}{4G_N}\int_{\mathcal{M}} d^2x\sqrt{-g}\left(\Psi^2 R + 2(\nabla\Psi)^2 + 2 - 2\Psi^2\Lambda - 4\pi\Psi^2 F^2\right) \\
&\quad + \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}}\sqrt{-\gamma}K.
\end{aligned}
$$

→2d dilaton gravity coupled to the 2d Maxwell field

- Near-horizon & near-extremal→expand Ψ^2 around the area of the horizon of the extremal BH

$$\Psi^2 = \frac{1}{4\pi}(\Phi_0 + \Phi)\,,\;\; \Phi_0 = 4\pi r_h^2$$

area of the extremal BH

with $\Phi/\Phi_0 \ll 1$

also wish to only capture small corrections to the extremal field strength $F_0^2 \equiv -Q_{\mathrm{ext}}^2/8\pi^2 r_h^4$

$$F_{ab} = (F_0)_{ab} + \tilde{F}_{ab} = 2\partial_{[a}(A_0)_{b]} + 2\partial_{[a}\tilde{A}_{b]}$$

Expand the action to the linear order both in Φ and $\tilde{F}$

"JT-like model"

$$I_{\text{JT-like}} = I_{\text{JT}} - \frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{-g}(\Phi_0 - \Phi)\Lambda_2 \\ - \frac{1}{4G_N}\int_{\mathcal{M}} d^2x\sqrt{-g}\left[(\Phi_0+\Phi)F_0^2 + 2\Phi_0 F_0^{\mu\nu}\tilde{F}_{\mu\nu}\right]$$

Solution:

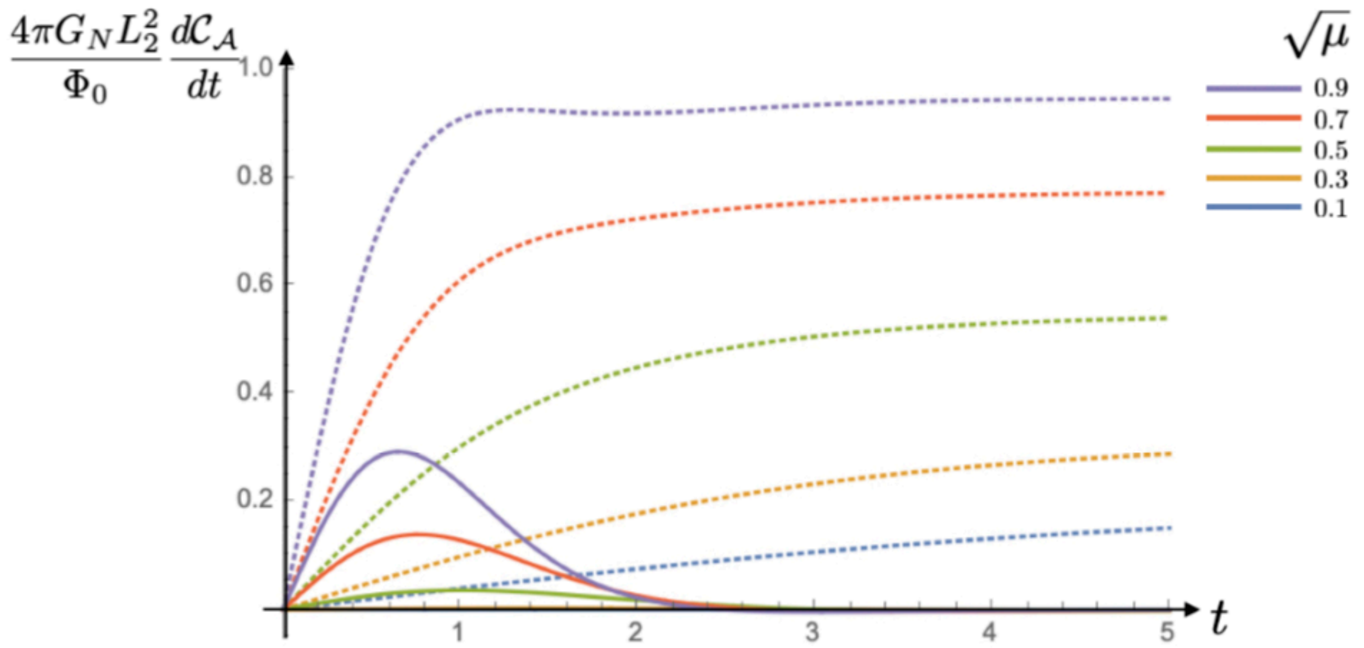
the metric & dilaton: same eqs as the JT model
→ AdS$_2$ solution

$$0 = R - 2\Lambda_2$$
$$0 = \nabla_\mu\nabla_\nu\tilde{\Phi} - g_{\mu\nu}\nabla^2\tilde{\Phi} - g_{\mu\nu}\Lambda_2\tilde{\Phi}$$

with a Maxwell field background

$$(F_0)_{\mu\nu} = \frac{Q_{\text{ext}}}{\Phi_0}\epsilon_{\mu\nu}\,,\ \ \tilde{F}_{\mu\nu} = -\frac{Q_{\text{ext}}}{\Phi_0^2}\tilde{\Phi}\epsilon_{\mu\nu}$$

Holographic Complexity in the "JT-like" model



On-shell actions of the JT & JT-like model differ by the 2d Maxwell boundary term

$$I_{\text{JT-like}}\big|_{\text{on-shell}} = I_{\text{JT}}\big|_{\text{on-shell}} - I_{\text{Max-boundary}}$$

→additional contribution to the complexity growth

$$\lim_{t\to\infty}\frac{d\mathcal{C}_\mathcal{A}{}^{\text{JT-like}}}{dt} = 2S_0T$$: agree with CV

Summary

- The JT model shows **vanishing growth rate** of action complexity at late times while the volume complexity grows linearly in time.

- 4d magnetic black holes (without Maxwell bdy term) show the similar late time behavior of action complexity to the JT model.

- 4d electric black holes (without Maxwell bdy term) and the corresponding 2d dilaton gravity theory show **linear growth rate** of action complexity at late times

Questions and Future Problem

Q. Action ``**complexity**'' is really complexity?

it stops growing in some cases….

If it is, which action should we use?

w/ or w/o the Maxwell bdy term?

Complexity depends on the **thermodynamic ensemble**?

→Can we built a circuit model which reproduces the C=A behavior in the fixed charge ensemble?

↑vanishing growth rate

Q.Can we define complexity in SYK model?