Conformal bootstrap: A biased review

Aninda Sinha

Indian Institute of Science

P(R. Gopakumar, K. Sen, A. Kaviraj, K. Ghosh, P. Dey, A. Zahed, L. F. Alday, P. Ferrero)

IPMU June `19

The Polyakov bootstrap

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In the late '60's, Migdal and Polyakov[64] developed a "bootstrap" formulation

of critical phenomena based on a skeleton Feynman graph expansion, in which all parameters including the expansion parameter inself would be determined self-consistently. They were unable to solve the bootstrap equations because of their complexity, although after the ε expansion about four dimensions was discovered, Mack showed that the bootstrap could be solved to lowest order in ε. If the 1971 renormalization group ideas had not been developed, the Migdal-Polyakov bootstrap would have been the most promising framework of its time for trying to further understand critical phenomena. However, the renormalization group methods have proved both easier to use and more versatile, and the bootstrap receives very little attention today.

Wilson-Nobel lecture 1982

PHYSICAL REVIEW D VOLUME 7, NUMBER 10 15 MAY 1973

Quantum Field-Theory Models in Less Than 4 Dimensions*

Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 9 November 1972)

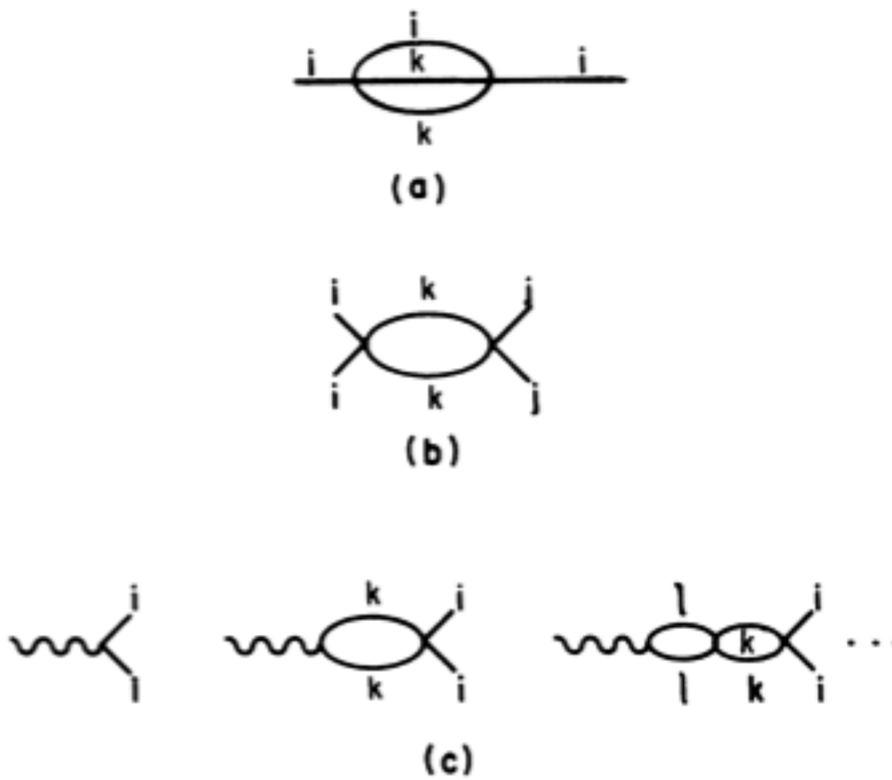


FIG. 1. (a) Diagram giving the lowest-order correction to the propagator. The two lines with internal index k form a loop; the sum over k gives a factor of N. (b) Diagram giving the leading correction to the four-point function (k is summed over). (c) Bubble graphs for vertex function involving ϕ^2 or $\bar{\psi}\psi$. The wavy line represents ϕ^2 or $\bar{\psi}\psi$; the straight lines refer to the elementary fields ϕ, ψ, or $\bar{\psi}$. The indices k and l are summed over.

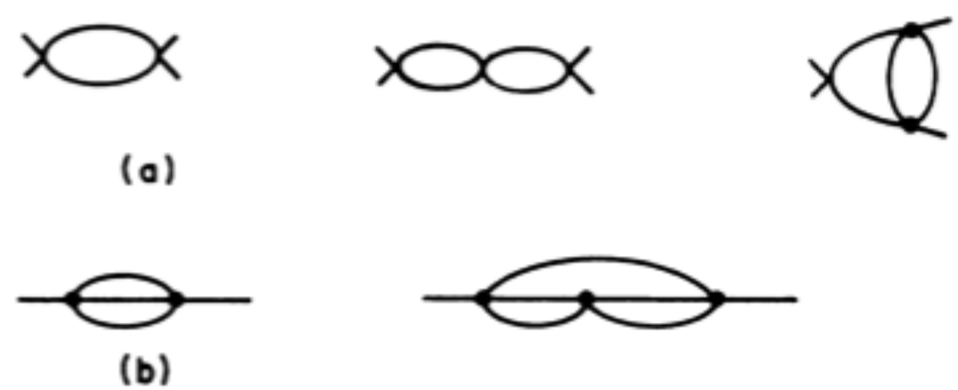


FIG. 2. (a) Diagrams determining u_R to order u_0^3. (b) Diagrams determining $D(q)$ to order u_0^2.

Epsilon expansion

Epsilon-expansion; Wilson; Wilson-Fisher; Wilson-Kogut; Polyakov;.Mack…..Rychkov, Tan

O(N) model

$$\int d^{4-\epsilon}x \, \left[(\partial_\mu \phi^i)^2 + \lambda(\phi^i\phi^i)^2\right]$$

- Wilson-Fisher fixed point.
- 3d Ising model (critical point of water)

  $N = 1, \epsilon = 1$
- 2d Ising model

  $N = 1, \epsilon = 2$
- XY model

  $N = 2, \epsilon = 1$

This is an asymptotic series!

$$\Delta_{\phi^2} = d - 2 + \frac{N+2}{N+8}\epsilon + \frac{N+2}{2(N+8)^3}(13N+44)\epsilon^2$$

Ising

$$d = 2 \to 1.136$$

$$\text{actual} = 1$$

$$d = 3 \to 1.45$$

$$\text{numerics} \approx 1.41$$

$$\text{expts} \approx 1.41$$

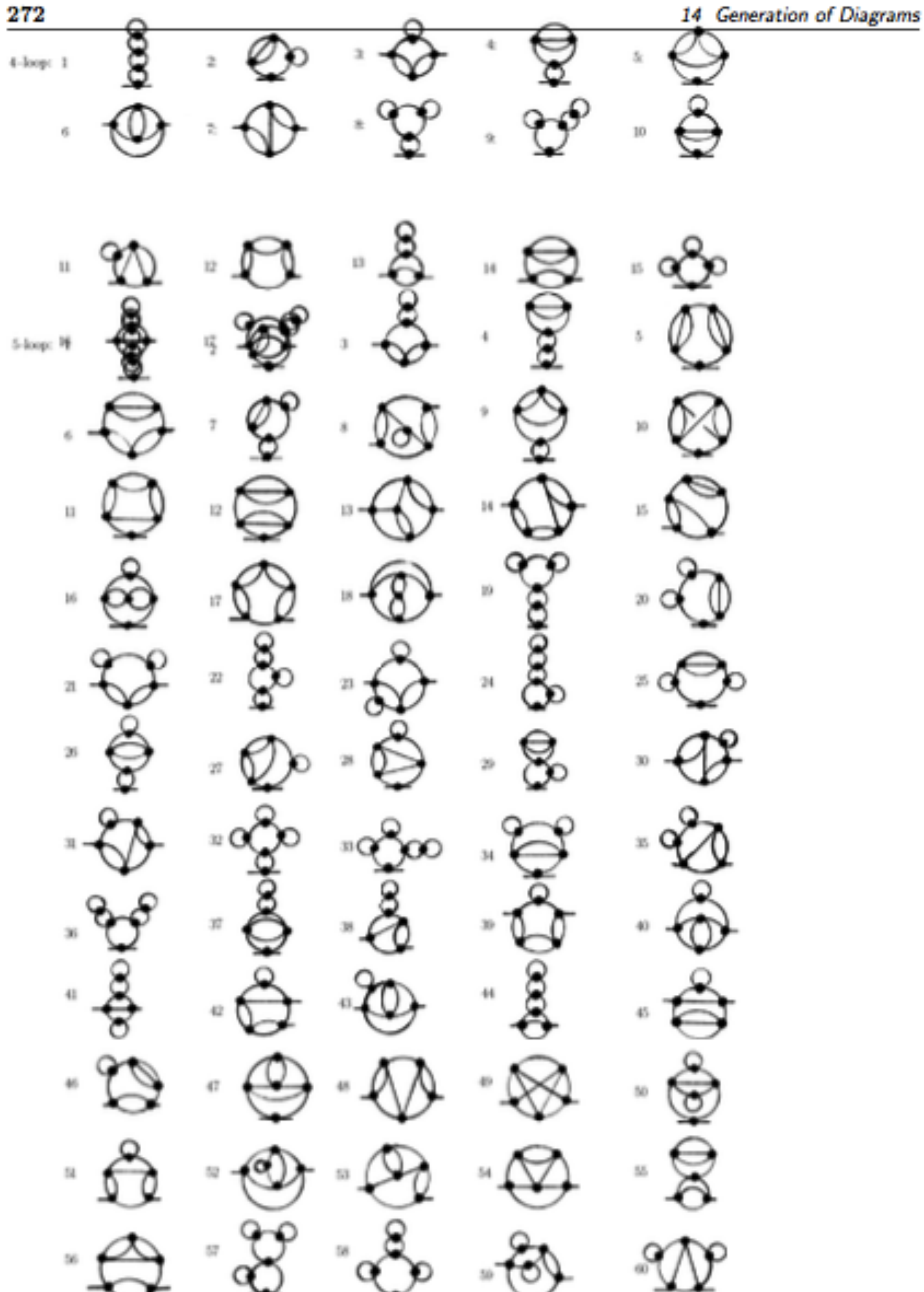
XY

$$d = 3 \to 1.54$$

$$\text{expt} \approx 1.51$$

International
space station
superfluid He
experiment

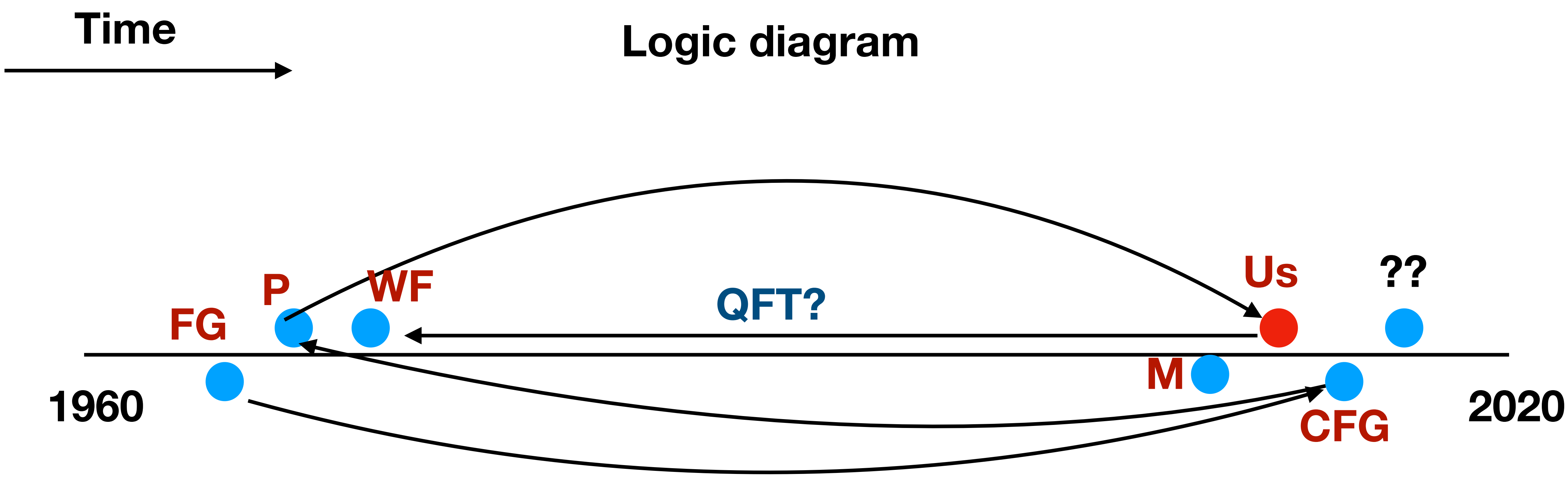
Wilson, Wilson-Fisher
1970's: RG



Your reaction at being told to verify this !!

Conformal scream

A new incarnation of the conformal bootstrap will enable us to produce the Wilson-Fisher results algebraically, including yielding new results for OPE. No Feynman diagrams will be needed.


Time
Logic diagram
QFT?
P
WF
FG
Us
??
M
CFG
1960
2020

Review of standard bootstrap

Operator Product Expansion

$$\phi(0)\phi(x) \sim \sum c_{\Delta,\ell}(x^2)^{\Delta_O/2-\Delta_\phi-\ell/2} x^{a_1}\cdots x^{a_\ell} O_{a_1\cdots a_\ell}(0)$$

- Operator relation.

Review of standard bootstrap

Operator Product Expansion

CFT data

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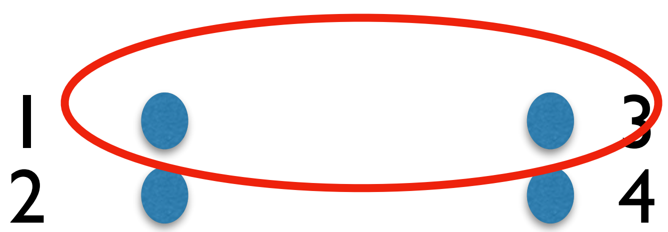
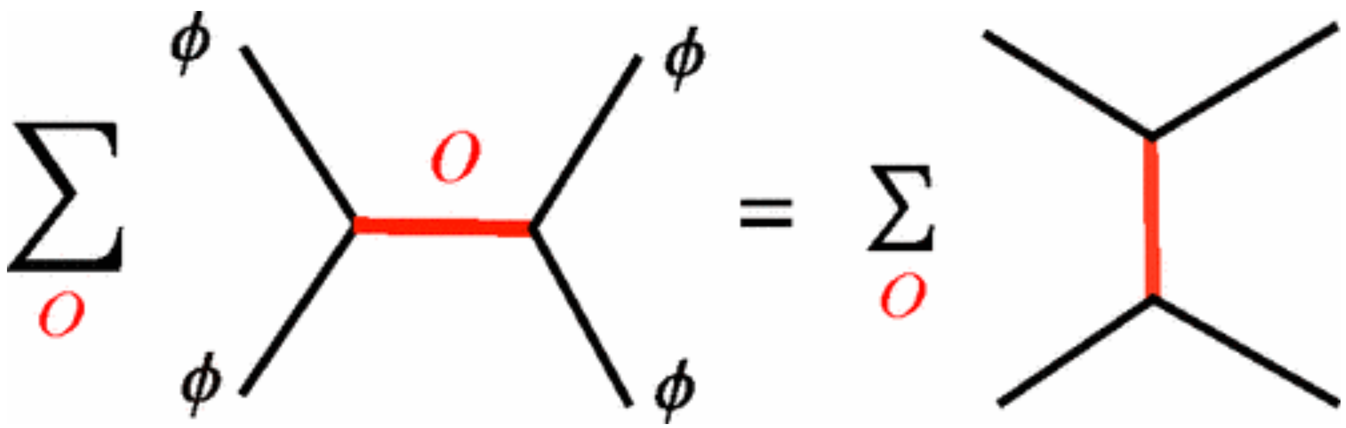


$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle\phi(x_1)\phi(x_4)\phi(x_3)\phi(x_2)\rangle$$



"direct-channel"

"crossed-channel"

Crossing

Early work: Ferrara, Gatto, Grillo; Polyakov; Modern revival: Rattazzi, Rychkov, Tonni, Vichi; Many, many others

"Cross-ratios"

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$u = z\bar{z}, v = (1-z)(1-\bar{z})$$

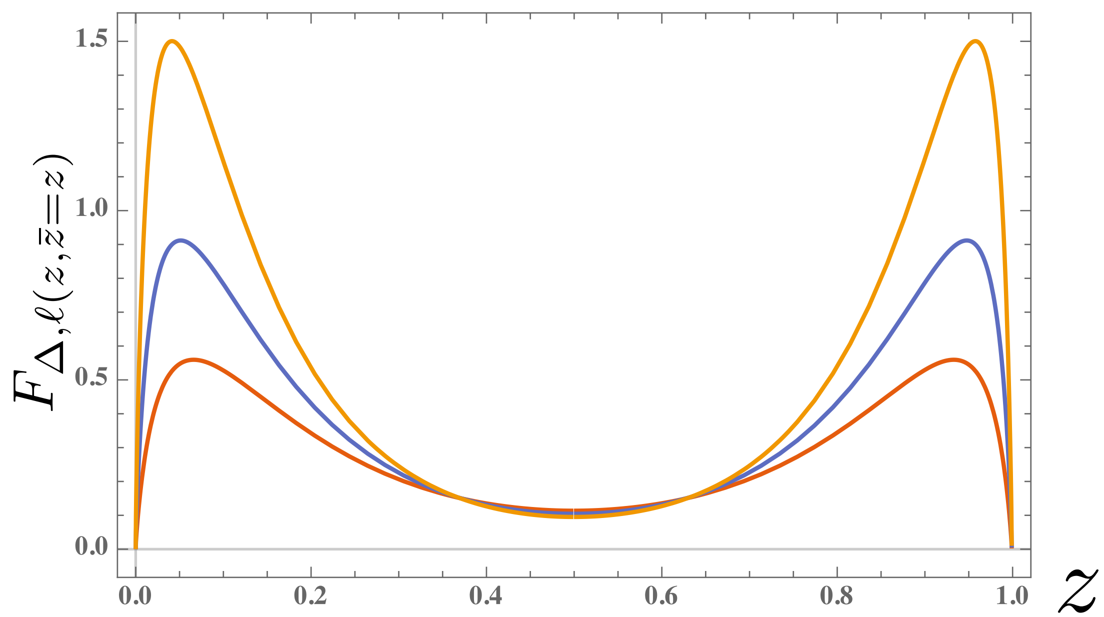
$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \mathcal{A}(u,v)$$

CROSSING EQUATION

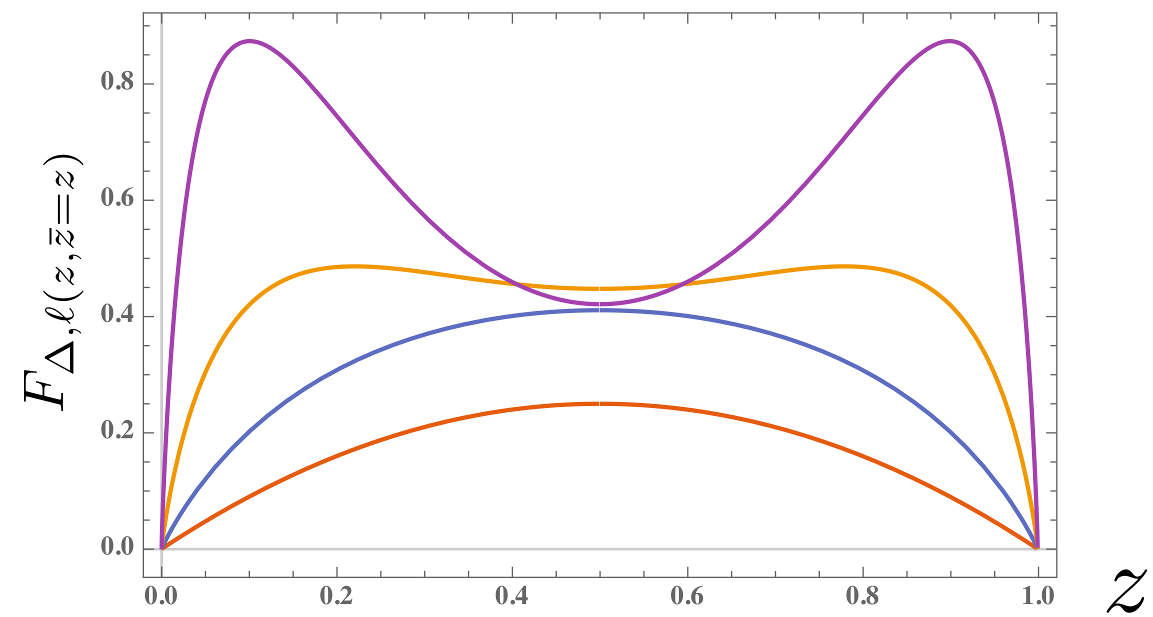
$$\mathcal{A}(u,v) = (\frac{u}{v})^{\Delta_\phi} \mathcal{A}(v,u)$$

NON-PERTURBATIVE!!

$$1 = \sum_{\Delta,\ell} c_{\Delta,\ell} F_{\Delta,\ell}(z,\bar{z})$$ **CROSSING EQUATION**



$$\ell \neq 0, \Delta \geq d - 2 + \ell$$

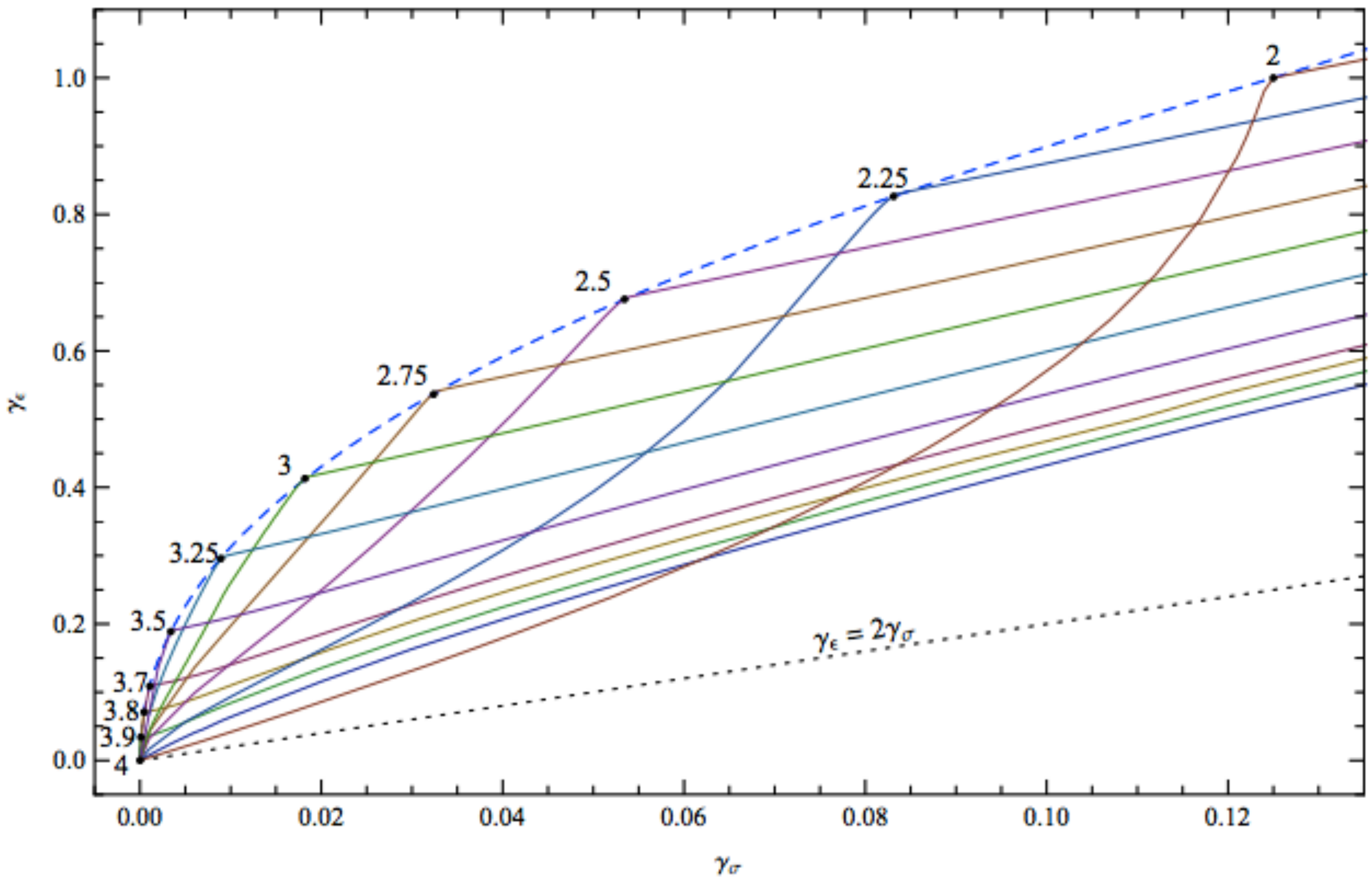


$$\ell = 0, \Delta \geq \frac{d-2}{2}$$

$$F_{\Delta,\ell} : \text{Combination of conformal blocks}$$

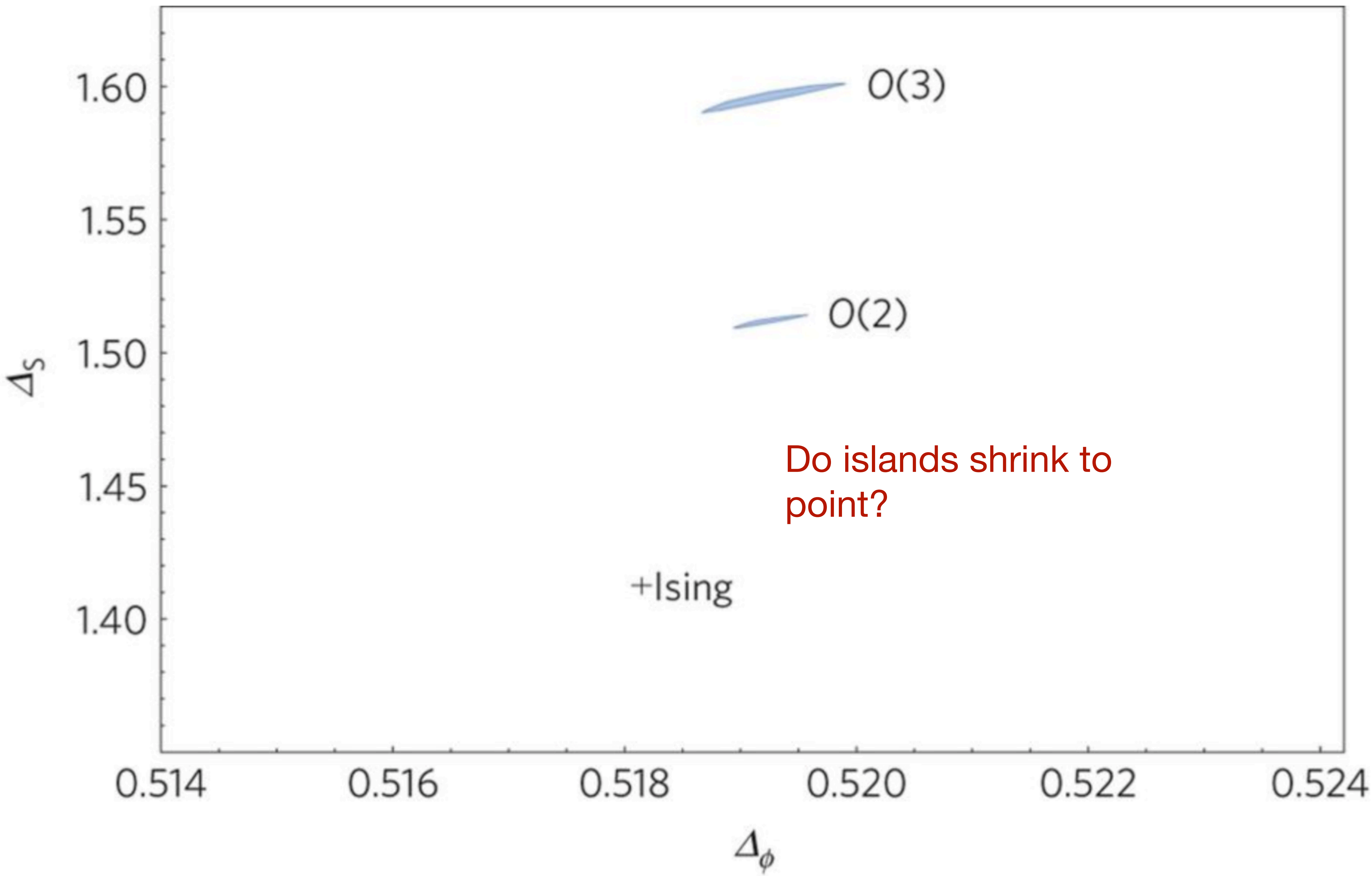
Observe shape around mid-point. With non-zero spins satisfying unitarity bounds, it is impossible to satisfy the bootstrap constraints. We need at least one scalar in the OPE with some bounded conformal dimension. Plot allowed dimension of lightest scalar in OPE vs dimension of external scalar.

Numerical bounds in fractional dimensions



1309.5089, El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi building on 2008 work by Rattazzi, Rychkov, Tonni and Vichi

Islands from multiple correlators

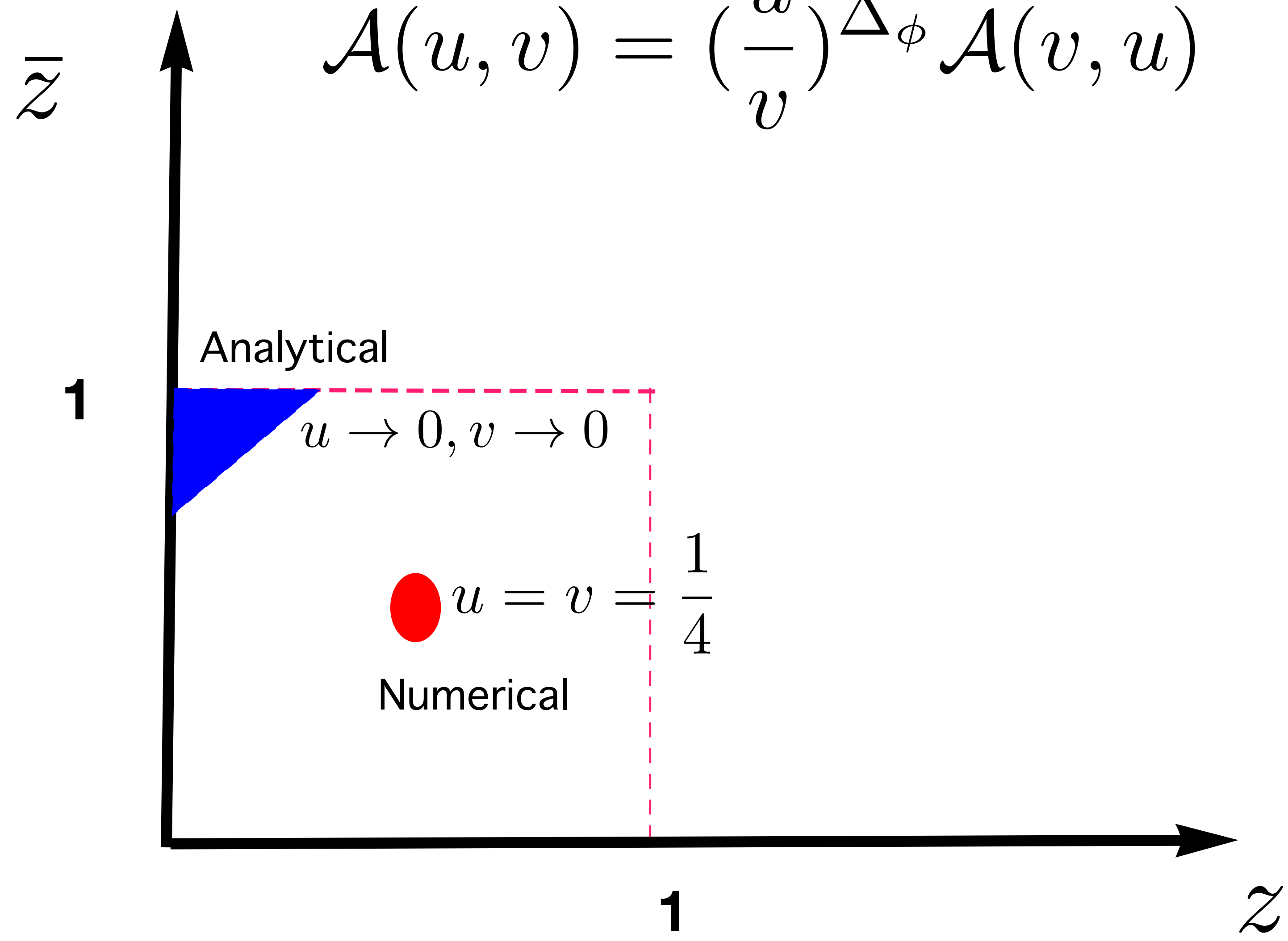


Do islands shrink to point?

Kos, Poland, Simmons-Duffin, Vichi 2016

Numerical vs Analytical

$$\mathcal{A}(u,v) = (\frac{u}{v})^{\Delta_\phi} \mathcal{A}(v,u)$$



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The simplest question: 2d Ising model

1d-bootstrap

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$$A(z) = \frac{1}{(1-z)^{\frac{1}{4}}} = \sum_{\Delta} C_{\Delta} z^{\Delta} {}_2F_1(\Delta,\Delta,2\Delta,z)$$

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$$C_{\Delta} = \left\{1, \frac{1}{4}, \frac{1}{32}, \frac{1}{384}, \frac{7}{10240}, \frac{29}{286720}, \frac{107}{4128768}, \frac{277}{60555264}, \frac{4183}{3598712832}, \frac{48337}{215922769920}, \frac{184837}{3262832967680}, \dots\right\}$$

The simplest question: 2d Ising model

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$$C_\Delta = \frac{\sqrt{\pi}(-1)^n 2^{-2(n+1)}\Gamma\left(-\frac{1}{4}\right)\Gamma(n)\,{}_3F_2\left(-\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{7}{4}-n,n+\frac{3}{4};1\right)}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{7}{4}-n\right)\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{3}{4}\right)}$$

$$C_{\Delta=n} \xrightarrow{n\to\infty} \frac{2^{\frac{3}{2}-2n}\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi}n\Gamma\left(\frac{1}{4}\right)}$$

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$$s \to 1/4 - s$$

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Mellin transform of block

$$F^{(2)}_{\hat{\Delta}}(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds \left(\frac{z}{1-z}\right)^s M_{\hat{\Delta}}(s)$$

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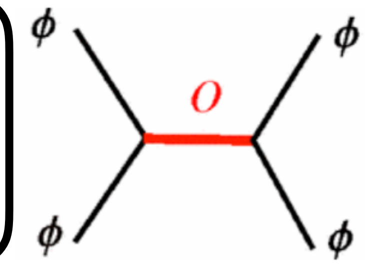

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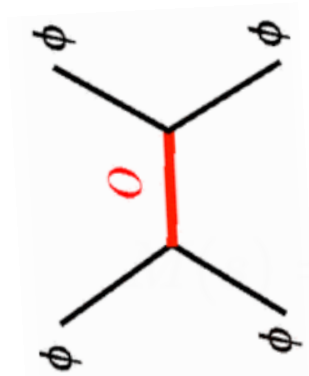

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$$\underbrace{\sum_{\Delta} C_{\Delta} M_{\Delta}(s)}_{simple\ poles} = \underbrace{\sum_{\Delta} C_{\Delta} M_{\Delta}(2\Delta_{\phi} - s)}_{double\ poles}$$

Close on right

Position space: power law

Position space: logs

$$M(s) = \frac{\Gamma(-\frac{1}{4} + s)\Gamma(-s)}{\Gamma(-\frac{1}{4})}$$

Symmetric under $s \to \frac{1}{4} - s$

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Symmetric under $s \to \dfrac{1}{4} - s$

1. Can we expand using a basis that makes this symmetry manifest: a crossing symmetric basis?

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Symmetric under $s \to \dfrac{1}{4} - s$

1. Can we expand using a basis that makes this symmetry manifest: a crossing symmetric basis?
2. Cannot just add the crossed block as it has double poles on the right which would be incompatible with s-channel OPE.

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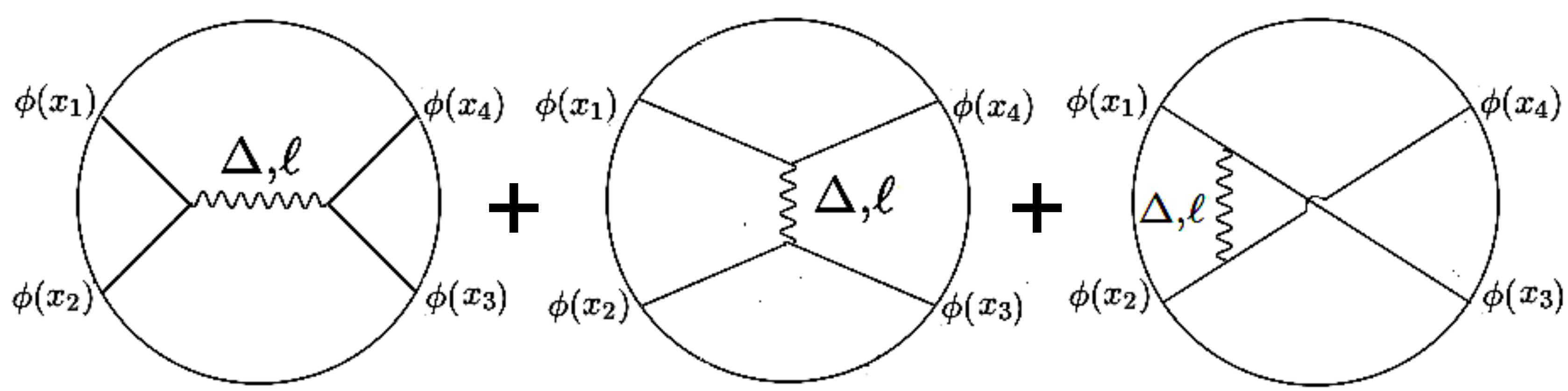
Symmetric under $s \to \dfrac{1}{4} - s$

1. Can we expand using a basis that makes this symmetry manifest: a crossing symmetric basis?
2. Cannot just add the crossed block as it has double poles on the right which would be incompatible with s-channel OPE.
3. This gives a hint: we should add these spurious poles also in the s-channel with the hope that together with the crossed channel, they will cancel.

Can we use a crossing symmetric Witten diagram-like basis? A potential advantage is that in order to get something that is manifestly crossing symmetric we will not have to sum over an infinite number of operators. Flip side: check consistency with OPE.

$$\mathcal{M}(s,t) = \sum_{\Delta,\ell} c_{\Delta,\ell}$$



Mellin amplitude ?=? Sum over exchange Witten diagrams

Polyakov-Mellin bootstrap

Non-Hamiltonian approach to conformal quantum field theory

A. M. Polyakov

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences

(Submitted July 9, 1973)
Zh. Eksp. Teor. Fiz. 66, 23–42 (January 1974)

The completeness requirement for the set of operators appearing in field theory at short distances is formulated, and replaces the *S*-matrix unitarity condition in the usual theory. Explicit expressions are obtained for the contribution of an intermediate state with given symmetry in the Wightman function. Together with the "locality" condition, the completeness condition leads to a system of algebraic equations for the anomalous dimensions and coupling constants; these equations can be regarded as sum rules for these quantities. The approximate solutions found for these equations in a space of $4-\epsilon$ dimensions give results equivalent to those of the Hamiltonian approach.

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What does this have to do with Witten diagrams?!!

- 1510.07770 (w K. Sen)
- 1609.00572, 1611.08407(w R. Gopakumar, A. Kaviraj and K. Sen), 1612.05032 (w A. Kaviraj and P. Dey), 1709.06110 (w P. Dey and Kausik Ghosh)
- 1809.10975 (w R. Gopakumar)

- This new approach is connected to using tree level exchange Witten diagrams as a kinematical basis.

- You should think about this as a crossing symmetric kinematical basis. Suggestive of a dual AdS description.

Hjalmar Mellin (Finnish mathematician)

$$f(x) = \int_{c-i\infty}^{c+i\infty} \tilde{f}(s) x^{-s} \, ds$$

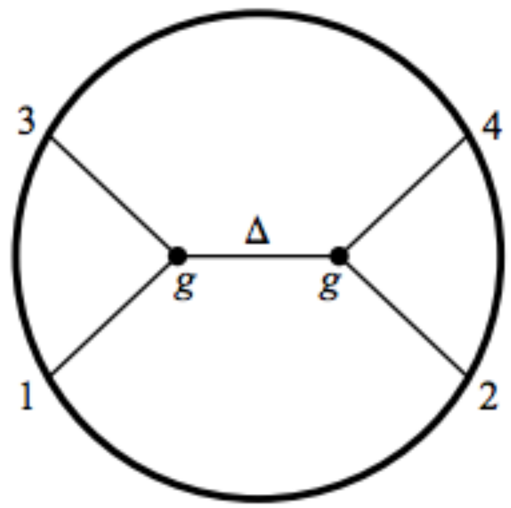
$$\mathcal{A}(u,v) = \int \frac{ds dt}{(2\pi i)^2} \Gamma(-t)^2 \Gamma(s+t)^2 \Gamma(\Delta_\phi - s)^2 \mathcal{M}(s,t) u^s v^t$$

$\mathcal{M}(s,t)$: Mellin amplitude

Witten diagrams in 1974

MELLIN SPACE

PRESENT DAY



$$= \frac{1}{2s-\Delta}\frac{\Gamma^2(\Delta_\phi+\frac{\Delta-2h}{2})}{\Gamma(1+\Delta-h)}\,{}_3F_2\left[\begin{matrix}1-\Delta_\phi+\frac{\Delta}{2}, 1-\Delta_\phi+\frac{\Delta}{2}, \frac{\Delta}{2}-s \\ 1+\frac{\Delta}{2}-s, 1+\Delta-h\end{matrix};1\right]$$

Mack; Penedones; Paulos

$$= \frac{1}{4\pi i \Gamma(\Delta_\phi - s)^2}\int_{-i\infty}^{i\infty} d\nu q[\nu]q[-\nu]$$

$$q[\nu] = \frac{\Gamma(\frac{h+\nu}{2}-s)\Gamma^2(\frac{2\Delta_\phi-h+\nu}{2})}{(\Delta-h)+\nu}$$

1974

$$\mathcal{M}(\nu) = q[\nu]q[-\nu]$$

Spectral function. Polyakov gave a different physical argument for the double poles. Exactly the same form!! Momentum/position space are not ideal to see the simplification we see in Mellin space.

Conformal block—Dolan & Osborn

Mack polynomials

$$G_{\Delta,\ell}(u,v) = \int_{-i\infty}^{i\infty} dsdt\, u^s v^t \Gamma^2(-t)\Gamma^2(s+t)\Gamma^2(\Delta_\phi - s)\frac{\Gamma(\frac{\Delta-\ell}{2}-s)\tilde{\Gamma}(\frac{(2h-\Delta-\ell)}{2}-s)}{\Gamma^2(\Delta_\phi - s)} P_{\Delta,\ell}(s,t)$$

$$\mathcal{A}(u,v) = \sum_{\Delta,\ell} C_{\Delta,\ell} G_{\Delta,\ell}(u,v)$$

$$d = 2h$$

Impose crossing symmetry as a constraint.

AdS-Witten diagram

$$\frac{1}{2s-\Delta}\frac{\Gamma^2(\Delta_\phi+\frac{\Delta-2h}{2})}{\Gamma(1+\Delta-h)}{}_3F_2\left[\begin{matrix}1-\Delta_\phi+\frac{\Delta}{2},1-\Delta_\phi+\frac{\Delta}{2},\frac{\Delta}{2}-s\\ 1+\frac{\Delta}{2}-s,1+\Delta-h\end{matrix};1\right]$$

Conformal block

$$\frac{\Gamma(\frac{\tau}{2}-s)\Gamma(\frac{\tilde{\tau}}{2}-s)}{\Gamma^2(\Delta_\phi-s)}$$

Do not look anything like each other!!

A quick derivation of the Witten diagram meromorphic piece

$$\frac{\Gamma(\tau/2-s)\Gamma(\tilde{\tau}/2-s)}{\Gamma^2(\Delta_\phi-s)}=\frac{B(\underbrace{\tau/2-s,}\Delta_\phi-\tau/2)\Gamma(\tilde{\tau}/2-s)}{\Gamma(\Delta_\phi-\tau/2)\Gamma(\Delta_\phi-s)}$$

$$=\sum_n \frac{(-1)^n(\Delta_\phi-\tau/2-n)_n}{(\tau/2+n-s)\Gamma(\Delta_\phi-\tau/2)}\frac{\Gamma(\tilde{\tau}/2-\tau/2-n)}{\Gamma(\Delta_\phi-\tau/2-n)}+\text{regular}$$

$$=\frac{\Gamma(h-\Delta)}{\tau/2-s}\frac{\sin^2\pi(\Delta_\phi-s)}{\sin^2\pi(\Delta_\phi-\frac{\tau}{2})}{}_3F_2\left[\begin{matrix}\tau/2-s,\tau/2-\Delta_\phi+1,\tau/2-\Delta_\phi+1\\ \tau/2-s+1,\Delta-h+1\end{matrix};1\right]$$

$$+\text{regular}$$

This gets multiplied by the Mack polynomial

A quick derivation of the Witten diagram meromorphic piece

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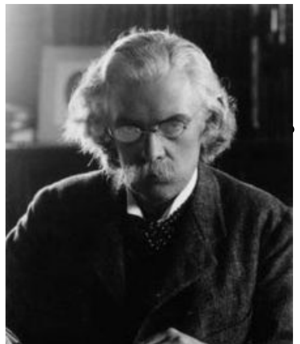
$+$ 

This gets multiplied by the Mack polynomial

$$\mathcal{M}^{(s)}_{\Delta,\ell}(s,t) = W^{(s)}_{\Delta,\ell} \boxed{\frac{\sin^2\pi(\Delta_\phi - s)}{\sin^2\pi(\Delta_\phi - \frac{\tau}{2})}} \rightarrow \exp(\pi|s|)$$

- Polyakov's observation (our modern interpretation Gopakumar, AS '18) is that to have a better behaviour at large imaginary s, it is better not to have this factor.

- However, we will now have spurious poles which will be inconsistent with OPE.

$$G_{\Delta,\ell}(u,v) = \int_{-i\infty}^{i\infty} ds dt\, u^s v^t \Gamma^2(-t)\Gamma^2(s+t)\Gamma^2(\Delta_\phi - s)\ \left(W^{(s)}_{\Delta,\ell}(s,t) + \rho(s,t)\right) \sin^2(\Delta_\phi - s)$$

$$w_{\Delta,\ell}(u,v) = \int_{-i\infty}^{i\infty} ds dt u^s v^t \Gamma^2(-t)\Gamma^2(s+t)\Gamma^2(\Delta_\phi - s) \left(W^{(s)} + W^{(t)} + W^{(u)} + \rho_c(s,t)\right)$$

Removal of zeroes introduces spurious poles. Demanding that these cancel gives consistency conditions.

Key point in the logic then....

- In the traditional approach we expand in terms of partial waves which are consistent with OPE.

Swap

- Impose crossing symmetry as constraint.

Key point in the logic then....

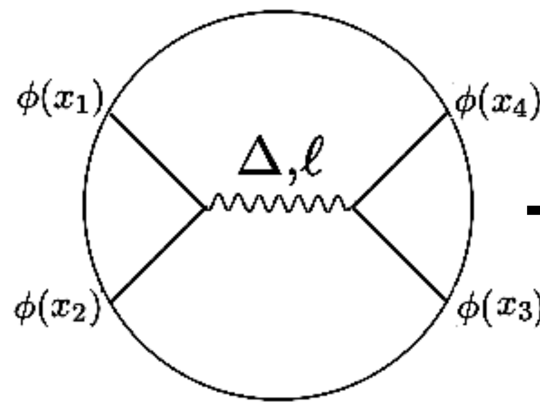
- In the traditional approach we expand in terms of partial waves which are consistent with OPE.
- Impose crossing symmetry as constraint.

Swap

Spurious pole cancellation conditions as consistency conditions

$$\mathcal{M}^{(s)}_{\Delta,s}(s,t) \to \left(W^{(s)}_{\Delta,\ell} + W^{(t)}_{\Delta,\ell} + W^{(u)}_{\Delta,\ell} + \rho_c(s,t) \right)$$

$$\mathcal{M}(s,t) = \sum_{\Delta,\ell} c_{\Delta,\ell}$$



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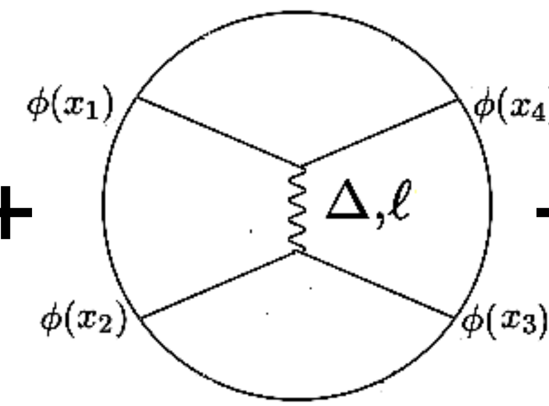

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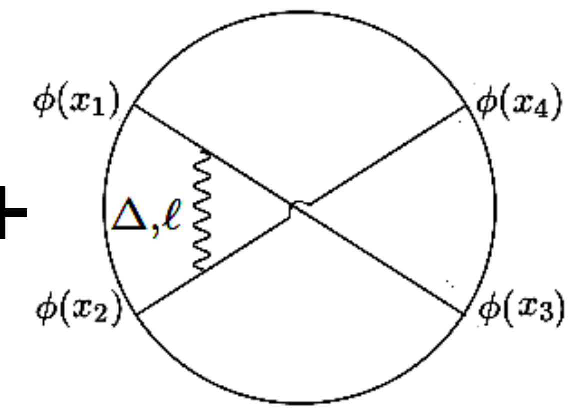

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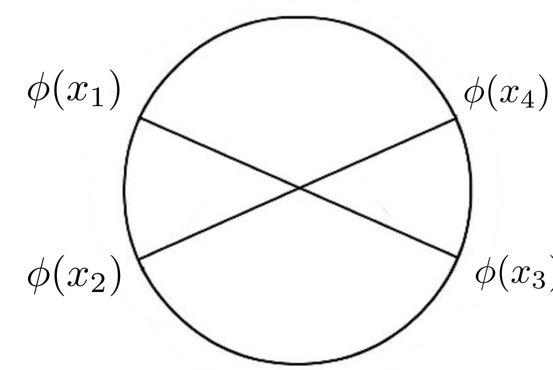
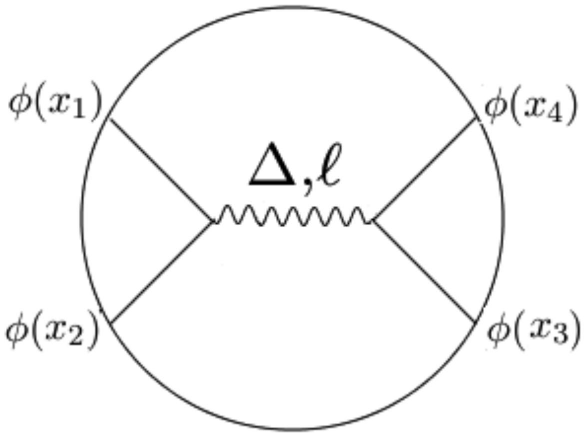




$$\sim \int \frac{dsdt}{(2\pi i)^2}\Gamma(-t)^2\Gamma(s+t)^2\Gamma(\Delta_\phi - s)^2 \frac{\sum_m Q^{\Delta}_{\ell,m}(t)}{s-\frac{\Delta-\ell}{2}-m}u^s v^t$$

$$\to u^{\Delta_\phi + r}\log u, u^{\Delta_\phi + r}$$

$$\to \text{incompatible with s} - \text{channel OPE}$$

$$\to \text{conditions needed to cancel}$$

NB: log u is genuinely spurious as it comes singly and not due to expanding some u-power

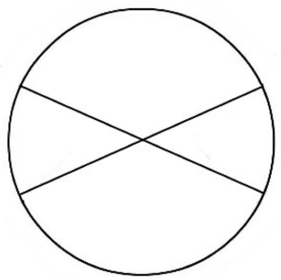
- Find it convenient to decompose the t dependence in terms of continuous Hahn polynomials (orthogonal polys in t).

$$\mathcal{M}(s,t) = \sum q_{\ell'}(s) Q^{2s+\ell'}_{\ell',0}(t)$$

$$q_{\ell'}(s = \Delta_\phi + r) = 0$$

$$\partial_s q_{\ell'}(s = \Delta_\phi + r) = 0$$

Holographic bootstrap[Gopakumar, AS;'18]



$$= \sum_{m,n} a_{mn}[s(s+t-\Delta_\phi)]^m [t(s+t)+s(s-\Delta_\phi)]^n$$

- Number of contact terms are exactly the same as the number of local AdS vertices that can be added for a given spin exchange.
- This provides a straightforward derivation of the results in Heemskerk, Penedones, Polchinski and Sully using our techniques.
- However, the point is that for a CFT an arbitrary set of contact terms cannot be added and further restrictions have to be put in place. This is the current challenge: what principle constrains the contact terms?

Wilson-Fisher epsilon expansion

Wilson-Fisher epsilon expansion

$$d = 4 - \epsilon$$

$$\Delta_\phi = 1 + \sum_n \delta_\phi^{(n)} \epsilon^n$$

$$\Delta_{\phi^2} = 2 + \sum_n \delta_0^{(n)} \epsilon^n$$

$$\Delta_{\phi^4} = 4 + \sum_n \delta_{4,0}^{(n)} \epsilon^n$$

$$\Delta_T = 4 - \epsilon$$

$$C_{\mathcal{O}} = \sum_{n=0} C_{\mathcal{O}}^{(n)} \epsilon^n$$

Never been computed using diagrams.

Kind of equations to solve

$$\frac{5}{16}\left(C_0^{(0)}(1+(\delta_0^{(1)})^2)-72C_2^{(0)}\delta_\phi^{(2)}\right)=0\,.$$

The difference between the $s=\Delta_\phi$ and $s=\Delta_\phi+1$ condition to leading order in ϵ leads to

$$\frac{1}{12}\left(-C_0^{(0)}(1+\delta_0^{(1)})(7+10\delta_0^{(1)})+72C_2^{(0)}\delta_\phi^{(2)}\right)=0\,.$$

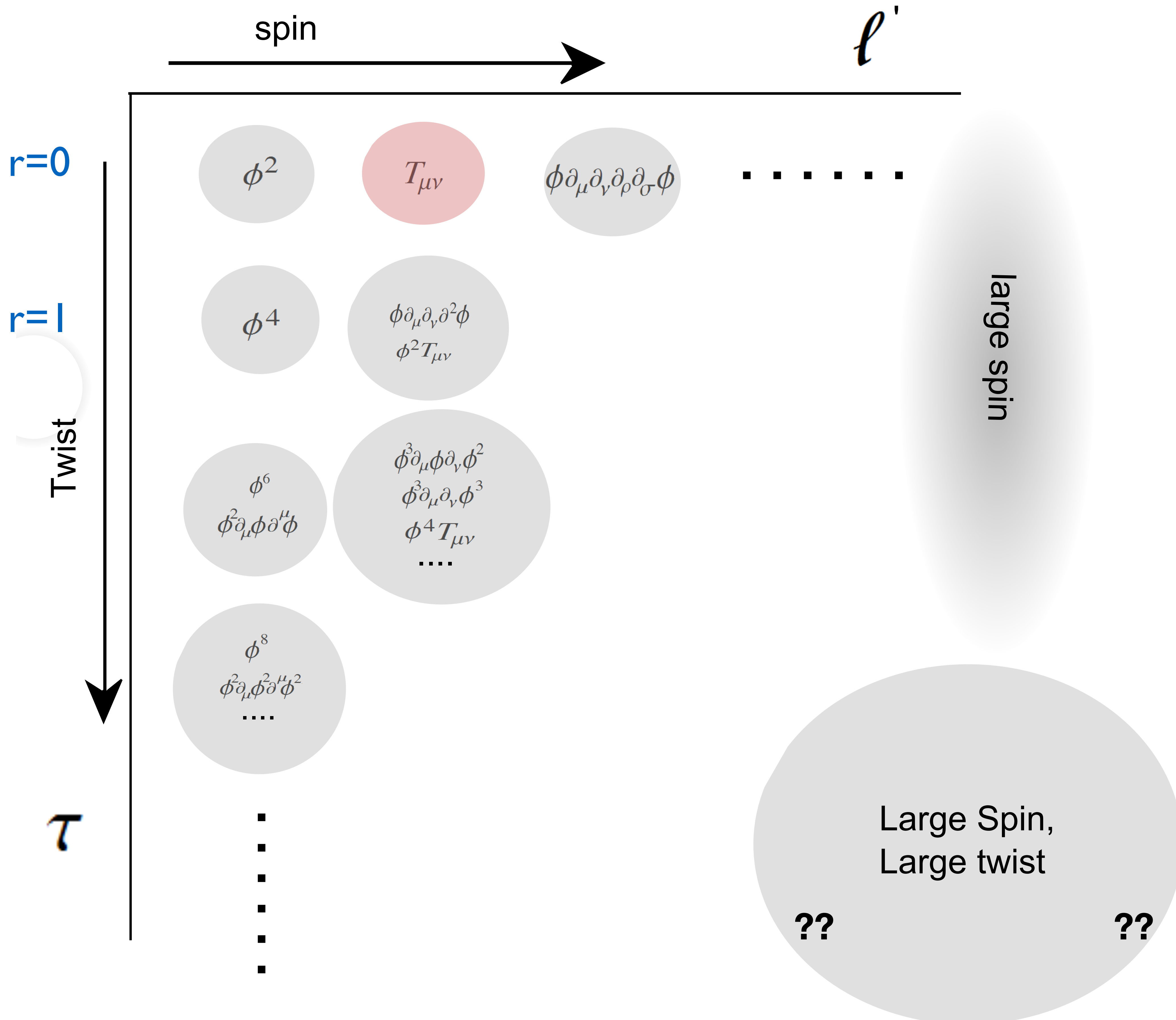
$$\frac{5}{16}\left(C_0^{(0)}(1+(\delta_0^{(1)})^2)-72C_2^{(0)}\delta_\phi^{(2)}\right)=0\,.$$

$$11+216\delta_0^{(2)}-3888\delta_\phi^{(3)}=0$$

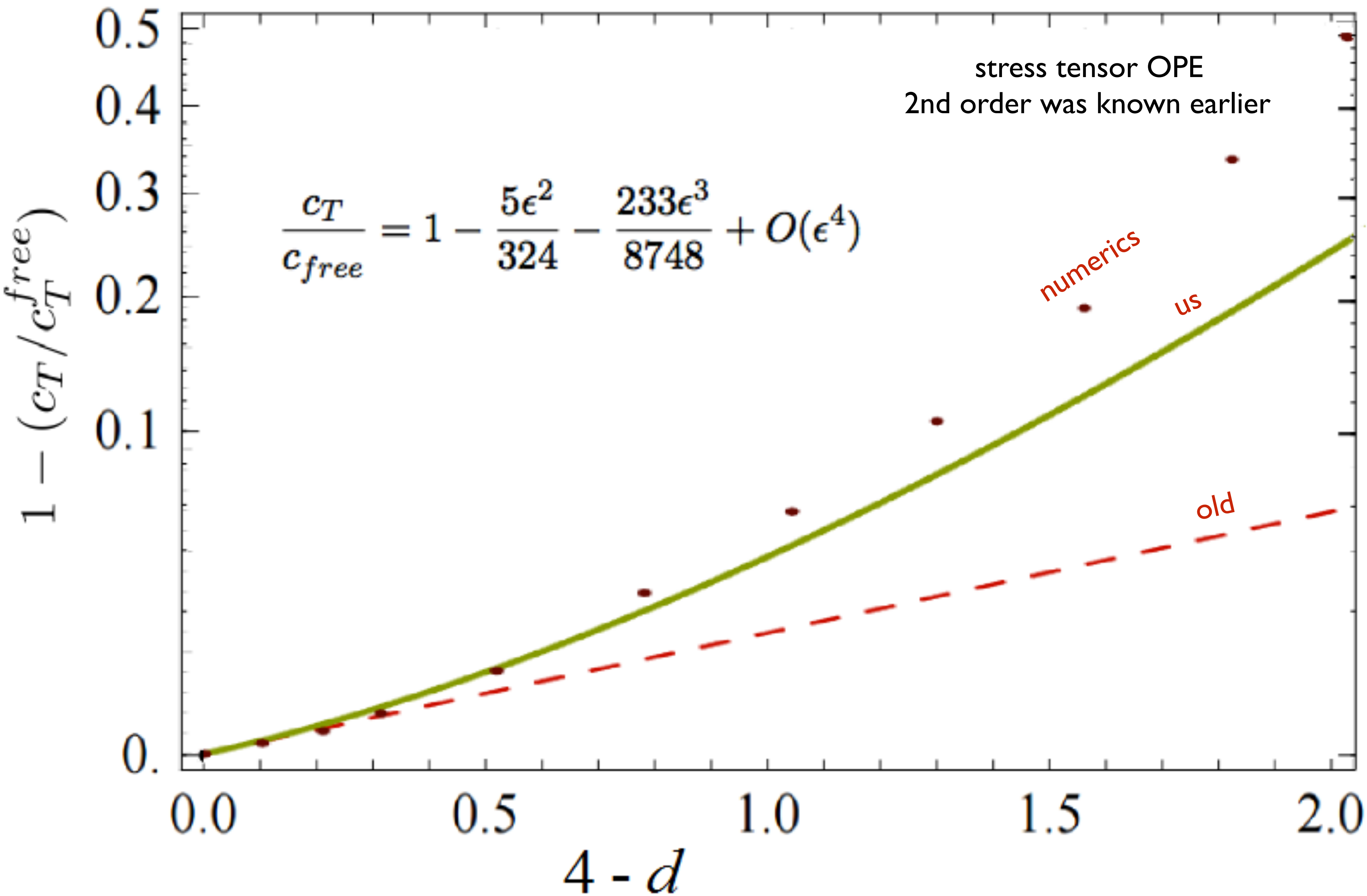
$$103+108\delta_{4,0}^{(1)}-1188\delta_0^{(2)}+3888\delta_\phi^{(3)}=0$$

Agrees with Feynman diagram higher loop results!! And gives new results for OPE coefficients which have not been computed so far.

Structure of equations



Sampling of results



ℓ	Δ_{numerics}	$\Delta_{\epsilon\text{-expansion}}$	% variation
4	5.02267	5.02495	0.0454992
6	7.02849	7.03091	0.0344653
8	9.03192	9.03332	0.0154195
10	11.0324	11.0345	0.0192799
12	13.0333	13.0353	0.015114
14	15.0338	15.0357	0.0124619
16	17.0343	17.036	0.0103496

Table 1: Comparison of Δ from ϵ expansion with numerical estimates

Numerics from Simmons-Duffin: 1612.08471

Spin ℓ	$f_{\phi\phi J_\ell}\|_{DSD}$	$f_{\phi\phi J_\ell}\|_{\epsilon=1}$	Percentage Deviation
$\ell=2$	0.326	0.328	0.33
$\ell=4$	0.069	0.070	0.86
$\ell=6$	1.57×10^{-2}	1.61×10^{-2}	2.55
$\ell=8$	3.69×10^{-3}	3.76×10^{-3}	2.04
$\ell=10$	8.76×10^{-4}	8.82×10^{-4}	0.79
$\ell=12$	2.10×10^{-4}	2.06×10^{-4}	1.86
$\ell=14$	5.06×10^{-5}	4.79×10^{-5}	5.52
$\ell=16$	1.22×10^{-5}	1.10×10^{-5}	10.6

Table 2 : Comparison of OPE

1609.00572 Gopakumar, Kaviraj, Sen, AS in PRL

$O(\epsilon^3)$

Sampling of results

Conductivity superfluid-insulator quantum critical point O(2)

$$\frac{\sigma(\infty)}{\sigma_Q} = 0.36$$

Witzcak-Krempa, Sorensen, Sachdev—Nature Physics 2013

$$\frac{\sigma(\infty)}{\sigma_Q} = \frac{\pi}{8}(1 - \frac{3}{100}\epsilon^2 - \frac{9}{200}\epsilon^3) = 0.363$$

P. Dey, A. Kaviraj, A.S.—1612.05032

Higher orders?

- Problem is that we will need to fix contact diagrams. Without contact diagrams, we would get an answer that does not agree with Feynman diagram results at $O(\epsilon^3)$ for φ^2 [Gopakumar, AS, 2018].

- A puzzle is that we do not see any reason a priori why we need to add these contact diagrams. One would have hoped that some inconsistency would tell us the need to add them. May be mixed correlators will help.

- Leaves open the interesting possibility that solutions other than pQFT exist: either ways would be important to rule in/rule out.

1d-fixing of contact terms

- Need to find a way to constrain contact terms in our basis—surely not arbitrary set of contact terms are possible.
- Recently, Paulos and Mazac (11/2018) have made an important observation which claims to solve this problem for d=1.
- They find constraints by demanding the completeness of the basis. Let me summarise their logic.

- The idea is to construct a “complete” set of functionals to act on the usual bootstrap equations (basically a kernel to integrate the u,v dependence). These functionals by design “isolate” the Mean Field solution by having zeros at the operator locations.

- The existence of these functionals is non-trivial. One can use these to act on the Witten diagram expansion. What happens is that for the boson case, one of the equations is ill-defined as sum over the spectrum does not converge. Regulating this is the same as adding a specific contact Witten diagram!

- In our language, this is simply to add a phi^4 contact interaction.

- Operationally, we simply work with subtracted pair of equations—effectively we lose one equation.

- In d=1, we can understand this by observing that derivative contact interactions would make anomalous dimensions grow "too quickly" and would need an infinite number of contact interactions to make finite.

[K. Ghosh, A. Zahed, L. F. Alday, P. Ferrero, work in progress]

What is the "simplest" bootstrap?

with L. F. Alday, P. Ferrero, K. Ghosh and A. Zahed in progress

- Turns out that the simplest bootstrap that we can hope to solve analytically with current techniques is not the 2d Ising model. For 2d Ising you need other methods.

- There is a 1d problem. We can consider bootstrap in 1d (3 generators, no boosts).

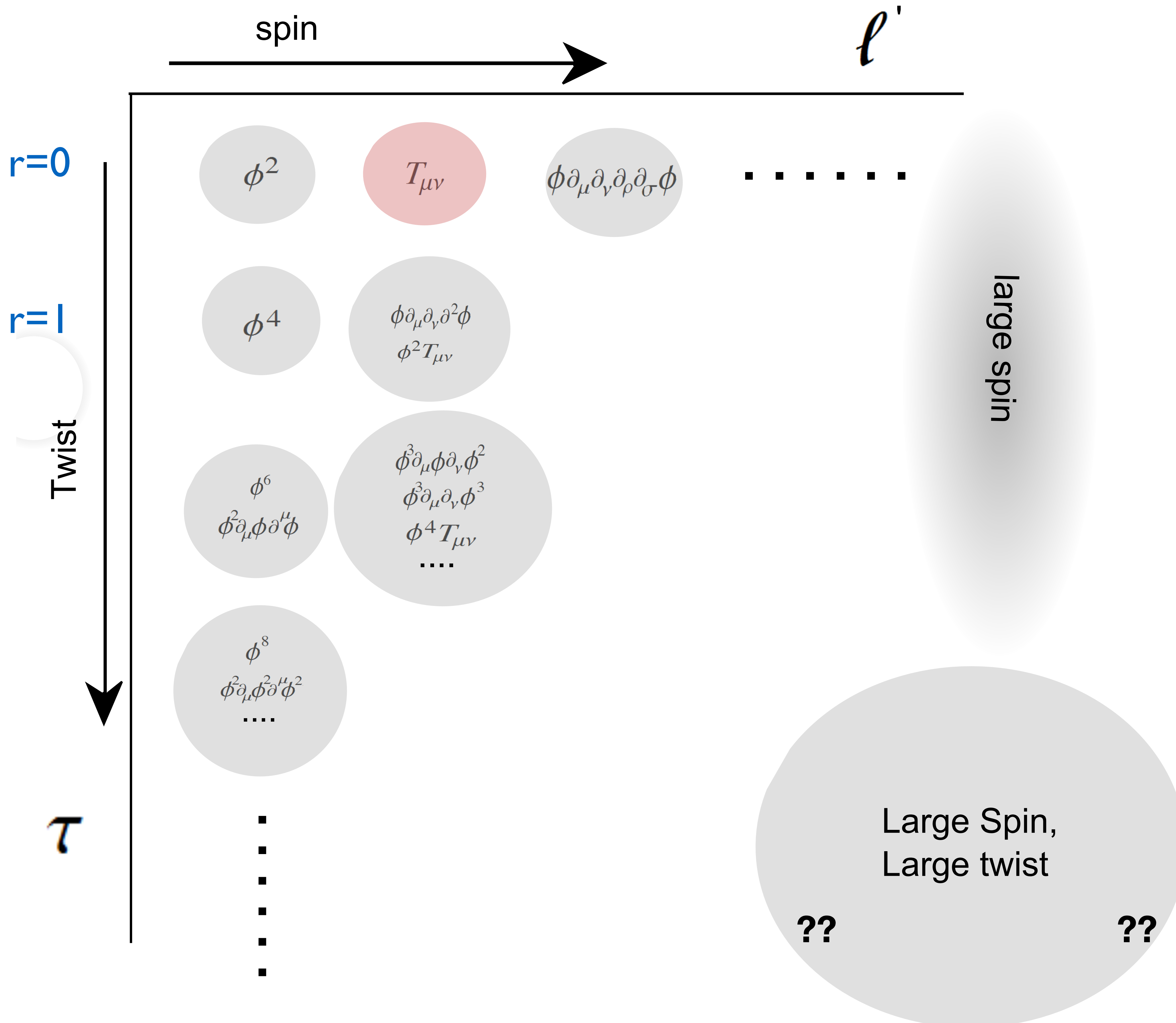
- You can think of this as the diagonal limit of the bootstrap problem in higher dimensions.
[Also Mazac, Paulos 2018]

Steps to set up

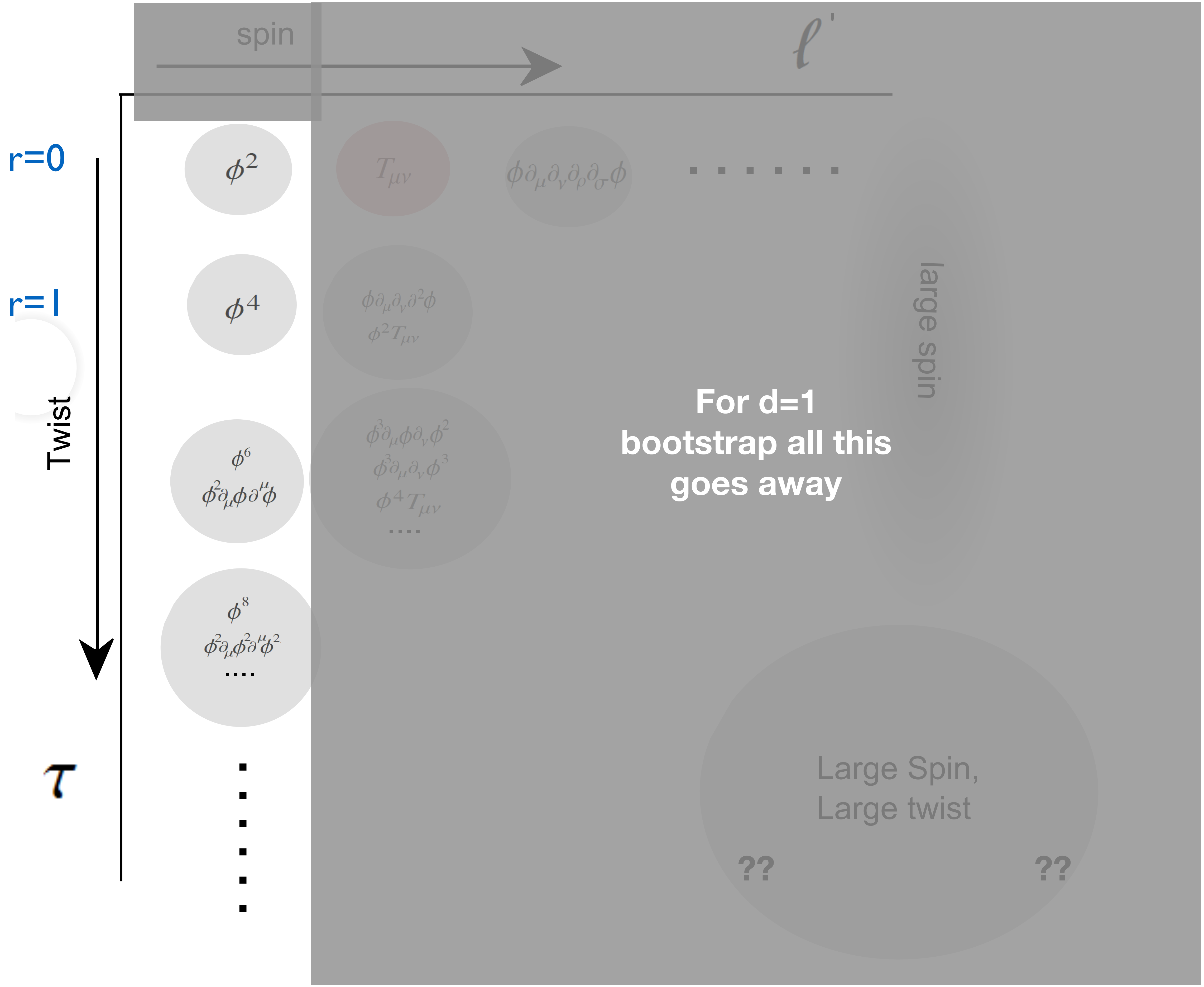
- Start with Polyakov-Mellin bootstrap.
- In 1d there is no spin. To deform away from free theory add scalar contact term.
- Solve!!

Equivalently we could have computed loops in AdS_2 but that is rather hard and even 2 loops have not been calculated!!

Structure of equations



Structure of equations



L here is the loop order in AdS_2

$$\Delta_n = 2\Delta_\phi + 2n + \sum_{L=0} \gamma_n^{(L)} g^{L+1}$$

$$C_n = C_n^{(GFF)} + \sum_{L=0} C_n^{(L)} g^{L+1}$$

Expect this to be true at any loop which is the reason no further contact terms are necessary.

$$\gamma_n \sim O(\frac{1}{n^2})$$

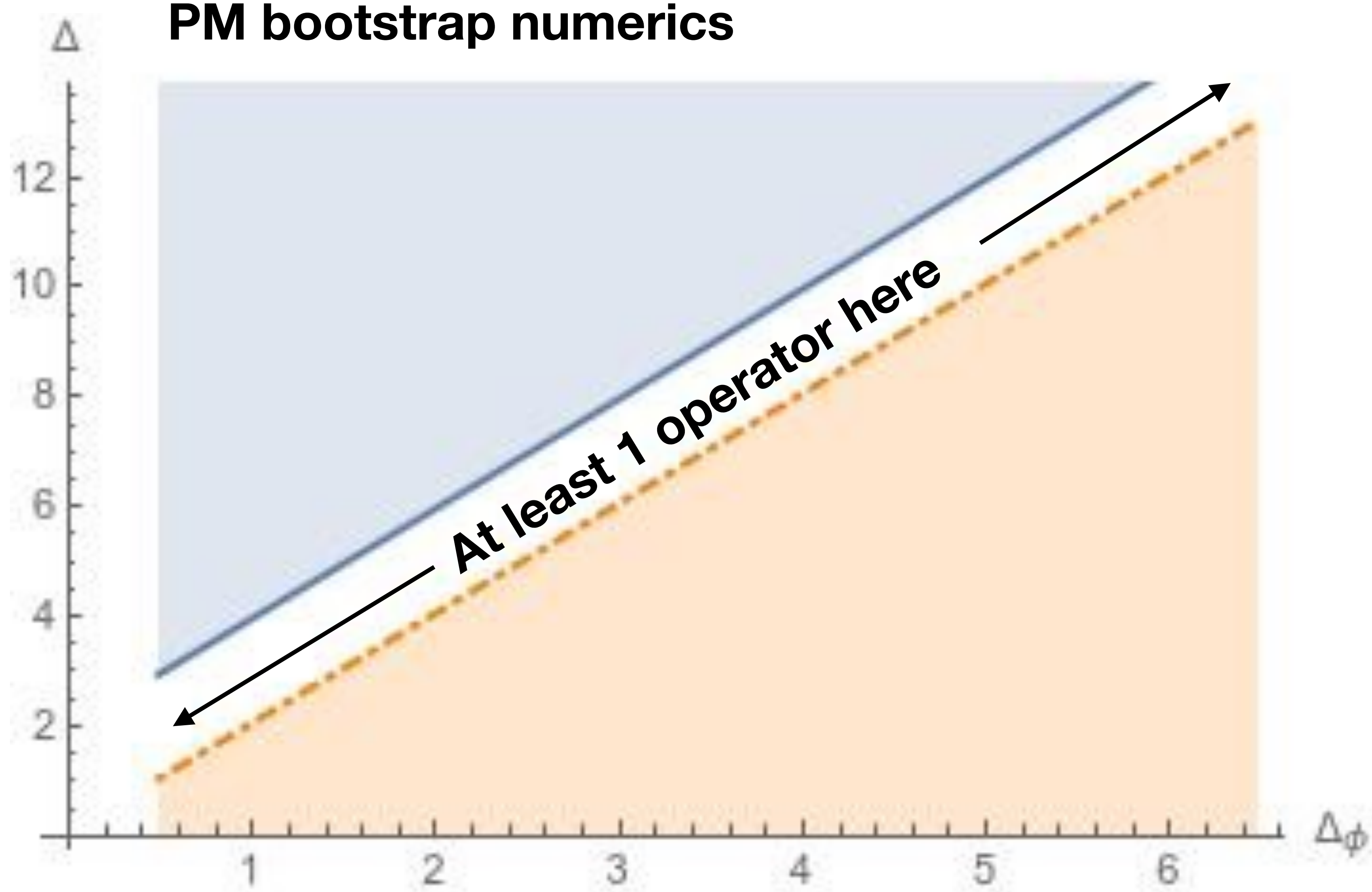
	$\Delta_\phi = 1$	$\Delta_\phi = \frac{15}{10}$	$\Delta_\phi = 2$	$\Delta_\phi = \frac{25}{10}$	$\Delta_\phi = 3$
$C_1^{(GFF)}$	1.2	2.57143	4.44444	6.81818	9.69231
$C_1^{(0)}$	-0.246667	-0.487147	-0.629136	-0.577572	-0.261218
$C_1^{(1)}$	-0.2487	-0.4413	-0.5112	-0.4374	-0.1752

	$\Delta_\phi = 1$	$\Delta_\phi = \frac{15}{10}$	$\Delta_\phi = 2$	$\Delta_\phi = \frac{25}{10}$	$\Delta_\phi = 3$
$\gamma_1^{(0)}$	0.166667	0.234375	0.28	0.3125	0.336735
$\gamma_1^{(1)}$	0.19796	0.24196	0.25656	0.26362	0.26795
$\gamma_1^{(2)}$	0.2570	0.1932	0.1822	0.1791	0.1789

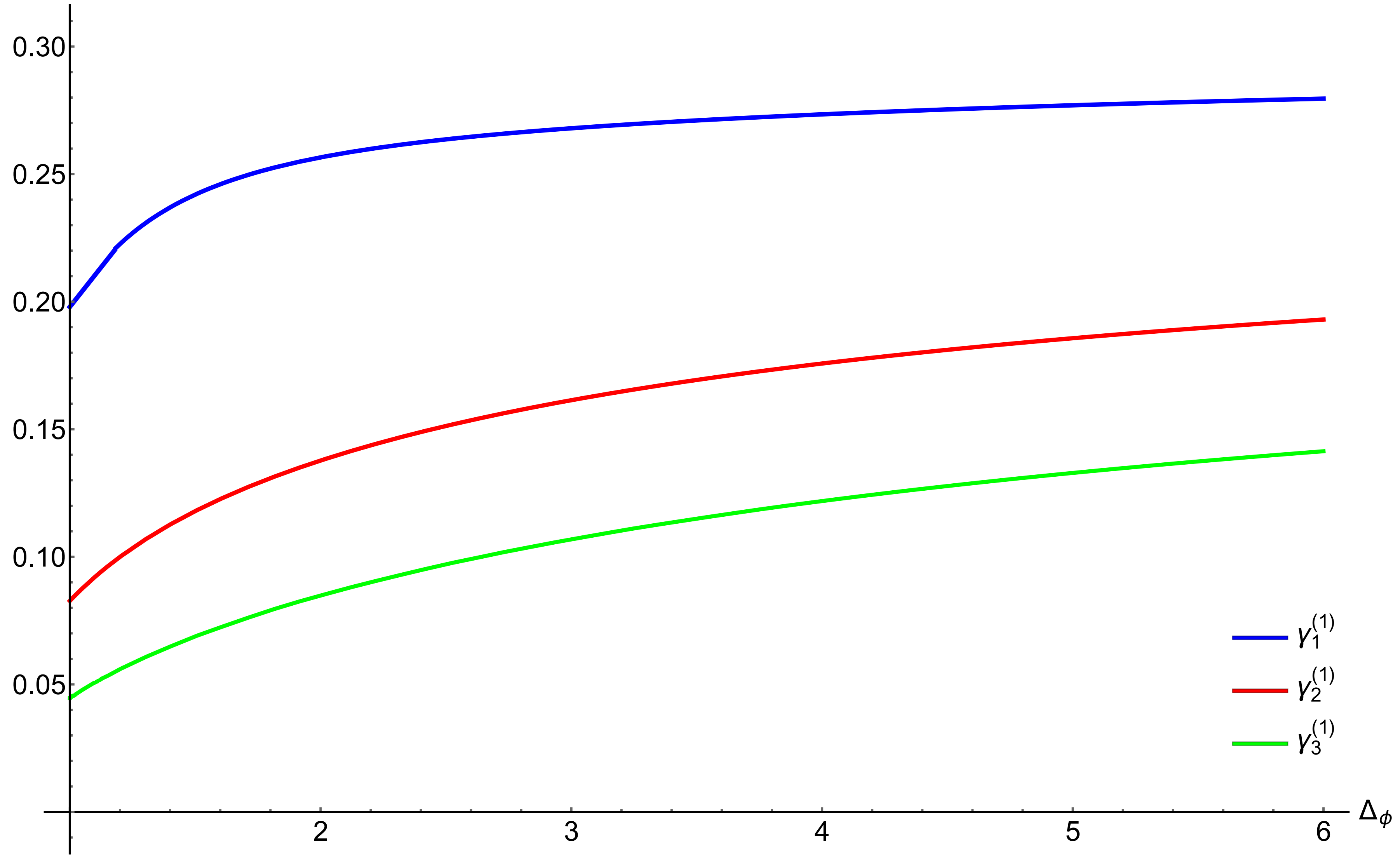
$$\Delta_n = 2\Delta_\phi + 2n + \sum_{L=0} \gamma_n^{(L)} g^{L+1}$$

$$C_n = C_n^{(GFF)} + \sum_{L=0} C_n^{(L)} g^{L+1}$$

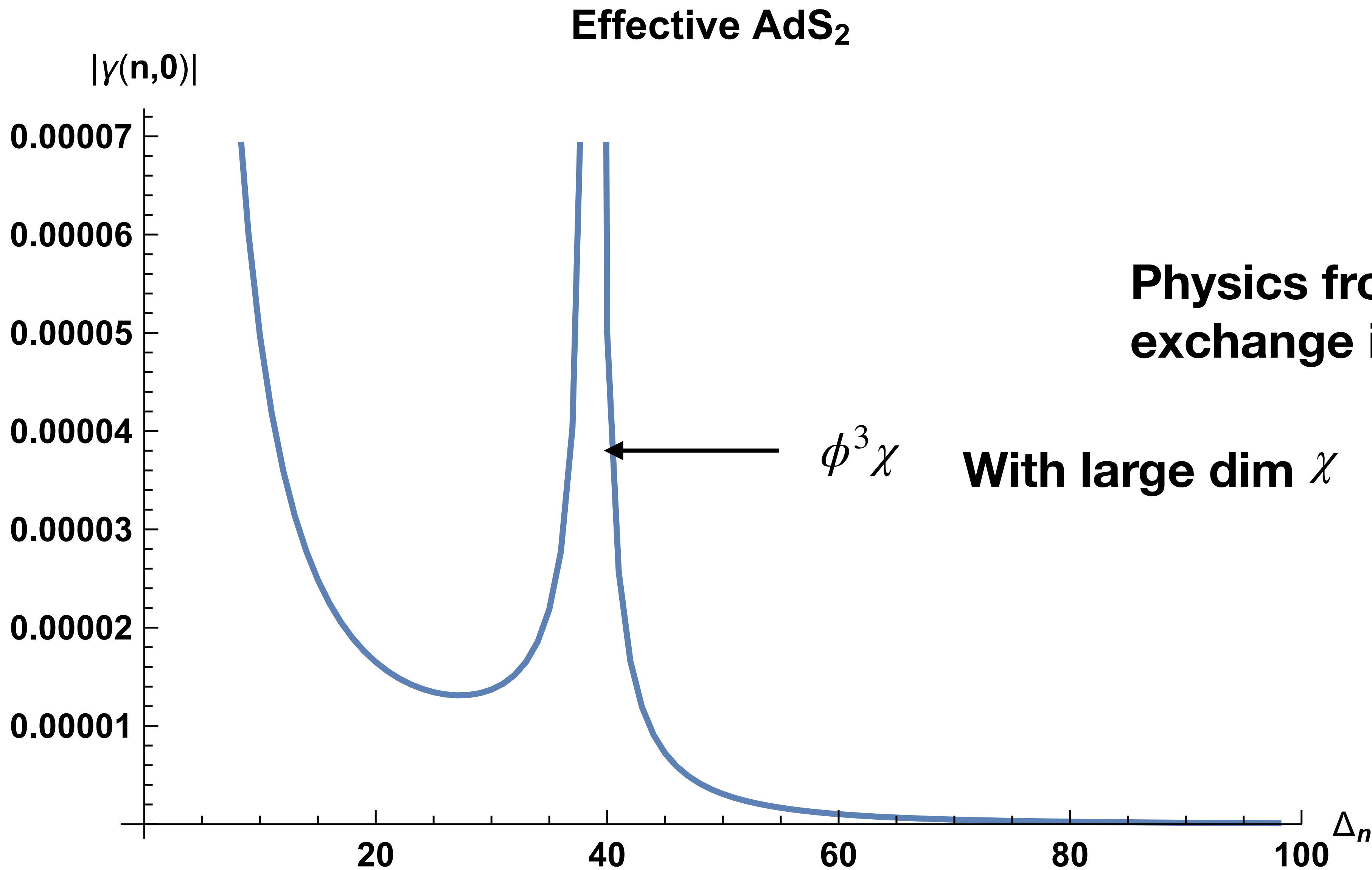
PM bootstrap numerics



Peculiarities: In contrast to usual numerics (only upper bound) we get an allowed band —more constraining. Curiously a similar feature exists for the cyclic polytope approach (yesterday's talk!).



Peculiarities: As a function of external operator dimension, the 1-loop anomalous dimensions seem to asymptote to a fixed value.



Physics from an exchange interaction

With large dim χ

Fitzpatrick et al 1007.0412—Effective field theory in AdS. Bootstrap produces exactly this thereby giving a derivation of EFT from bootstrap.

[K. Ghosh, A. Zahed, L. F. Alday, P. Ferrero, work in progress]

Way ahead

- Lots of open questions. Need to understand the crossing symmetric contact diagrams in higher d.

- A systematic perturbative method seems to cry out for a way to project out higher twist operators as well—what is so special about double trace (twist) ops?

- Bootstrapping composite operators in any approach seems to be lacking a systematic algorithm.

Rich mathematics ahead

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- The intermediate building blocks appear to be very rich in Mathematics. The key element appears to be very well poised $_7F_6$ hypergeometric functions [Gopakumar, AS, 2018].

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$$W(a;b,c,d,e,f) \equiv$$

$$ {}_7F_6\left(\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e, & f \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{matrix};1\right)$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1+a-c-d)\Gamma(1+a-b-d)\Gamma(1+a-b-c)\Gamma(1+a-e-f)}$$

$$\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\sigma \, \frac{\Gamma(-\sigma)\Gamma(1+a-b-c-d-\sigma)\Gamma(b+\sigma)\Gamma(c+\sigma)\Gamma(d+\sigma)\Gamma(1+a-e-f+\sigma)}{\Gamma(1+a-e+\sigma)\Gamma(1+a-f+\sigma)} .$$

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- Very well poised ${}_7F_6$ hypergeometric functions are generalised 6j symbols for non-compact groups [eg Raynal 1979].

Rich mathematics ahead

- The intermediate building blocks appear to be very rich in Mathematics. The key element appears to be very well poised ${}_7F_6$ hypergeometric functions [Gopakumar, AS, 2018].

- Very well poised ${}_7F_6$ hypergeometric functions are generalised 6j symbols for non-compact groups [eg Raynal 1979].

- Will have more role to play eventually—curiously one of Ramanujan's work in his "lost notebook" was on very well poised ${}_7F_6$!

Thank you.

Backup slides

Large spin asymptotics

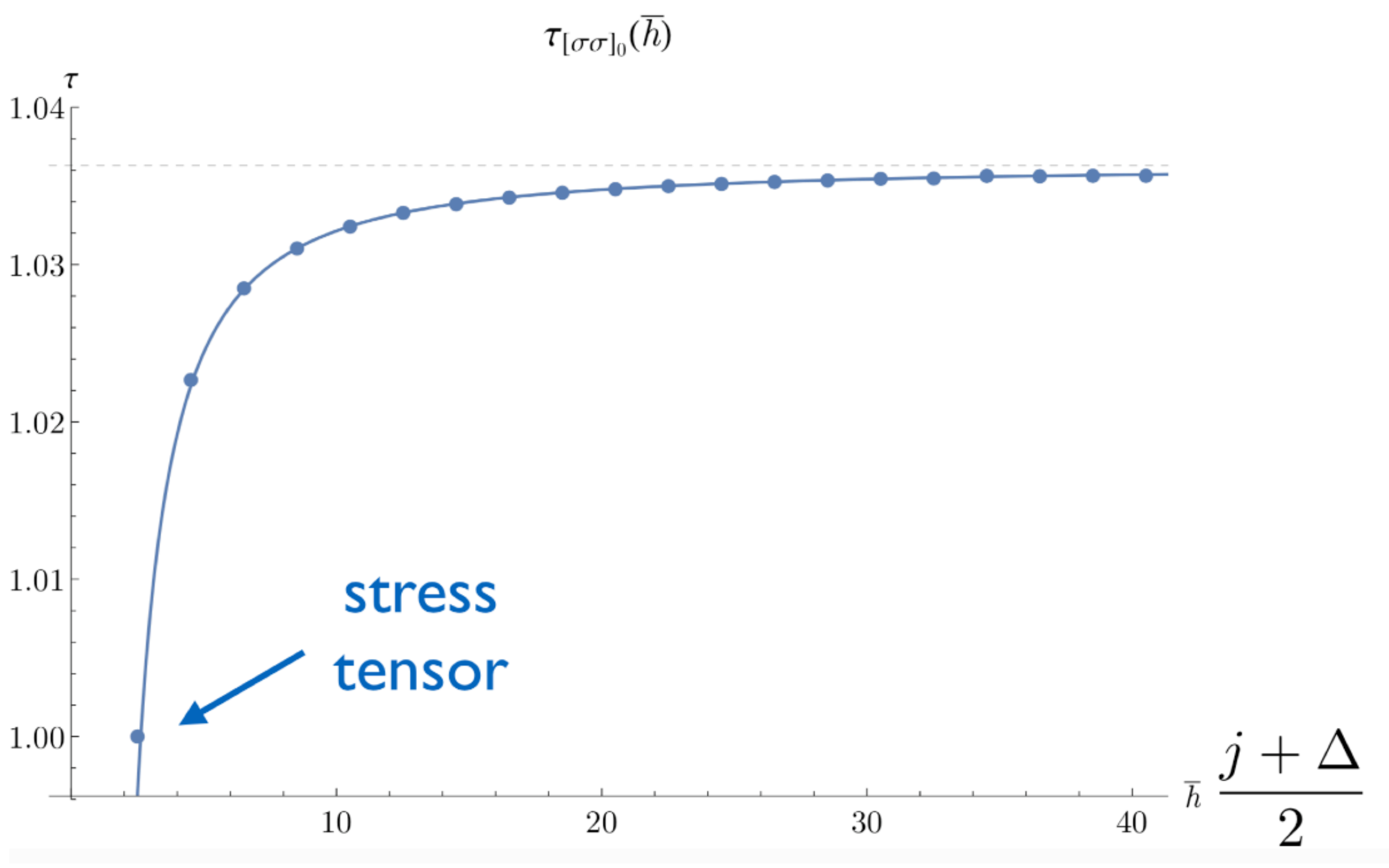
$$\sum_{\Delta,\ell} c_{\Delta,\ell} u^{\frac{\Delta-\ell}{2}} \left[{}_2F_1(\frac{\Delta+\ell}{2}, \frac{\Delta+\ell}{2}, \Delta+\ell, 1-v) + O(u) \right]$$

$$= c_m v^{\frac{\tau_m}{2}} \left[{}_2F_1(\frac{\Delta_m+\ell_m}{2}, \frac{\Delta_m+\ell_m}{2}, \Delta_m+\ell_m, 1-u) + \cdots \right]$$

- Assume operator of minimum twist
- In small u,v limit, lhs has log v while rhs has log u. To produce log u, we can expand power of u on lhs by assuming a small anomalous dimension
- A finite number of log v's cannot resum to a power of v. So we need infinite number of operators.
- To produce identity operator we will need double trace ops $\Delta = 2\Delta_\phi + \ell$

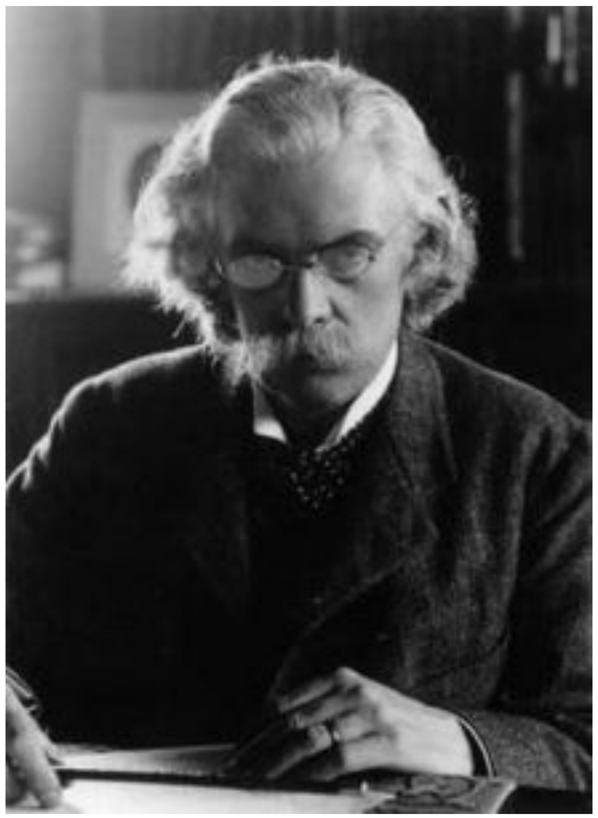
$$u^{(\Delta-\ell)/2} = u^{\Delta_\phi+\gamma_\ell} = u^{\Delta_\phi}(1 + \gamma_\ell \log u + \frac{1}{2}\gamma_\ell^2 (\log u)^2 + \cdots)$$

- By focusing on the u=0, v=0 limit, we can develop a systematic large spin asymptotic expansion for the anomalous dimensions of double trace operators which are supported by a single leading twist operator in the crossed channel.[Komargodski-Zhiboedov; Fitzpatrick et al; Alday et al; Kaviraj, Sen, AS]



3d Ising model data from Simmons-duffin 2016 and asymptotic expansion in large spin (LSPT) from Alday-Zhiboedov 2015. Seems to point at an analytic formula in spin. Conformal Froissart Gribov formula by Caron-huot 2017.

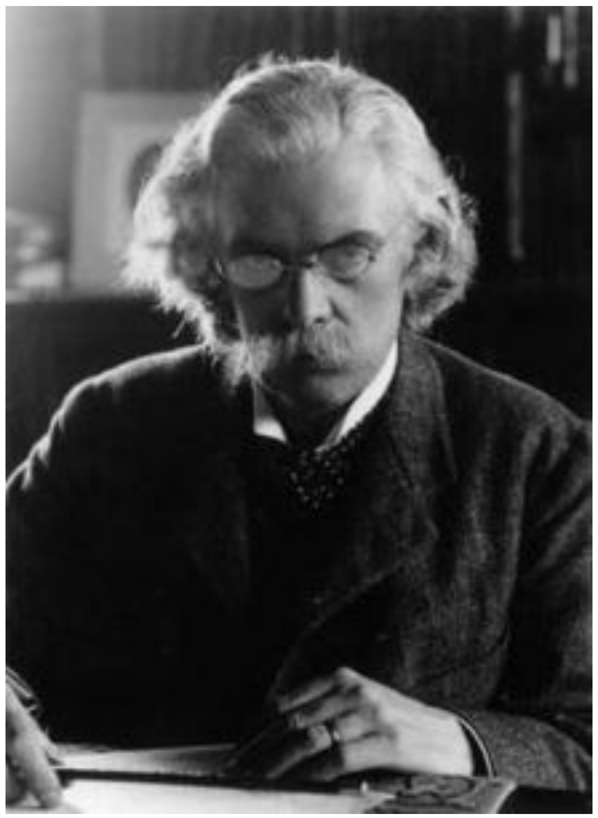
Enter Mellin's advisor!



Gosta Mittag-Leffler
(Swedish mathematician)

Thm: Existence of meromorphic functions with prescribed poles

Enter Mellin's advisor!



Mellin's advisor!

Gosta Mittag-Leffler
(Swedish mathematician)

Thm: Existence of meromorphic functions with prescribed poles

Ex I

Construct a function with residue n, poles at z=n for all positive integers n.

$$\sum_n \frac{n}{z-n}$$

Naive guess: Does not work as sum does not converge.

$$\sum_n \frac{n}{z-n} \quad (\frac{z}{n})^2$$

This works. Same as adding polynomials in z. [not unique]

Ex2

$$\Gamma(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(x+n)} \qquad + \Gamma_1(x)$$

Entire function

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(x+n)} \frac{\Gamma(y)}{\Gamma(y-n)}$$

no extra entire
function piece

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famous formula responsible for the birth of string theory!

no extra entire function piece

Thank you for listening!