Characterization of quantum chaos using the quantum Lyapunov spectrum and two-point functions: the case of the Sachdev-Ye-Kitaev model as an example

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Danshita, Hanada, and MT, PTEP 2017, 083I01 (arXiv:1606.02454)

Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, and MT, JHEP 1705, 118 (2017) (arXiv:1611.04650)

Hanada, Shimada, and MT, Phys. Rev. E **97**, 022224 (2018) (arXiv:1702.06935)

García-García, Loureiro, Romero-Bermudez, and MT, PRL **120**, 241603 (2018) (arXiv:**1707.02197**)

Gharibyan, Hanada, Swingle, and MT, JHEP 1904, 082 (2019) (arXiv:**1809.01671**), submitted (arXiv:**1902.11086**)

Collaborators in this work

arXiv:1809.01671
arXiv:1902.11086

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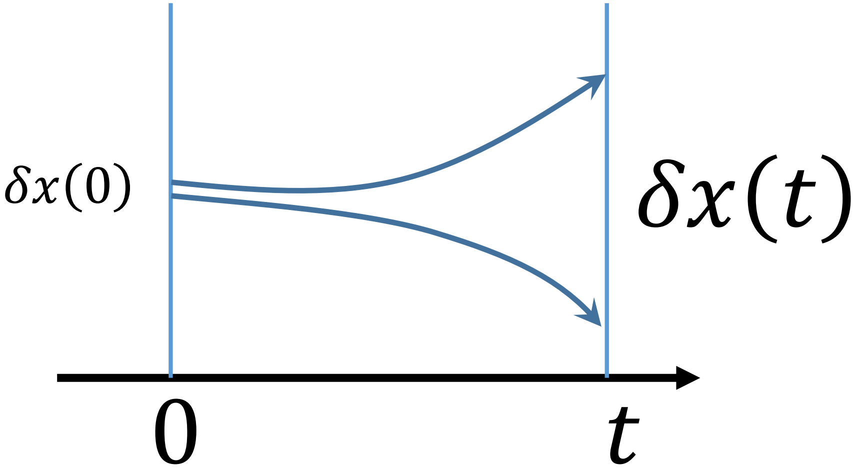
Plan of the talk

- Characterization of many-body quantum chaos
- The Sachdev-Ye-Kitaev model
- **The quantum Lyapunov spectrum**
- **The singular values of two-point correlators**
- The case of the XXZ spin chain
- Summary

Chaos in deterministic classical dynamics

- Sensitivity to initial conditions: exponential growth of initial perturbation



"butterfly effect"

Bounded, nonperiodic dynamics with nonlinearity

What happens in quantum mechanics?

How to characterize quantum chaos?

$\hat{H}$ = const.

$$i\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \qquad |\psi(t)\rangle = \hat{\mathrm{T}}\exp\left[-i\int_0^t \hat{H}(t')dt\right]|\psi(t=0)\rangle = \exp\left(-i\hat{H}t\right)|\psi(t=0)\rangle$$

Linear dynamics

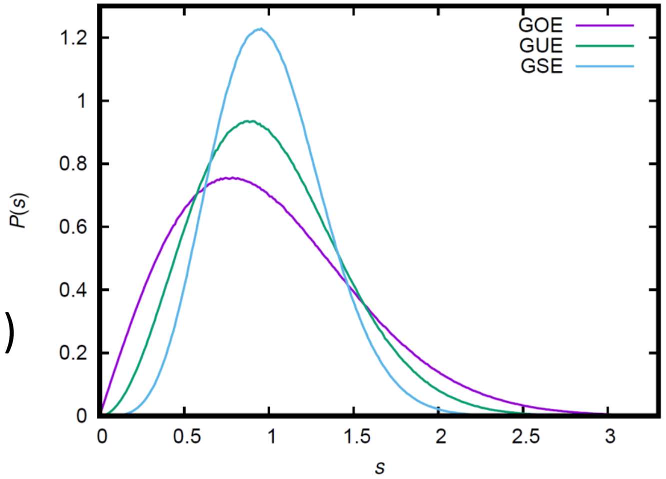
Unitary time evolution

- Long time: energy level statistics

Correlation between levels, as in random matrices

$P(s)$: normalized level separation distribution

Uncorrelated: Poisson (e^{-s})



cf. Bohigas-Giannoni-Schmit conjecture

- Short time: out-of-time correlator

Classically,

$$\{x_i(t), p_j(0)\}_{\mathrm{PB}}{}^2 = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)^2 \to e^{2\lambda_{\mathrm{L}}t} \text{ for large } t$$

Quantum version:

$$\text{OTOC: } C_T(t) = \left\langle \left|\left[\hat{W}(t), \hat{V}(t=0)\right]\right|^2 \right\rangle$$
$$= \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle + \cdots$$

Numerically ➔ limited to very small systems

➔ Hard to see exponential time dependence

Characterization of quantum many-body chaos

- Random-matrix like energy level correlation

- Exponential Lyapunov growth of out-of-time-order correlators (OTOC)

$$\langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle \sim C + \#\, e^{2\lambda_{\mathrm{L}} t}$$

Example: the Sachdev-Ye-Kitaev model

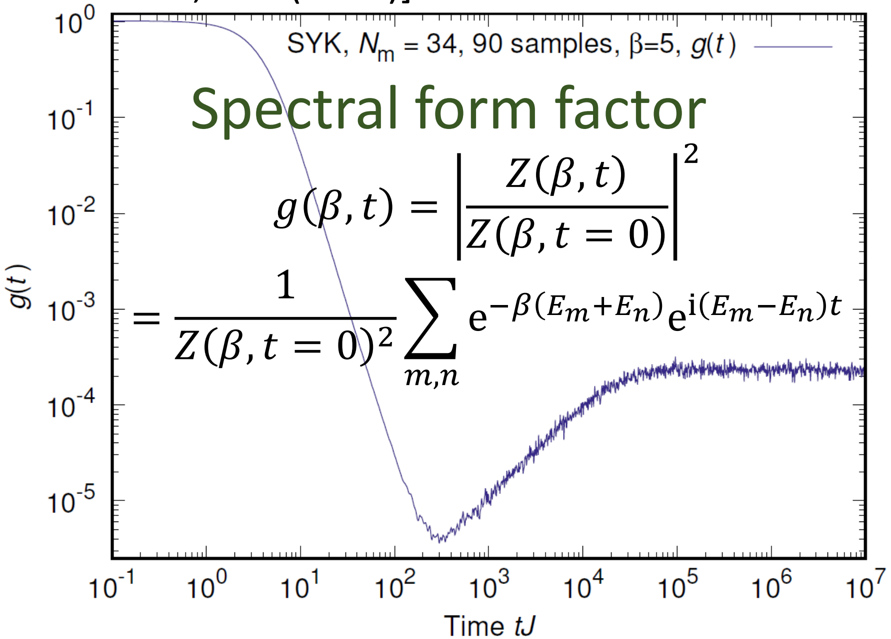
$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

$\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$

J_{abcd} : Gaussian random ($\langle {J_{abcd}}^2\rangle = J^2 = 1$)

[Kitaev 2015]

[Cotler, MT et al., JHEP **1705**, 118 (2017)]



Spectral form factor

$$g(\beta,t) = \left|\frac{Z(\beta,t)}{Z(\beta,t=0)}\right|^2 = \frac{1}{Z(\beta,t=0)^2}\sum_{m,n} e^{-\beta(E_m+E_n)} e^{i(E_m-E_n)t}$$

N mod 8	RMT
0	GOE
2	GUE
4	GSE
6	GUE

Lyapunov exponent

$$\lambda_{\mathrm{L}} = \frac{2\pi k_{\mathrm{B}} T}{\hbar}$$ in low T limit

(Maldacena-Shenker-Stanford chaos bound)

The Sachdev-Ye-Kitaev model

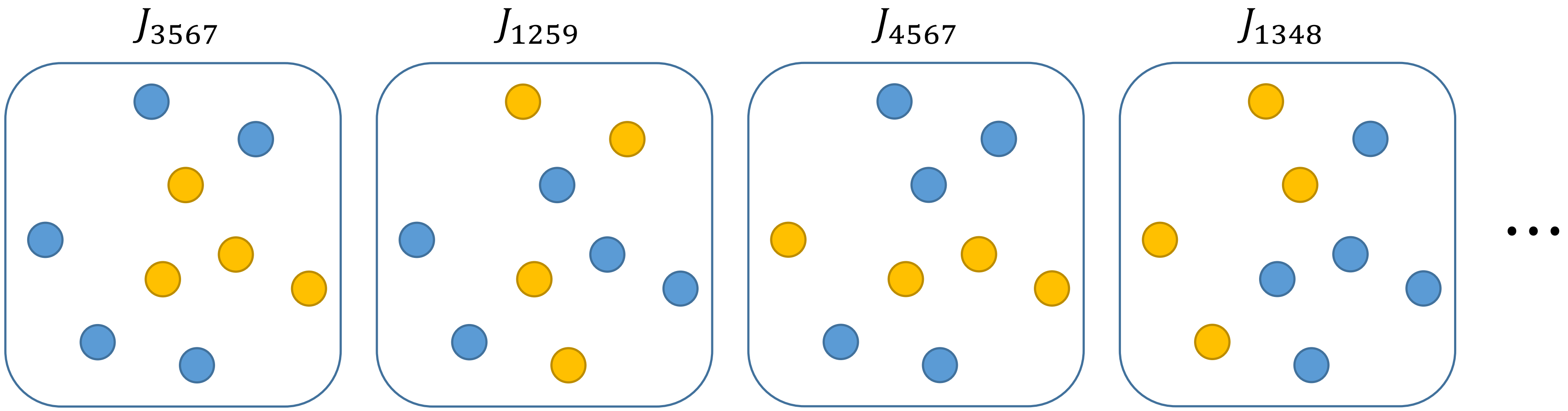
$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

cf. Sachdev-Ye model (1993)

$\hat{\chi}_{a=1,2,\ldots,N}$: N Majorana fermions ($\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$)

J_{abcd} : Gaussian random couplings ($\langle {J_{abcd}}^2 \rangle = J^2 = 1$)

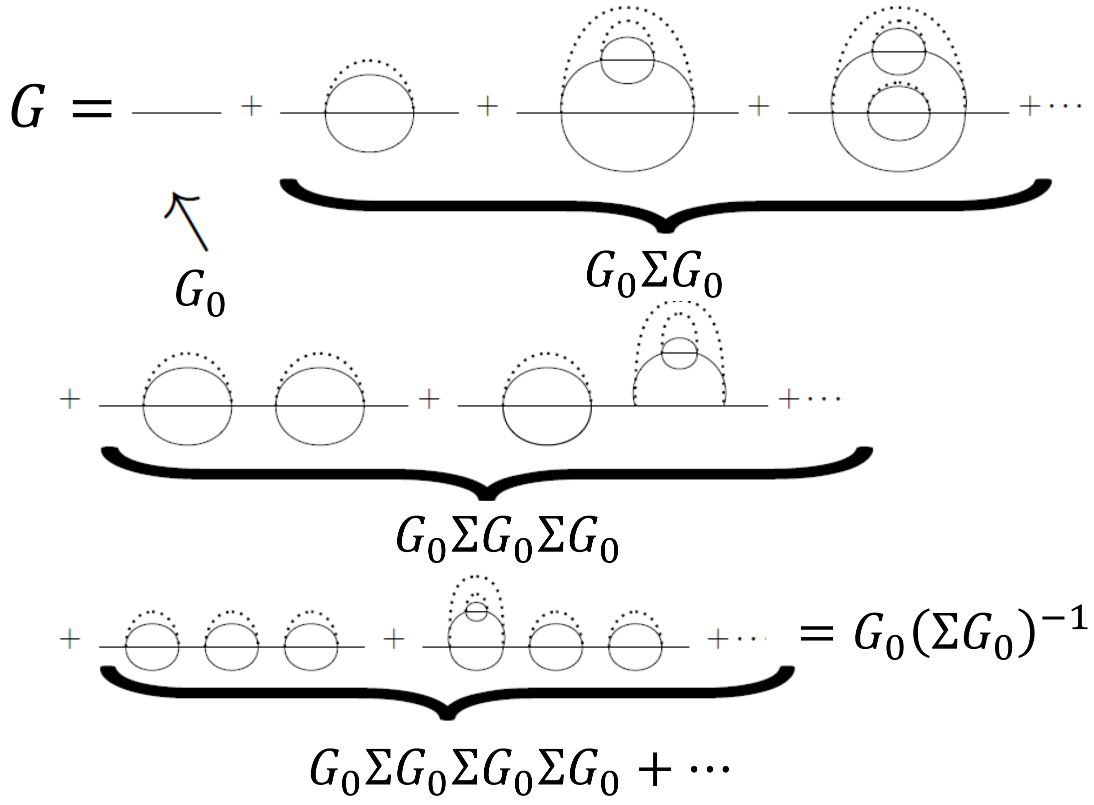
[A. Kitaev, talks at KITP (2015)]



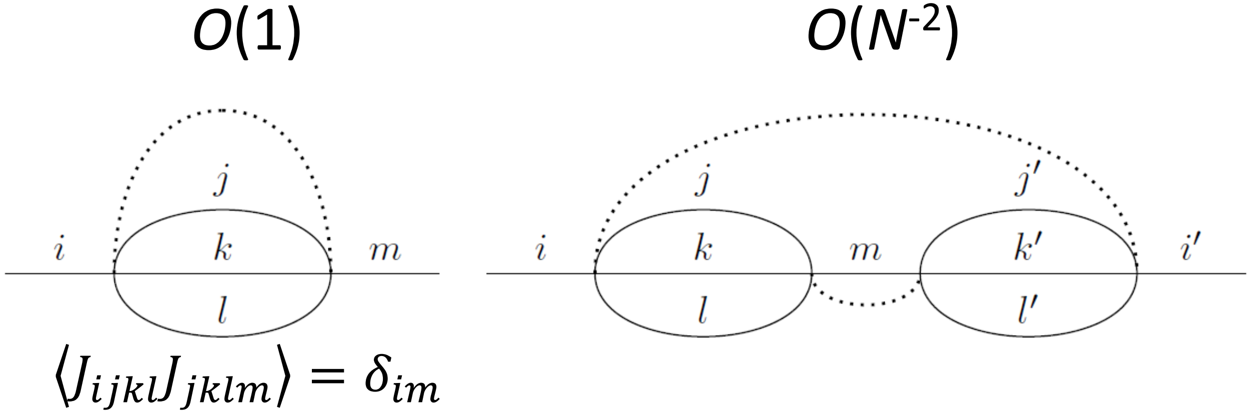
The SYK model

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

Analytically solvable in $N \gg 1$ limit



Figures from [I. Danshita, MT, and M. Hanada: Butsuri **73**(8), 569 (2018)]

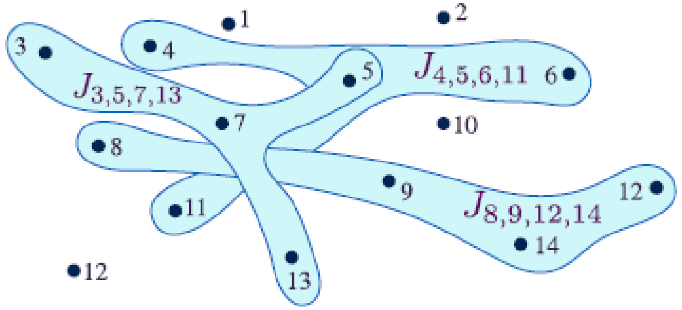


Only "melon-type" diagrams survive

Satisfies the "chaos bound" $\lambda_{\mathrm{L}} \le \frac{2\pi k_{\mathrm{B}} T}{\hbar}$ in the $T \to 0$ limit

Holographic connection to gravity?

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$



$$\mathcal{Q} = \frac{1}{N} \sum_i \left\langle c_i^\dagger c_i \right\rangle .$$

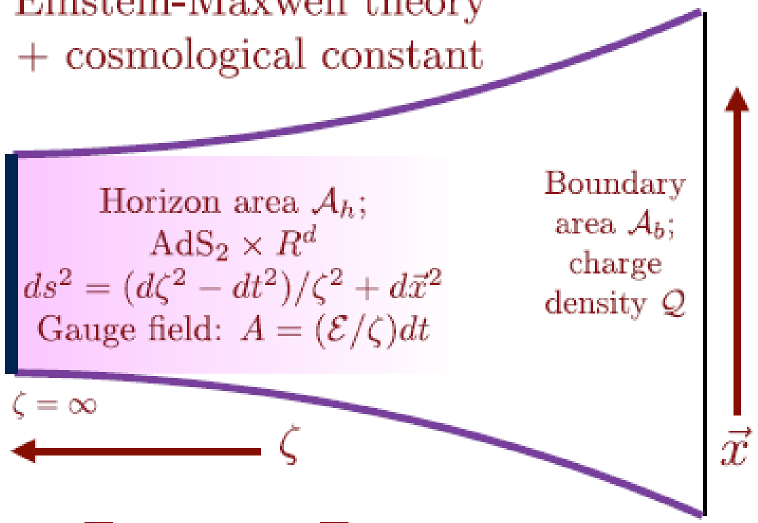
$$-\left\langle c_i(\tau) c_i^\dagger(0) \right\rangle \sim \begin{cases} -\tau^{-1/2}, & \tau > 0 \\ e^{-2\pi\mathcal{E}}\, |\tau|^{-1/2}, & \tau < 0. \end{cases}$$

Known "equation of state" determines $\mathcal{E}$ as a function of $\mathcal{Q}$

Microscopic zero temperature entropy density $\mathcal{S}$ obeys

$$\frac{\partial \mathcal{S}}{\partial \mathcal{Q}} = 2\pi\mathcal{E}$$

Einstein-Maxwell theory + cosmological constant



$$\mathcal{L} = \overline{\psi}\Gamma^\alpha D_\alpha \psi + m\overline{\psi}\psi$$

$$-\left\langle \psi(\tau)\overline{\psi}(0) \right\rangle \sim \begin{cases} -\tau^{-1/2}, & \tau > 0 \\ e^{-2\pi\mathcal{E}}\, |\tau|^{-1/2}, & \tau < 0. \end{cases}$$

"Equation of state" relating $\mathcal{E}$ and $\mathcal{Q}$ depends upon the geometry of spacetime far from the AdS_2

Black hole thermodynamics (classical general relativity) yields

$$\frac{\partial \mathcal{S}_{\rm BH}}{\partial \mathcal{Q}} = 2\pi\mathcal{E}$$

[S. Sachdev, Phys. Rev. X **5**, 041025 (2015)]

Sachdev-Ye model

- Strongly interacting random systems: model with analytical solutions?

[S. Sachdev and J. Ye, PRL **70**, 3339 (1993)] **cond-mat/9212030** *(Submitted on 21 Dec 1992)*
N SU(M) spins $\hat{S}$ with **all-to-all random coupling** J_{ij} (notation below: from [Sachdev, PRX 2015])

$$H = \frac{1}{(NM)^{1/2}} \sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{M} J_{ij} c^\dagger_{i\alpha} c_{i\beta} c^\dagger_{j\beta} c_{j\alpha}, \qquad \frac{1}{M} \sum_{\alpha} c^\dagger_{i\alpha} c_{i\alpha} = \mathcal{Q},$$

- Non-Fermi liquid with nonzero entropy at $T \rightarrow 0$

  Local dynamic spin susceptibility $$\bar{\chi}(\omega) = X \left[\ln\left(\frac{1}{|\omega|}\right) + i\frac{\pi}{2}\mathrm{sgn}(\omega) \right] + \cdots,$$

cf. Dynamic neutron scattering experiments on disordered antiferromagnets
[B. Keimer et al. PRL 1991 (LSCO); S.M. Hayden et al. PRL 1991 (LBCO);
C. Broholm et al. PRL 1990 (Kagome planes of Cr^{3+} ions in $Sr(Cr,Ga)_{12}O_{19}$)]

Proposals for experimental realization

s: molecular levels

$$\hat{H}_\mathrm{m} = \sum_{s=1}^{n_s}\left\{\nu_s \hat{m}_s^\dagger \hat{m}_s + \sum_{i,j} g_{s,ij}\left(\hat{m}_s^\dagger \hat{c}_i\hat{c}_j - \hat{m}_s \hat{c}_i^\dagger \hat{c}_j^\dagger\right)\right\}.$$

$|\nu_s| \gg |g_{s,ij}|$

Modified SYK model:

$$\hat{H}_\mathrm{eff} = \sum_{s,i,j,k,l} \frac{g_{s,ij}\,g_{s,kl}}{\nu_s}\hat{c}_i^\dagger\hat{c}_j^\dagger\hat{c}_k\hat{c}_l.$$

[I. Danshita, M. Hanada, MT: arXiv:1606.02454;
PTEP **2017**, 083I01 (2017)]
(also a proceedings manuscript arXiv:1709.07189)

Setup:

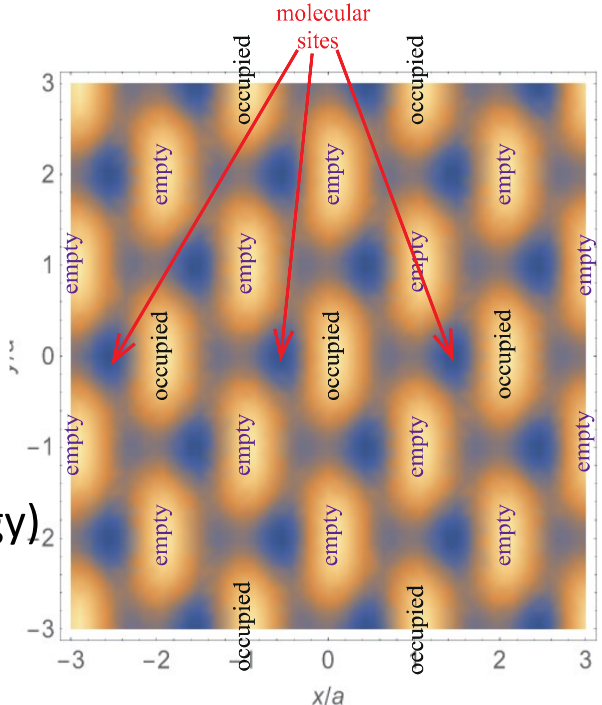
A double-well optical lattice

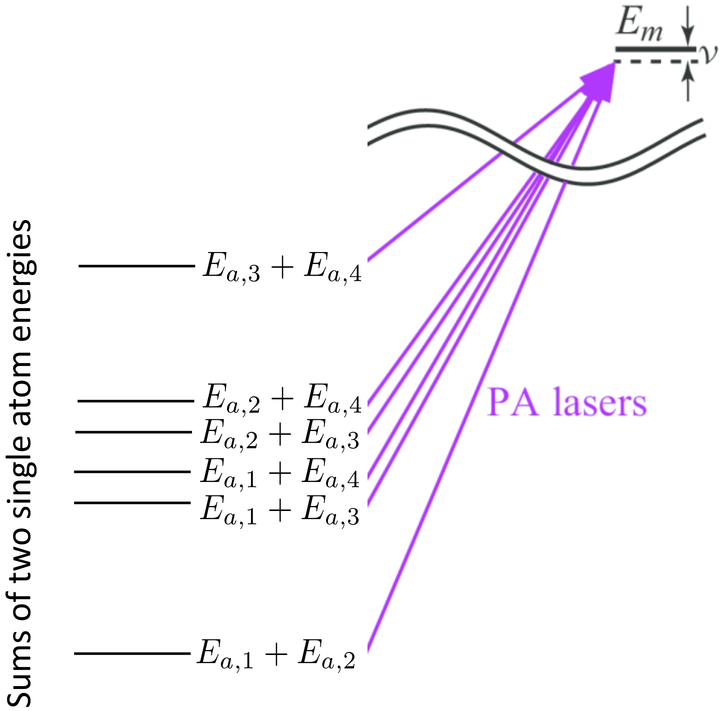
(no degeneracy in the band levels)

with

^{6}Li

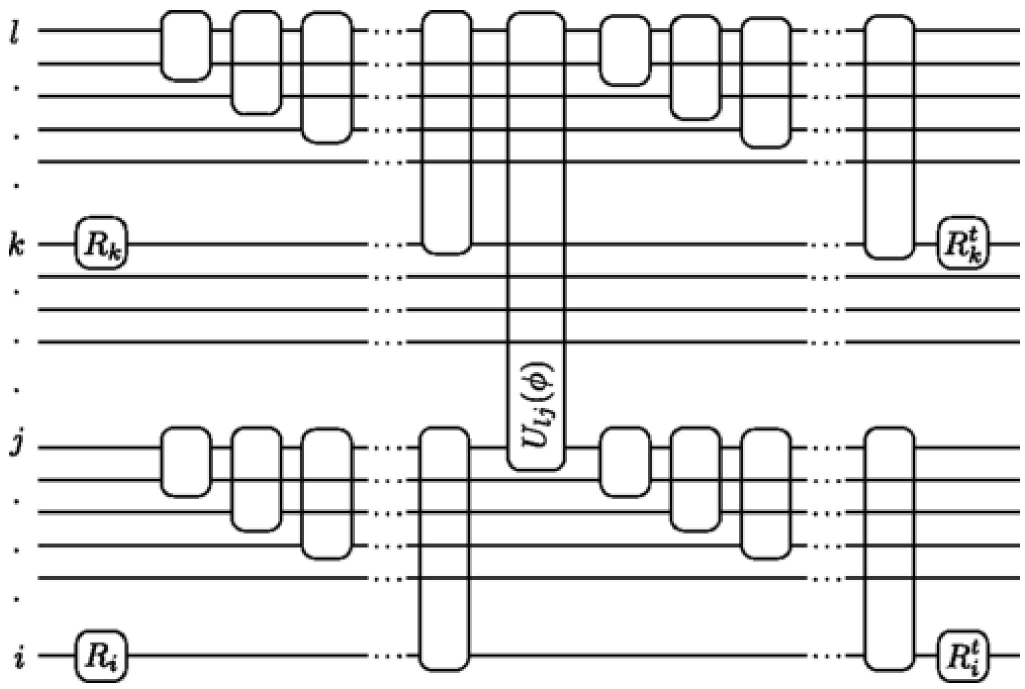
(large recoil energy)





Proposals for experimental realization

arXiv:1607.08560

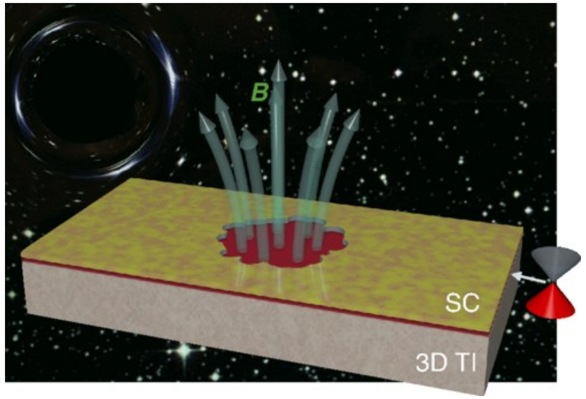


- Quantum circuit

L. García-Álvarez, I. L. Egusquiza, L. Lamata, A. del Campo, J. Sonner, and E. Solano, "Digital Quantum Simulation of Minimal AdS/CFT", PRL **119**, 040501 (2017)

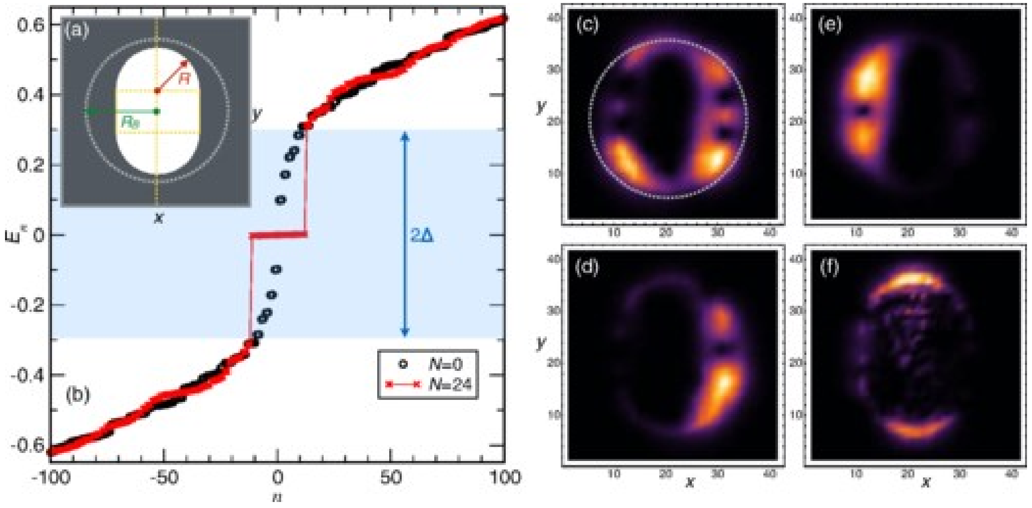
Proposals for experimental realization

arXiv:1702.04426



N quanta of magnetic flux through a nanoscale hole

Inhomogeneous wave functions
due to the irregular shape of the hole

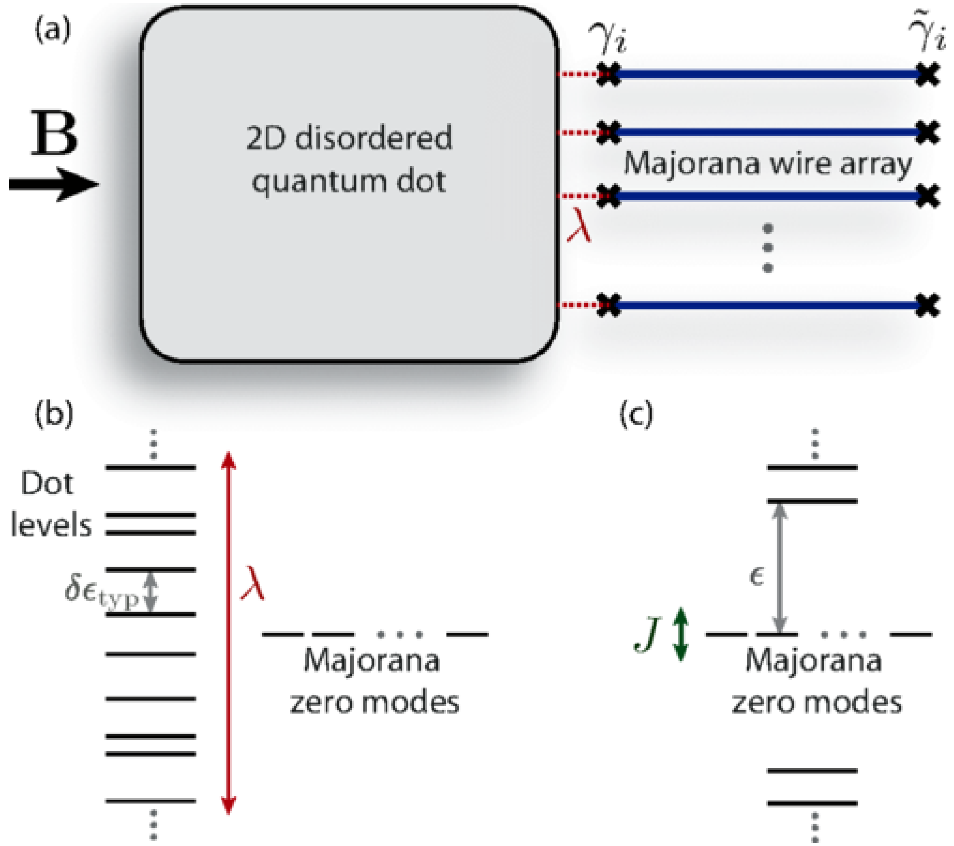


Zero energy states: Majorana fermions

D. I. Pikulin and M. Franz,
“Black Hole on a Chip: Proposal for a Physical Realization of the Sachdev-Ye-Kitaev model in a Solid-State System”,
PRX **7**, 031006 (2017)

Proposals for experimental realization

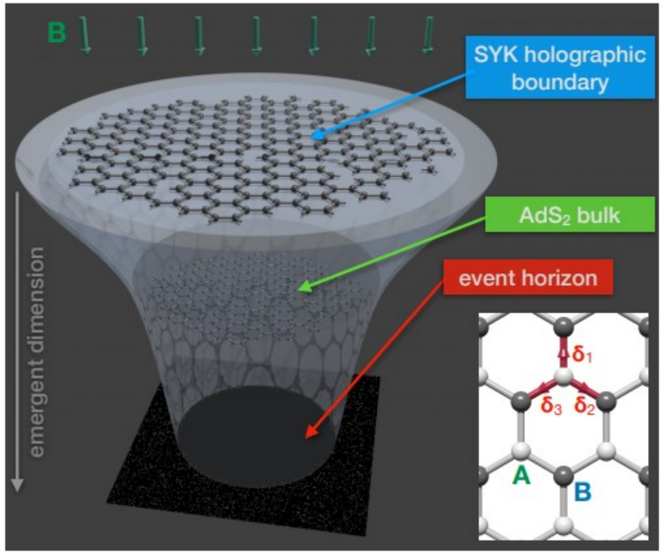
arXiv:1703.06890



Aaron Chew, Andrew Essin, and Jason Alicea, "Approximating the Sachdev-Ye-Kitaev model with Majorana wires", PRB **96**, 121119(R) (2017)

Proposals for experimental realization

arXiv:1802.00802



Anffany Chen, R. Ilan, F. de Juan, D.I. Pikulin, M. Franz, "Quantum holography in a graphene flake with an irregular boundary", arXiv:1802.00802 [PRL **121**, 036403 (2018)]

Review Article | Published: 29 November 2018

Mimicking black hole event horizons in atomic and solid-state systems

Marcel Franz & Moshe Rozali

Nature Reviews Materials **3**, 491–501 (2018) | Download Citation

Sachdev-Ye-Kitaev model

N Majorana- or Dirac- fermions randomly coupled to each other

[Majorana version]

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

[A. Kitaev: talks at KITP
(Feb 12, Apr 7 and May 27, 2015)]

[Dirac version]

$$\hat{H} = \frac{1}{(2N)^{3/2}} \sum_{ij;kl} J_{ij;kl} \hat{c}_i{}^{\dagger} \hat{c}_j{}^{\dagger} \hat{c}_k \hat{c}_l$$

[A. Kitaev's talk]
[S. Sachdev: PRX **5**, 041025 (2015)]

- Solvable in the large *N* limit, Sachdev-Ye "spin liquid" ground state
- Nearly conformal symmetric at low temperature ("emergent …")
- Connection to topological phases of matter
- Holographically corresponds to a quantum black hole?
- Realizes the Maldacena-Shenker-Stanford chaos bound $\lambda_{\mathrm{L}} = 2\pi k_{\mathrm{B}} T/\hbar$

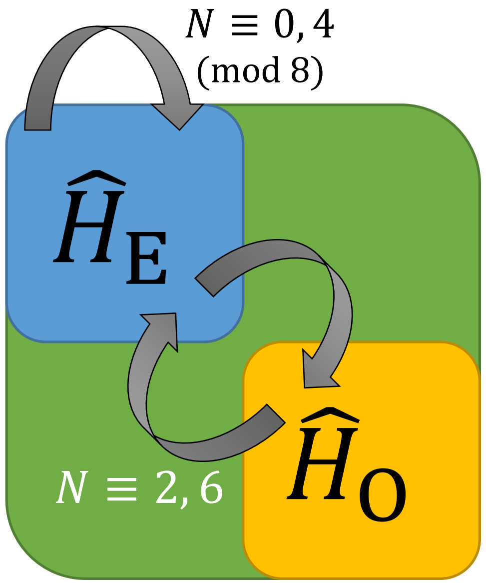
Classification and random matrix theory

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

SPT phase classification for class BDI:
$\mathbb{Z} \rightarrow \mathbb{Z}_8$ due to interaction
[L. Fidkowski and A. Kitaev, PRB 2010, PRB 2011]

Introduce $N/2$ complex fermions $\hat{c}_j = \frac{(\hat{\chi}_{2j-1} + \mathrm{i}\hat{\chi}_{2j})}{\sqrt{2}}$

$\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$ respects the complex fermion parity
Even ($\hat{H}_\mathrm{E}$) and odd ($\hat{H}_\mathrm{O}$) sectors: $L = 2^{N/2-1}$ dimensions



N mod 8	0	2	4	6
η	-1	+1	+1	-1
$\hat{X}^2$	**+1**	+1	**-1**	-1
$\hat{X}$ maps H_E to	H_E	H_O	H_E	H_O
Class	**AI**	**A+A**	**AII**	**A+A**
Gaussian ensemble	**GOE**	**GUE**	**GSE**	**GUE**

$$\hat{X} = \hat{K}\prod_{j=1}^{N/2}(\hat{c}_j^\dagger + \hat{c}_j)$$

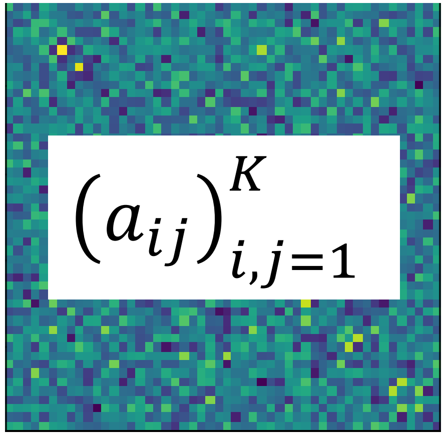
$$\hat{X}\hat{c}_j\hat{X} = \eta\hat{c}_j^\dagger$$

[You, Ludwig, and Xu, PRB 2017]

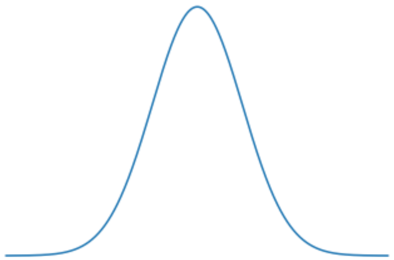
Sparse, but energy spectral statistics strongly resemble that of the corresponding (dense) Gaussian ensemble

[Cotler, …, MT, JHEP 2017]

Gaussian random matrices



$a_{ij} = a_{ji}^{*}$



Gaussian distribution

Density $\propto e^{-\frac{\beta K}{4}\mathrm{Tr}H^2} = \exp\left(-\frac{\beta K}{4}\sum_{i,j}^{K}\left|a_{ij}\right|^2\right)$

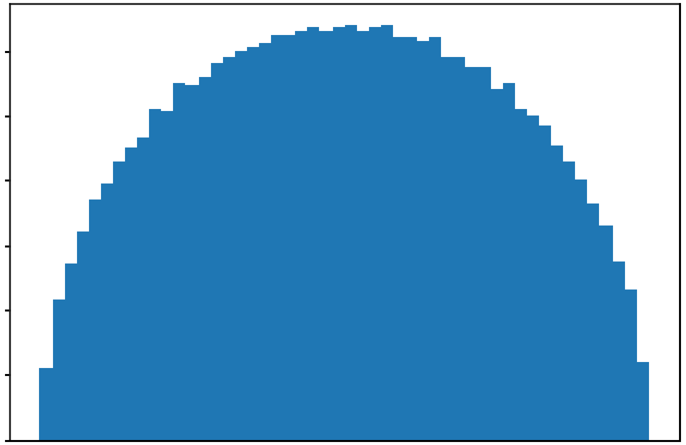
[F. J. Dyson, J. Math. Phys. **3**, 1199 (1962)]

Real (β=1): Gaussian Orthogonal Ensemble (GOE)

Complex (β=2): G. Unitary E. (GUE)

Quaternion (β=4): G. Symplectic E. (GSE)

Eigenvalue distribution: semi-circle law



$e_1 \leq e_2 \leq \cdots \leq e_L$

Δ: averaged level separation near e_j

Level repulsion

Joint distribution

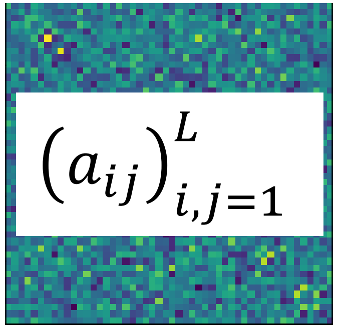
$$p(e_1, e_2, \ldots, e_K) \propto \prod_{1\leq i<j\leq K}\left|e_i - e_j\right|^{\beta}\prod_{i=1}^{K} e^{-\beta K {e_i}^2/4}$$

- $P(s)$: Distribution of normalized level separation $s = \frac{e_{j+1}-e_j}{\Delta(\bar{e})}$

GOE/GUE/GSE: $P(s) \propto s^{\beta}$ at small s, has e^{-s^2} tail

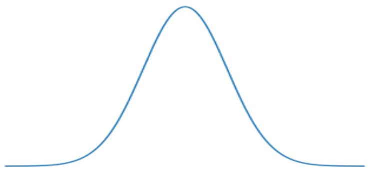
Uncorrelated: $P(s) = e^{-s}$ (Poisson distribution)

Gaussian random matrices



$\left(a_{ij}\right)_{i,j=1}^{L}$

$a_{ij} = a_{ji}^{*}$



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Density $\propto e^{-\frac{\beta K}{4}\mathrm{Tr}H^2} = \exp\left(-\frac{\beta K}{4}\sum_{i,j}^{K}\left|a_{ij}\right|^2\right)$

Real (β=1): Gaussian Orthogonal Ensemble (GOE)
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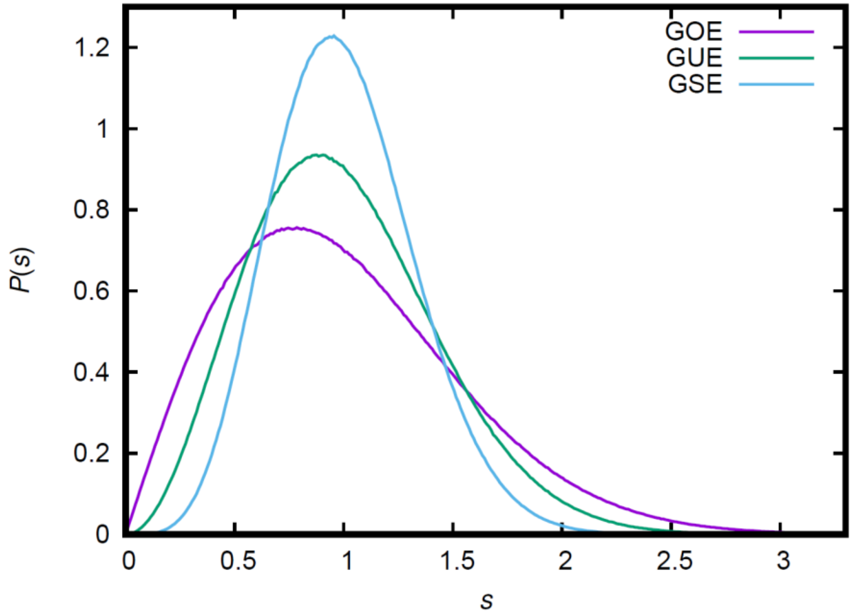
Level repulsion

Joint distribution

$$p(e_1, e_2, \ldots, e_K) \propto \prod_{1\le i<j\le K} \left|e_i - e_j\right|^{\beta} \prod_{i=1}^{K} e^{-\beta K e_i^2/4}$$



- $P(s)$: Distribution of normalized level separation $s = \frac{e_{j+1}-e_j}{\Delta(\bar{e})}$

  GOE/GUE/GSE: $P(s) \propto s^{\beta}$ at small s, has e^{-s^2} tail
  Uncorrelated: $P(s) = e^{-s}$ (Poisson distribution)

- $\langle r\rangle$: Average of neighboring gap ratio

  Uncorrelated: $2\log 2 - 1 \approx 0.386$
  GOE/GUE/GSE: larger (*e.g.* 0.599 for GUE [Y. Y. Atas *et al.* PRL 2013])

$$r = \frac{\min(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}$$

➔ SYK model results: indistinguishable from corresponding Gaussian ensemble

Density of states

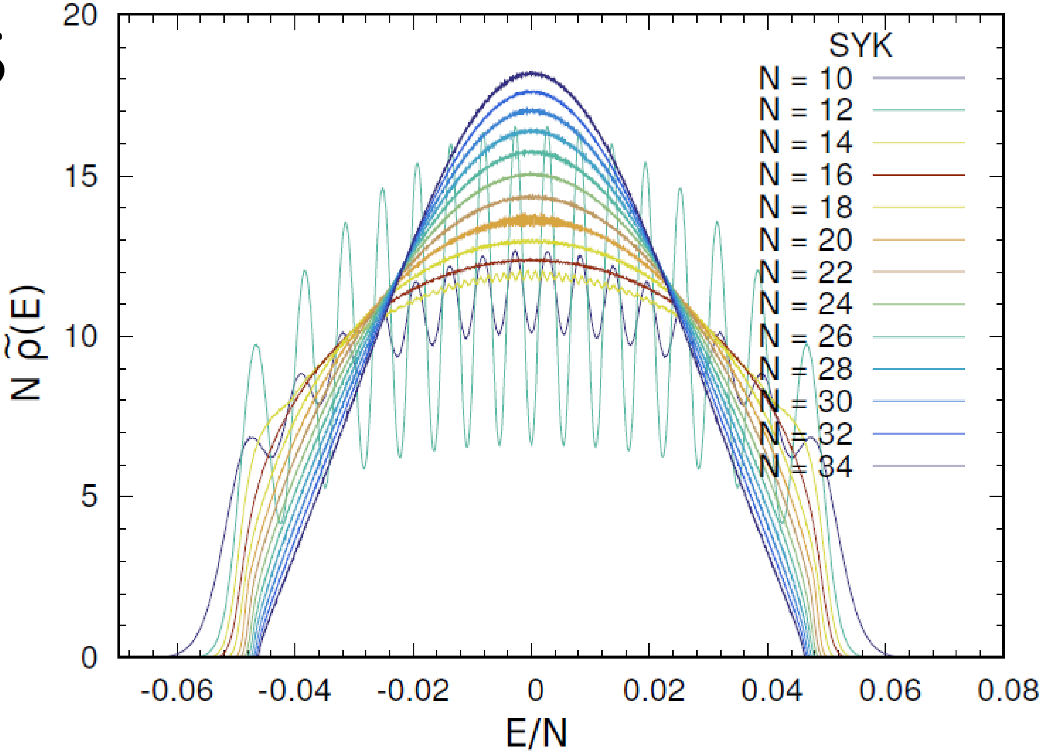


Figure 15. Normalized density of states $\tilde{\rho}(E)$ for the SYK model with $N = 10, 12, \ldots, 34$. The bin width is $10^{-3}J$. Notice that the energy is measured in units of NJ. The numbers of samples are 21600000 ($N = 10$), 10800000 ($N = 12$), 5400000 ($N = 14$), 1200000 ($N = 16$), 600 000 ($N = 18$), 240 000 ($N = 20$), 120 000 ($N = 22$), 48 000 ($N = 24$), 10 000 ($N = 26$), 3 000 ($N = 28$), 1 000 ($N = 30$), 516 ($N = 32$), 90 ($N = 34$).

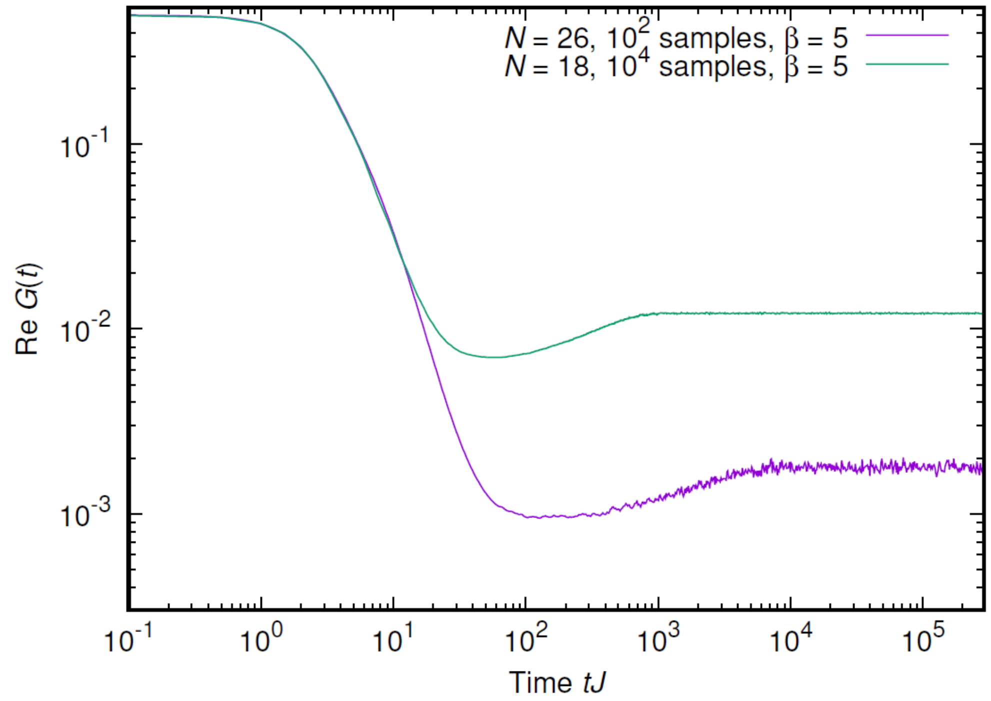
[Cotler, MT et al., JHEP05(2017)118]

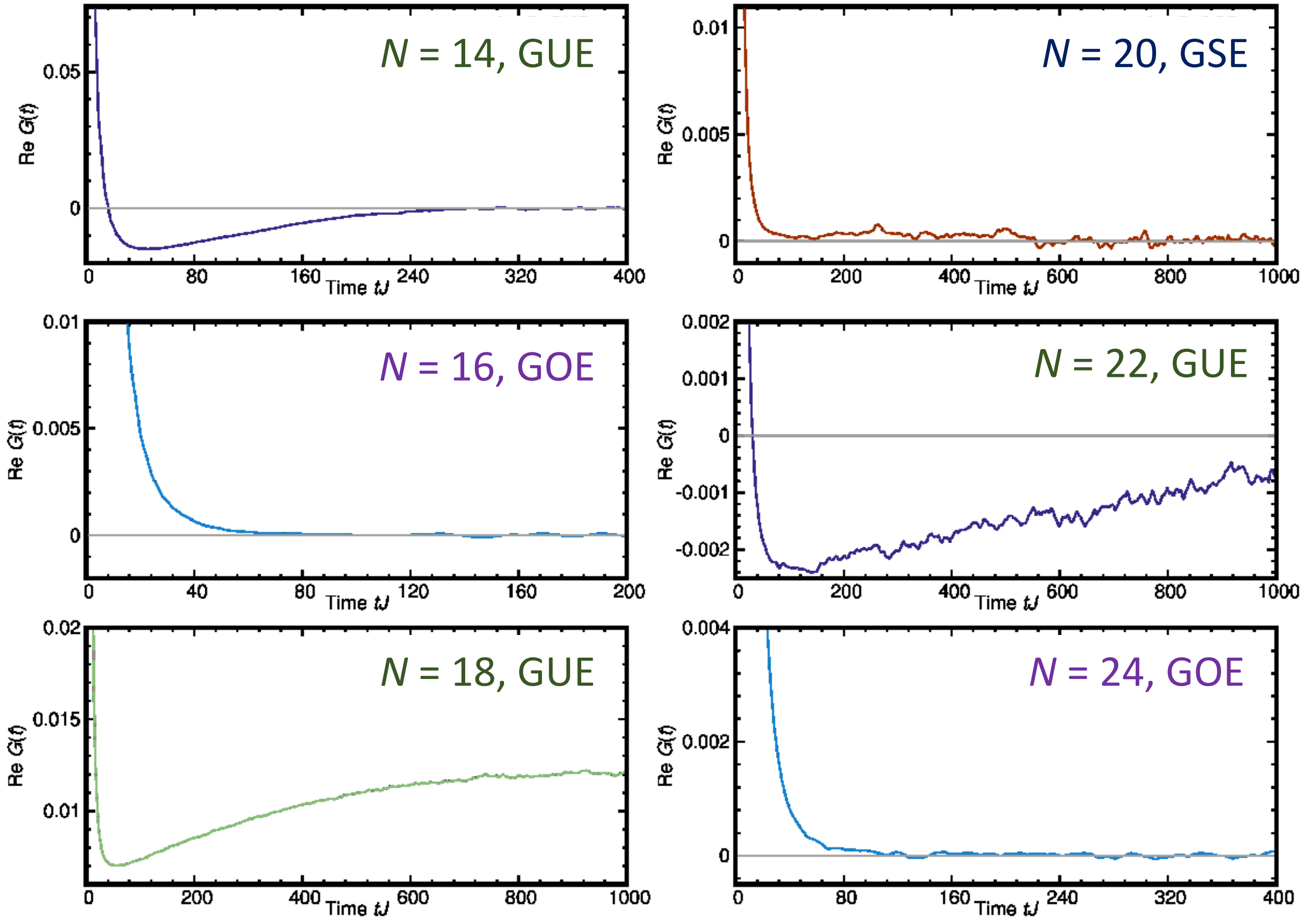
[Cotler, MT et al., JHEP05(2017)118]

Correlation function

$$G(t) = \langle \hat{\chi}_a(t)\hat{\chi}_a(0)\rangle_\beta = \frac{1}{Z(\beta)}\sum_{m,n} \mathrm{e}^{-\beta E_m}|\langle m|\hat{\chi}_a|n\rangle|^2 \mathrm{e}^{\mathrm{i}(E_m-E_n)t}$$

Dip-ramp-plateau structure for $N \equiv 2 \pmod 8$





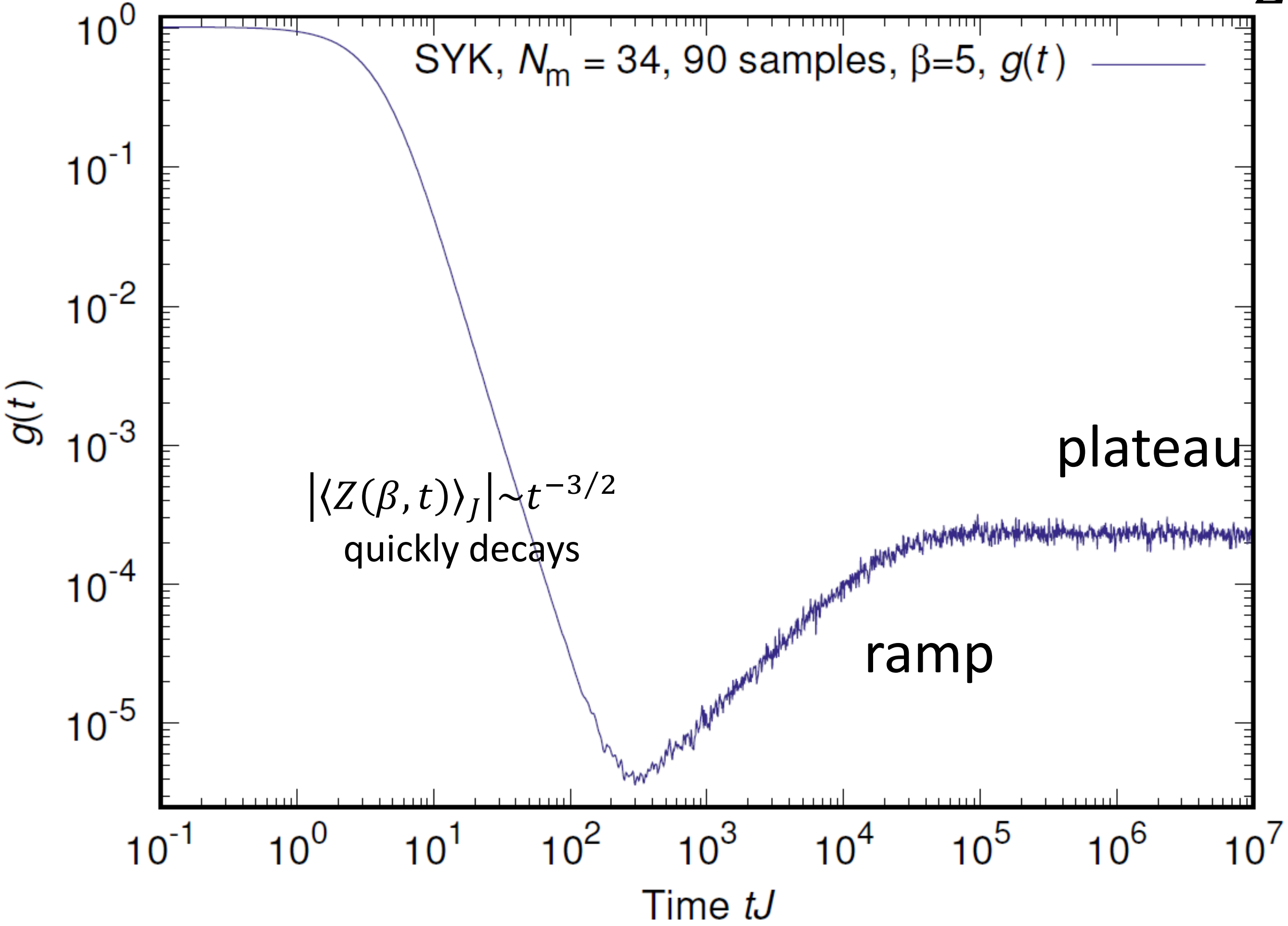
N mod 8	0	2	4	6
$\hat{X}$ maps H_E to	H_E	H_O	H_E	H_O
$\langle\mathrm{even}\vert\chi\vert\mathrm{odd}\rangle$		finite		0
Gaussian ensemble	GOE	GUE	GSE	GUE

Spectral form factor

$$G(t) = \langle \hat{\chi}_a(t)\hat{\chi}_a(0)\rangle_\beta = \frac{1}{Z(\beta, t=0)}\sum_{m,n} \mathrm{e}^{-\beta E_m}|\langle m|\hat{\chi}_a|n\rangle|^2 \mathrm{e}^{\mathrm{i}(E_m-E_n)t}$$

$$g(\beta,t) = \left|\frac{Z(\beta,t)}{Z(\beta,t=0)}\right|^2 = \frac{1}{Z(\beta,t=0)^2}\sum_{m,n} \mathrm{e}^{-\beta(E_m+E_n)}\mathrm{e}^{\mathrm{i}(E_m-E_n)t}$$

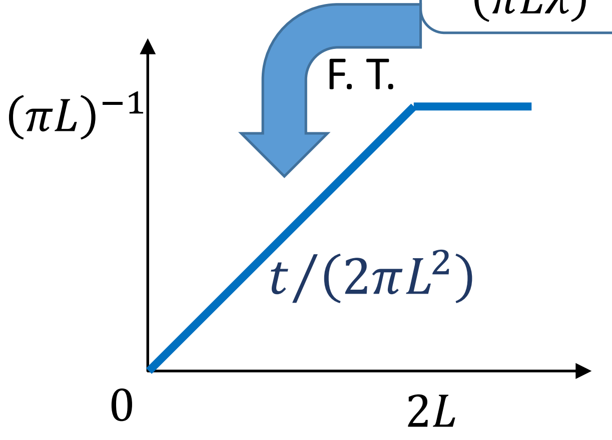
$$Z(\beta,t) = Z(\beta+\mathrm{i}t) = \mathrm{Tr}\left(\mathrm{e}^{-\beta\hat{H}-\mathrm{i}\hat{H}t}\right)$$



$$g_\mathrm{c}(\beta,t) = \frac{\langle |Z(\beta,t)|^2\rangle_J - |\langle Z(\beta,t)\rangle_J|^2}{\langle Z(\beta)\rangle_J^{\,2}}$$

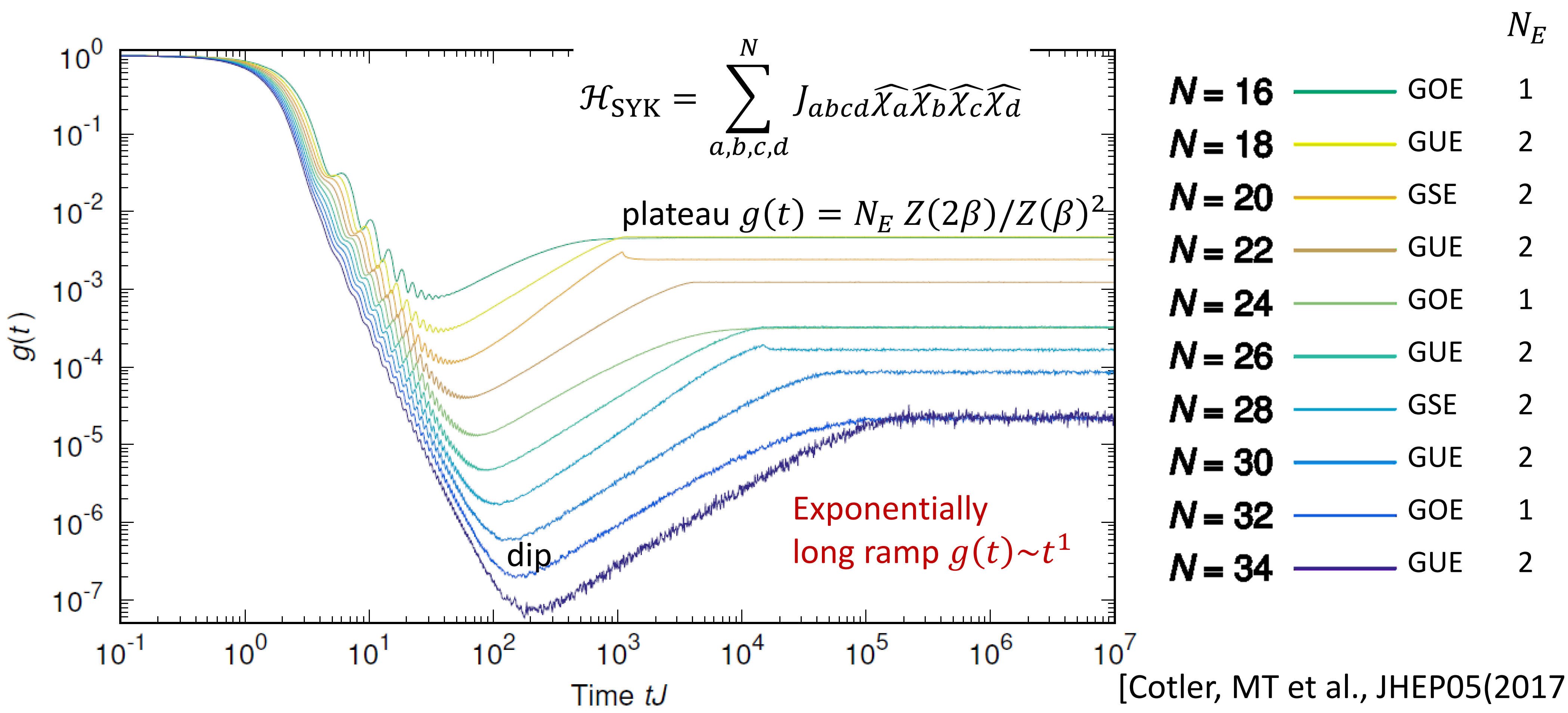
$$\sim \iint d\lambda_1 d\lambda_2 \langle \delta\rho(\lambda_1)\delta\rho(\lambda_2)\rangle e^{\mathrm{i}t(\lambda_1-\lambda_2)}$$

$$R(\lambda) = \langle \delta\rho(\lambda_1)\delta\rho(\lambda_1-\lambda)\rangle = -\frac{\sin^2 L\lambda}{(\pi L\lambda)^2} + \frac{1}{\pi L}\delta(\lambda)$$



in RMT

N dependence of the spectral form factor



[Cotler, MT et al., JHEP05(2017)118]

Characterization of quantum many-body chaos

- Random-matrix like energy level correlation

- Exponential Lyapunov growth of out-of-time-order correlators (OTOC)
$$\langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle \sim C + \#\, e^{2\lambda_{\rm L} t}$$

Example: the Sachdev-Ye-Kitaev model

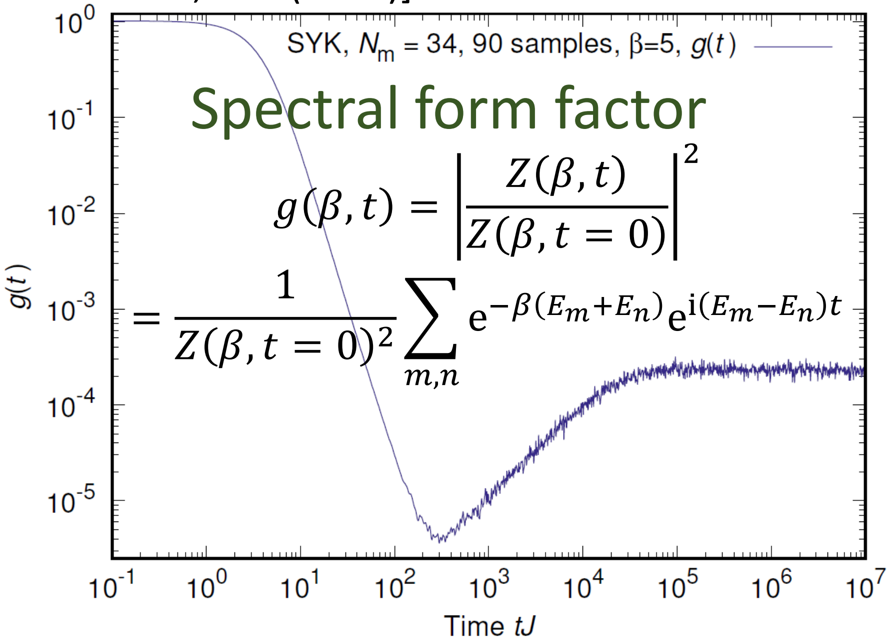
$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

$\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$

J_{abcd} : Gaussian random ($\langle {J_{abcd}}^2\rangle = J^2 = 1$)

[Kitaev 2015]

[Cotler, MT et al., JHEP **1705**, 118 (2017)]



Spectral form factor

$$g(\beta,t) = \left|\frac{Z(\beta,t)}{Z(\beta,t=0)}\right|^2 = \frac{1}{Z(\beta,t=0)^2}\sum_{m,n} e^{-\beta(E_m+E_n)}e^{i(E_m-E_n)t}$$

N mod 8	RMT
0	GOE
2	GUE
4	GSE
6	GUE

Lyapunov exponent

$$\lambda_{\rm L} = \frac{2\pi k_{\rm B}T}{\hbar} \text{ in low } T \text{ limit}$$

(Maldacena-Shenker-Stanford chaos bound)

We propose two new characterizations of quantum chaos

- Quantum Lyapunov spectrum: Quantum version of finite-time Lyapunov spectrum

$\hat{M}_{ab}(t)$: (anti)commutator of $\hat{O}_a(t)$ and $\hat{O}_b(0)$

$$\hat{L}_{ab}(t) = \sum_{j=1}^{N} \hat{M}_{ja}(t)^\dagger \hat{M}_{jb}(t)$$

$\left\{\lambda_k(t) = \frac{\log s_k(t)}{2t}\right\}$ for singular values $\{s_k(t)\}_{k=1}^{N}$ of $N \times N$ matrix $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$.

arXiv:1809.01671

- Two-point correlations:

$G_{ab}^{(\phi)} = \langle\phi|\hat{O}_a(t)\hat{O}_b(0)|\phi\rangle$ as matrix, log (singular values)

arXiv:1902.11086

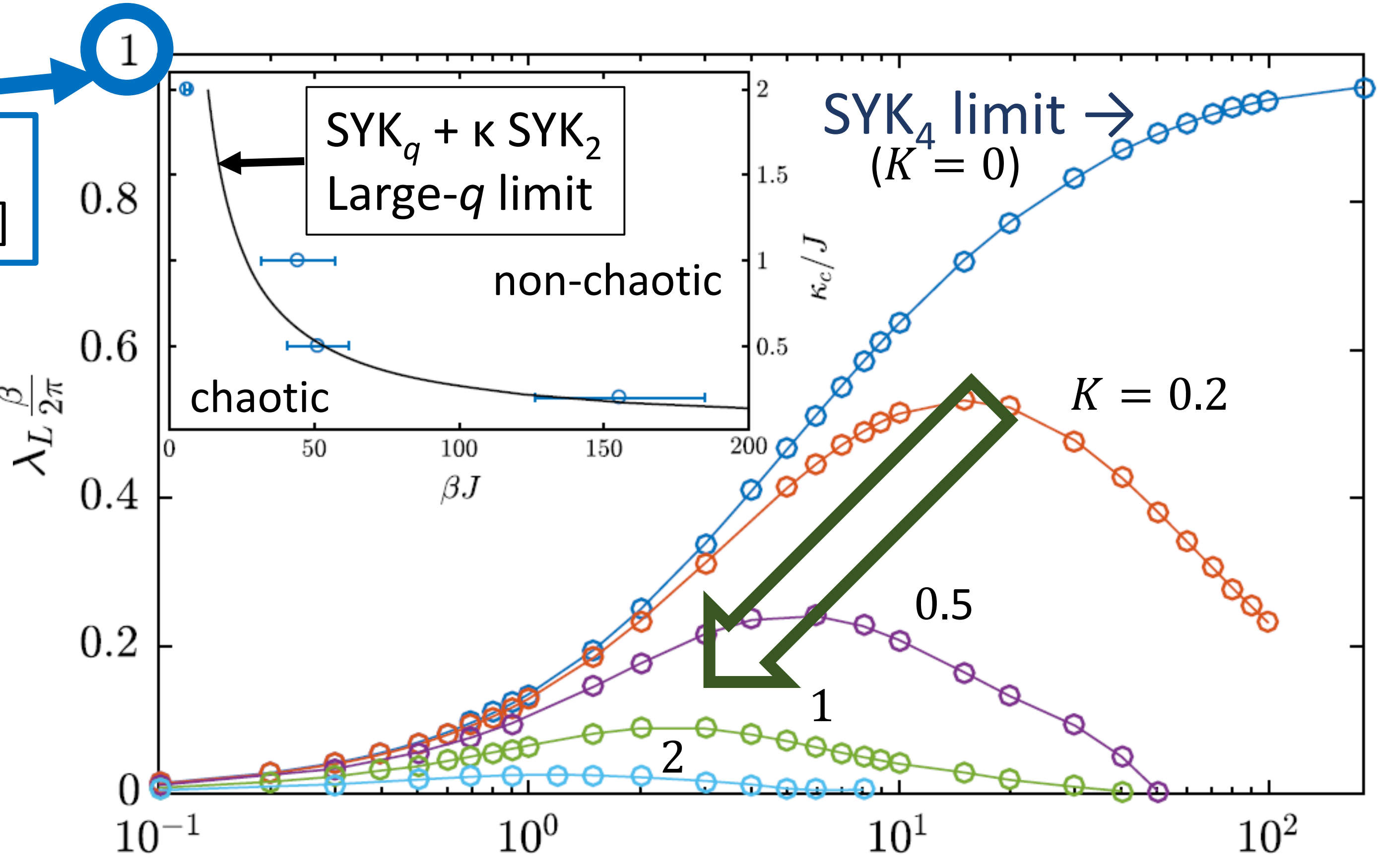
Modified SYK model: Large-N calculation for OTOC

$$\hat{H} = \overbrace{\sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}^{\text{SYK}_4} + i\overbrace{\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b}^{\text{SYK}_2}$$

K_{ab}: standard deviation $\frac{K}{\sqrt{N}}$

Chaos bound [Maldacena, Shenker, and Stanford 2016]

normalized Lyapunov exponent



$\beta = 1/(k_B T)$ inverse temperature

A. M. Garcia-Garcia, B. Loureiro, A. Romero-Bermudez, and MT, PRL **120**, 241603 (2018)

Deviation from the chaos bound as SYK$_2$ component is introduced

1. Quantum Lyapunov spectrum

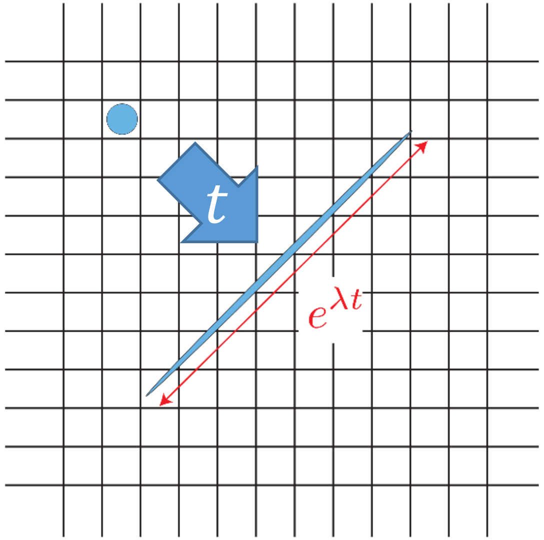
OTOCs have been intensively studied:

$$F(t) = \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle$$

- Measurement protocols
  - [B. Swingle, G. Bentsen, M. Schleier-Smith, P. Hayden, PRB **94**, 040302 (2016)] and experimental proposal papers for the SYK model
- Experimental measurements
  - trapped ions [M. Gärttner et al. Nat. Phys. **13**, 781 (2017) 1608.08938]
  - NMR [J. Li et al. PRX **7**, 031011 (2017) 1609.01246]
- Quantum information (scrambling, …)
- Many-body localization
- Fluctuation-dissipation theorem
  - [N. Tsuji, T. Shitara, and M. Ueda, PRE **97**, 012101 (2018)]

Q. Which operators should we use?

Lyapunov growth of phase space



Coarse-grained phase space

- Just one direction?
- If more than one, what are relations between λ?

[M. Hanada, H. Shimada, and M. Tezuka, PRE **97**, 022224 (2018)]

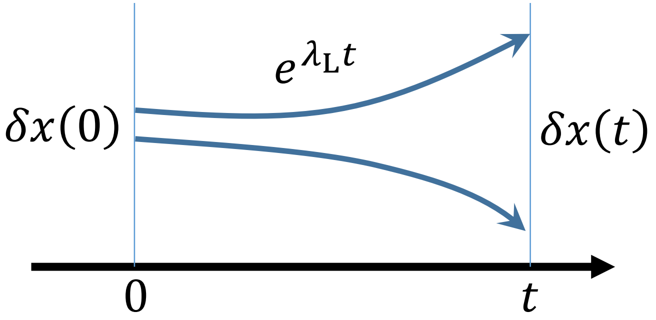
Observation for classical chaos

Classical system with K degrees of freedom

Deviation at t initial infinitesimal deviation

$$\delta x_i(t) = M_{ij}\delta x_j(0)$$

$$M_{ij} = \frac{\delta x_i(t)}{\delta x_j(0)} = \{x(t), p(0)\}_{\mathrm{PB}}$$



(Usually $t \to \infty$ limit is taken for obtaining λ_{L})

$$L = \left(\frac{\delta x_i(t)}{\delta x_j(0)}\right)^2 \qquad \{x(t), p(0)\}_{\mathrm{PB}}{}^2 = \left(\frac{\partial x(t)}{\partial x(0)}\right)^2 \to e^{2\lambda_{\mathrm{L}} t}$$

We consider finite t

Singular values of M_{ij}: $\{a_k(t)\}_{k=1}^{K}$

Time-dependent Lyapunov spectrum

$$\left\{\lambda_k(t) = \frac{\log a_k(t)}{t}\right\}_{k=1,2,\ldots,K}$$

obeys random matrix-like statistics

in several chaotic systems

- Logistic map
- Lorenz attractor
- D0 brane matrix model (without fermions)

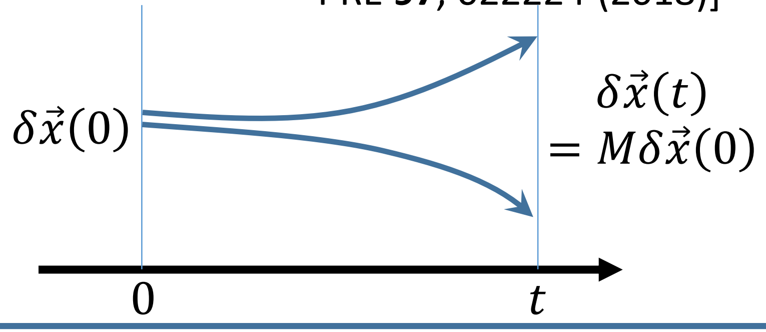
Quantum Lyapunov spectrum

Gharibyan, Hanada, Swingle, and MT, JHEP04(2019)082 (arXiv:**1809.01671**)

Finite-time **classical Lyapunov spectrum**: obeys RMT statistics for chaos

[Hanada, Shimada, and MT: PRE **97**, 022224 (2018)]

Singular values of $M_{ij} = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)$ **at finite *t***: $\{s_k(t)\} = \{e^{\lambda_k t}\}$



$L = \{x_i(t), p_j(0)\}_{\mathrm{PB}}{}^2 = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)^2 \to e^{2\lambda_{\mathrm{L}} t}$ for large t

OTOC: $C_T(t) = \left\langle \left| \left[\hat{W}(t), \hat{V}(t=0) \right] \right|^2 \right\rangle = \langle \hat{W}^\dagger(t) \hat{V}^\dagger(0) \hat{W}(t) \hat{V}(0) \rangle + \cdots$

Quantum Lyapunov spectrum: Define $\hat{M}_{ab}(t)$ as (anti)commutator of $\hat{O}_a(t)$ and $\hat{O}_b(0)$

$$\hat{L}_{ab}(t) = \left[\hat{M}(t)^\dagger \hat{M}(t)\right]_{ab} = \sum_{j=1}^{N} \hat{M}_{ja}(t)^\dagger \hat{M}_{jb}(t)$$

For $N \times N$ matrix $\langle \phi | \hat{L}_{ab}(t) | \phi \rangle$, obtain singular values $\{s_k(t)\}_{k=1}^{N}$.

The Lyapunov spectrum is defined as $\left\{\lambda_k(t) = \frac{\log s_k(t)}{2t}\right\}$.

Quantum Lyapunov spectrum for SYK model + modification

$$\hat{H} = \sum_{1 \le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1 \le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

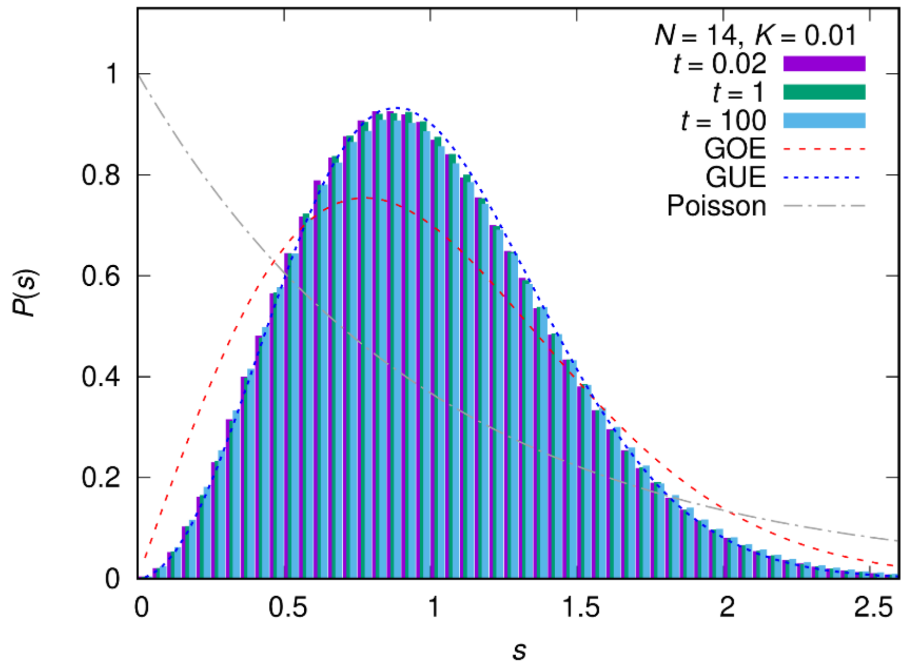
J_{abcd}: s. d. $= \frac{\sqrt{6}}{N^{3/2}}$

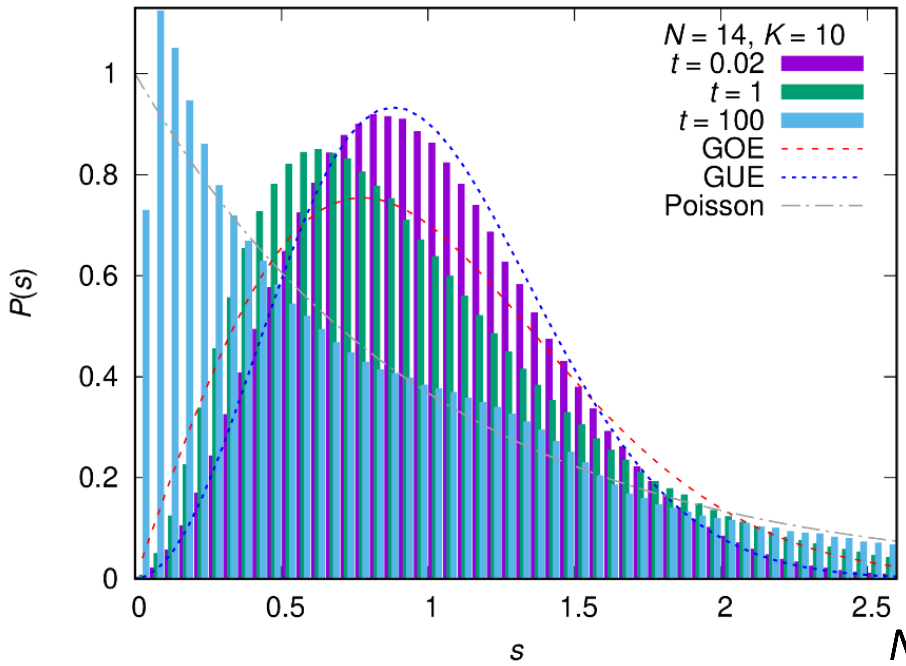
K_{ab}: s. d. $= \frac{K}{\sqrt{N}}$

- Define $\hat{L}_{ab}(t) = \sum_{j=1}^{N} \hat{M}_{ja}(t)\hat{M}_{jb}(t)$ for time-dependent anticommutator $\hat{M}_{ab}(t) = \{\hat{\chi}_a(t), \hat{\chi}_b(0)\}$.
- Obtain the singular values $\{a_k(t)\}_{k=1}^{K}$ of $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$
- Quantum Lyapunov spectrum: $\left\{\lambda_k(t) = \frac{\log a_k(t)}{2t}\right\}_{k=1,2,\dots,K}$ (also dependent on state ϕ)

Other possibilities: see Rozenbaum-Ganeshan-Galitski, 1801.10591; Hallam-Morley-Green: 1806.05204

Spectral statistics of quantum Lyapunov spectrum: SYK





arXiv:**1809.01671**

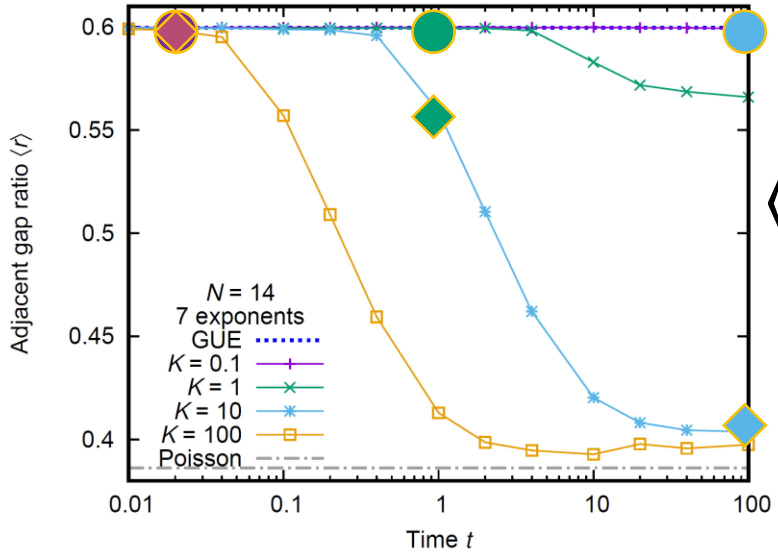
Energy eigenstates

$N/2$ larger exponents

$K = 0.01$ (●):

Remains GUE for long time

Exponents are nearly constant until the singular values of $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$ saturate: Lyapunov growth



$K = 10$ (◆):

Approaches Poisson

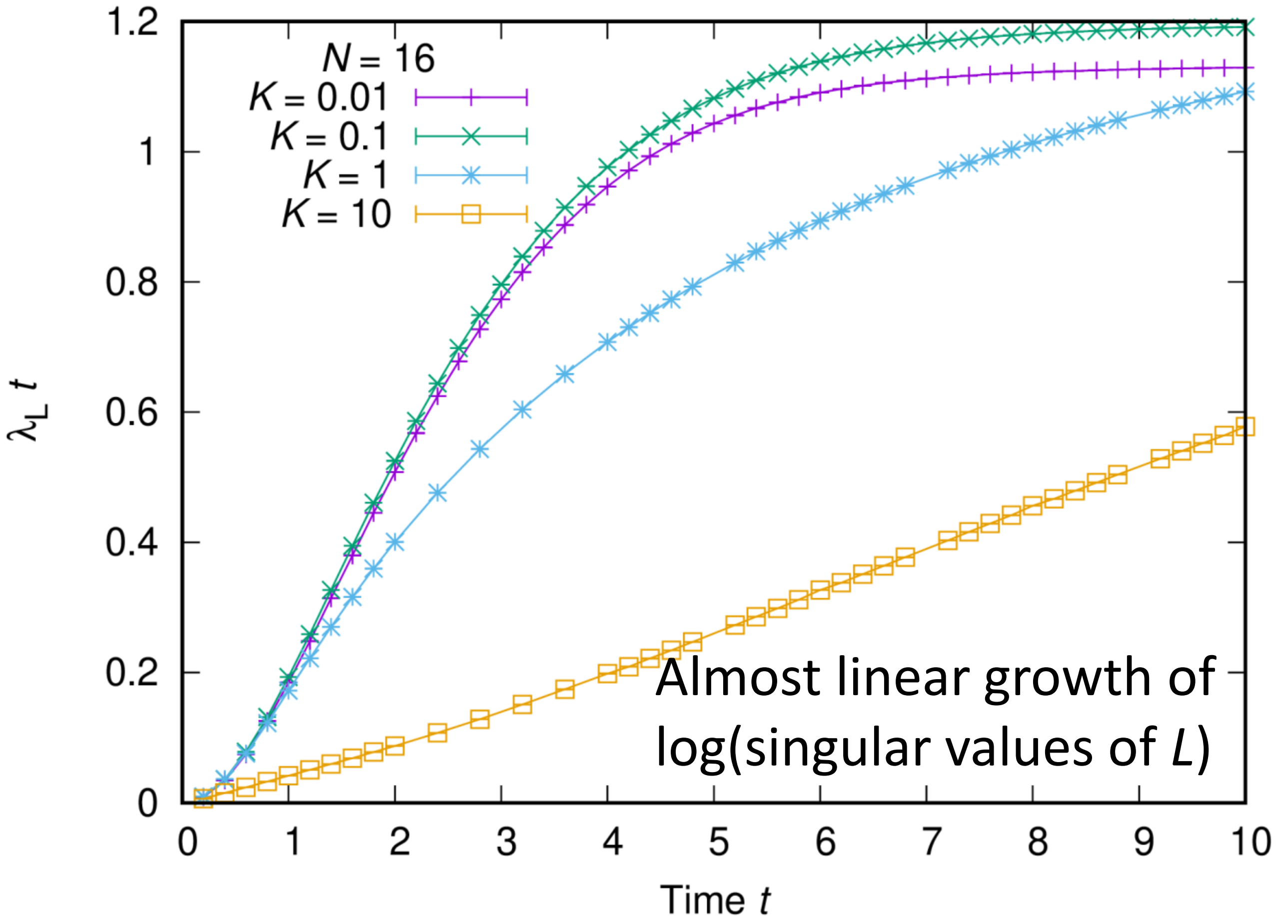
$\langle r \rangle$: average of

$$\frac{\min(\epsilon_{i+1}-\epsilon_i\,,\epsilon_{i+2}-\epsilon_{i+1})}{\max(\epsilon_{i+1}-\epsilon_i\,,\epsilon_{i+2}-\epsilon_{i+1})}$$

(fixed-i unfolding: unfold each gap $\lambda_{i+1} - \lambda_i$ using its average)

Gharibyan, Hanada, Swingle, and MT, JHEP04(2019)082 (arXiv:**1809.01671**)

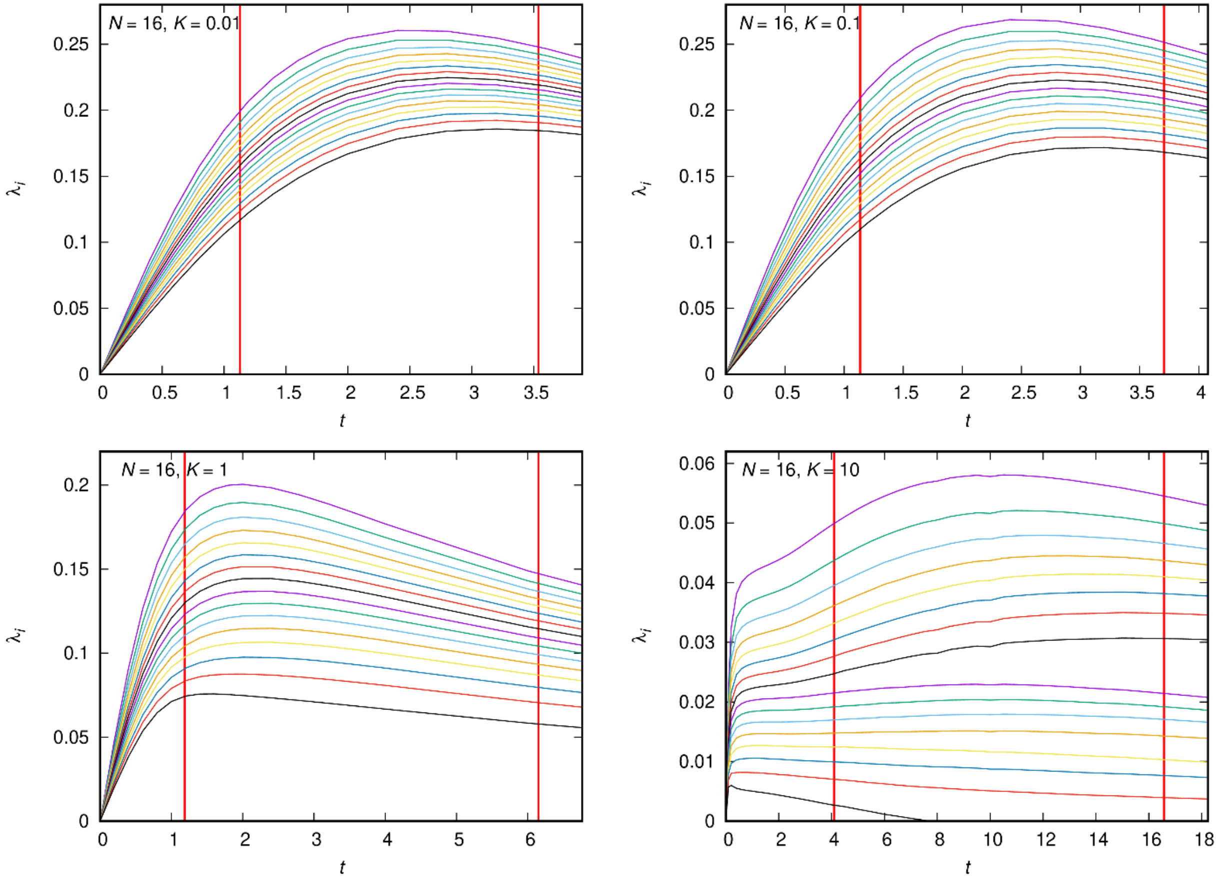
Growth of (largest Lyapunov exponent)*time



Gharibyan, Hanada, Swingle, and MT, JHEP04(2019)082 (arXiv:**1809.01671**)

Full Lyapunov spectrum

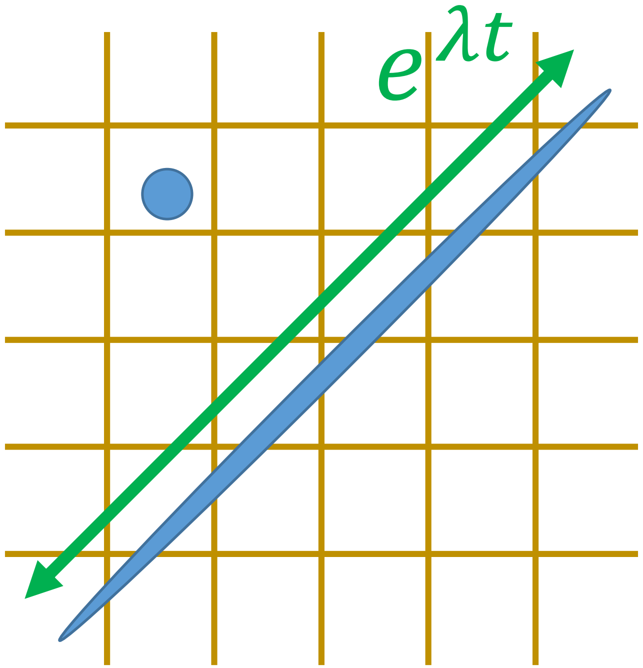
Sample- and state-averaged



Close to constant between red lines
(20 % and 80 % of the saturated value of $\lambda_N t$)

Kolmogorov-Sinai entropy

Coarse-grained entropy
= log(# of cells covering the region)
~ (sum of positive λ) t

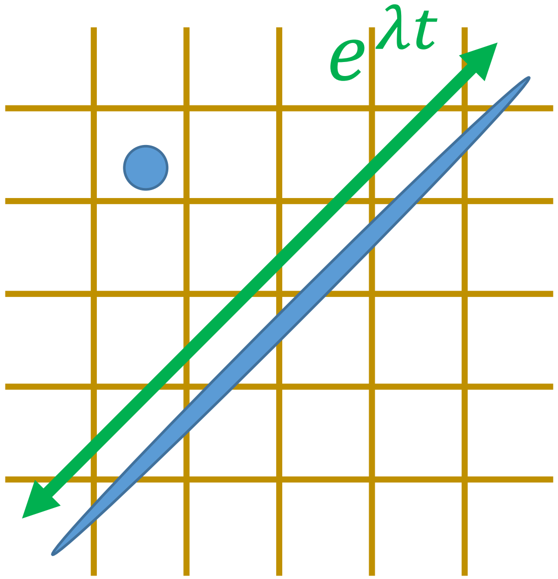


Kolmogorov-Sinai entropy h_{KS}
= (sum of positive λ)
= entropy production rate

arXiv:**1809.01671**

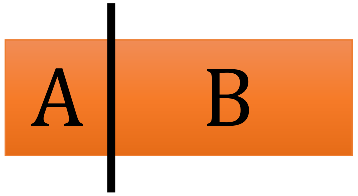
Kolmogorov-Sinai entropy vs entanglement entropy production

Coarse-grained entropy
= log(# of cells covering the region)
~ (sum of positive λ) t



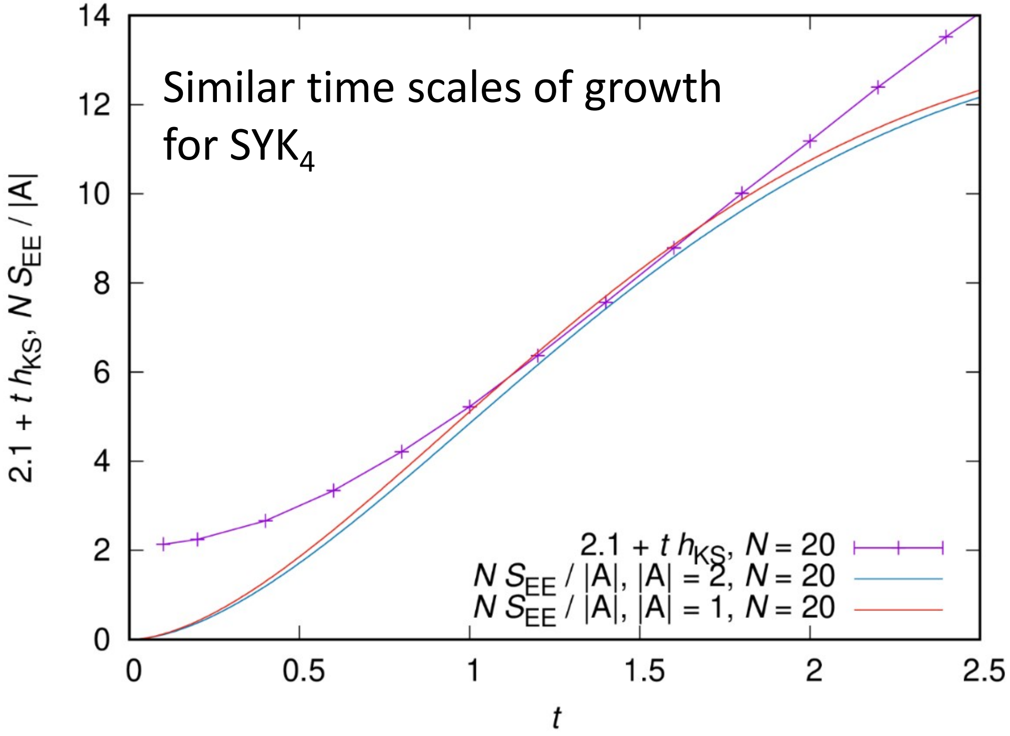
Kolmogorov-Sinai entropy h_{KS}
= (sum of positive λ)
= entropy production rate

Initial state with $S_{\mathrm{EE}} = 0$:
$|\psi(t=0)\rangle = |\underbrace{000\ldots000}_{N/2}\rangle$ in the complex fermion basis $\hat{c}_j = \frac{(\chi_{2j-1} + \mathrm{i}\chi_{2j})}{\sqrt{2}}$



$\rho_{\mathrm{A}}(t) = \mathrm{Tr}_{\mathrm{B}}\,\rho(t),\ \rho(t) = |\psi(t)\rangle\langle\psi(t)|$

$$S_{\mathrm{EE}}(t) = -\mathrm{Tr}\,\rho_{\mathrm{A}}(t)\log\big(\rho_{\mathrm{A}}(t)\big)$$

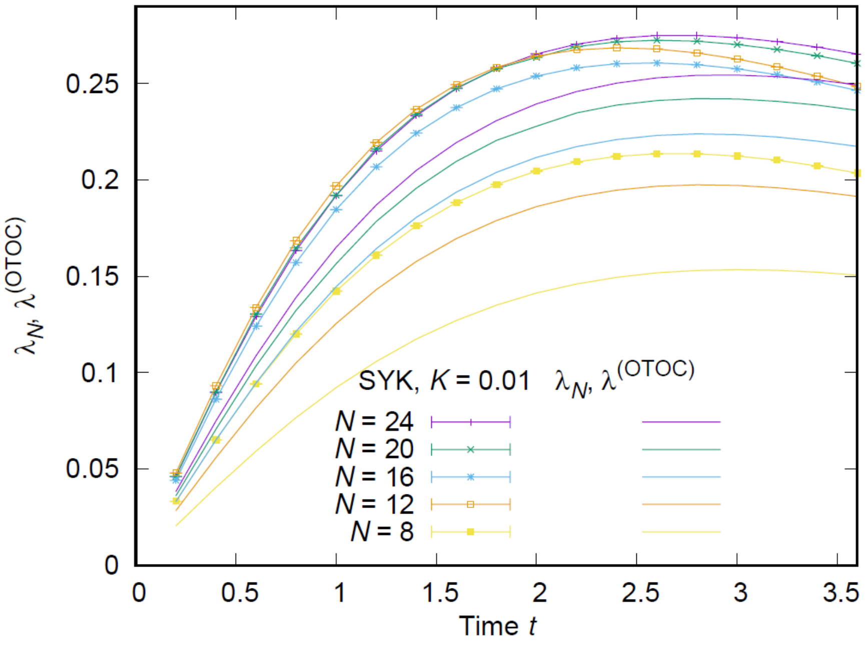


Fastest entropy production?

SYK$_4$ limit

- λ_N and $\lambda_{\mathrm{OTOC}} = \frac{1}{2t}\log\left(\frac{1}{N}\sum_{i=1}^{N} e^{2\lambda_i t}\right)$ approach each other; difference decreases as $1/N$
- Same for λ_N and λ_1:

  all exponent → single peak
- All saturate the MSS bound at strong coupling (low *T*) limit
- Growth rate of entanglement entropy $\sim h_{\mathrm{KS}} =$ sum of positive (all) λ_i

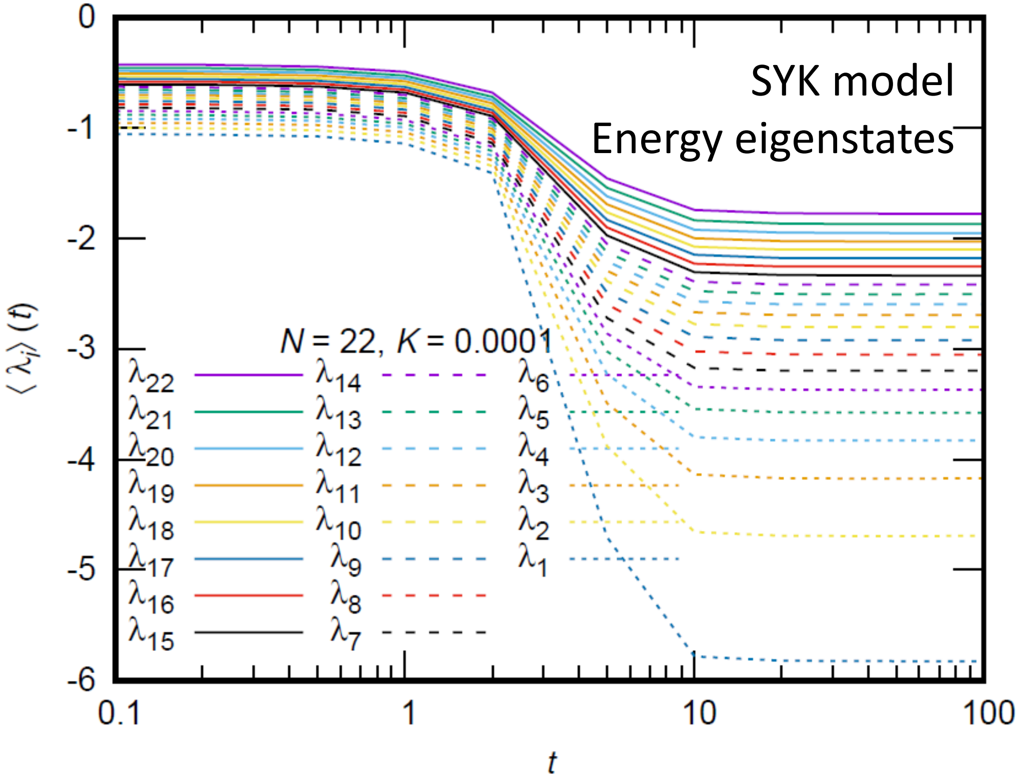


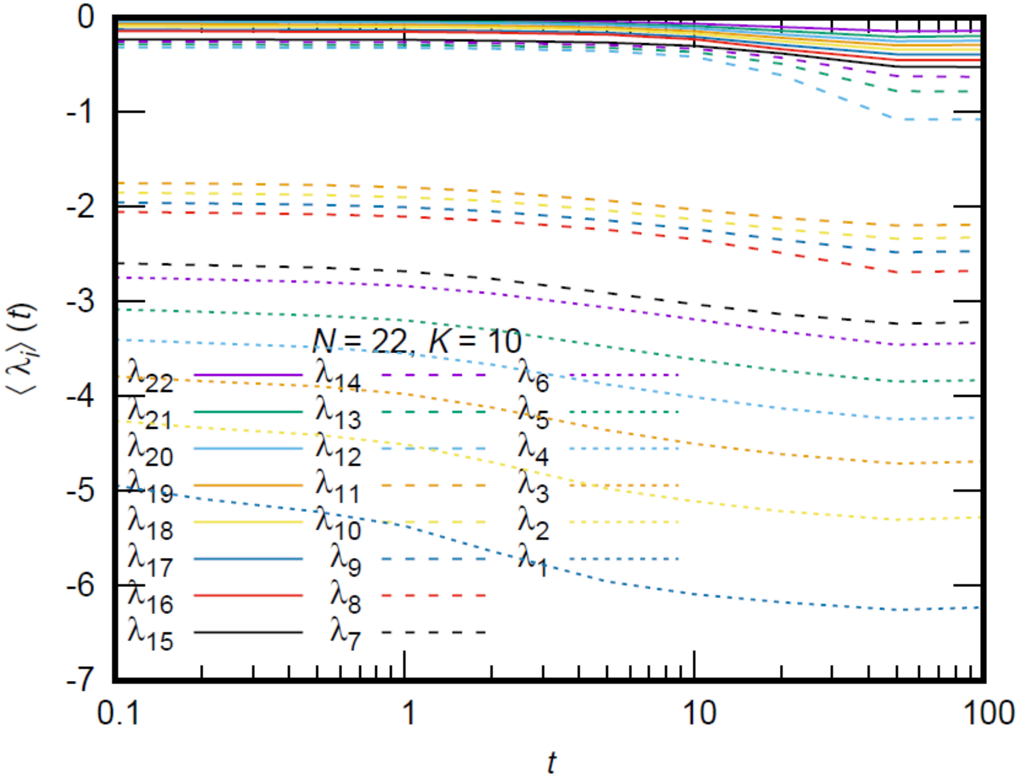
➔ [conjecture] SYK model: not only the fastest scramblers, but also fastest entropy generators

2. Singular value statistics of two-point functions

$$G_{ab}^{(\phi)} = \langle \phi | \hat{\chi}_a(t) \hat{\chi}_b(0) | \phi \rangle$$

$$\lambda_j = \log\left[\text{singular values of}\left(G_{ab}^{(\phi)}\right)\right]$$





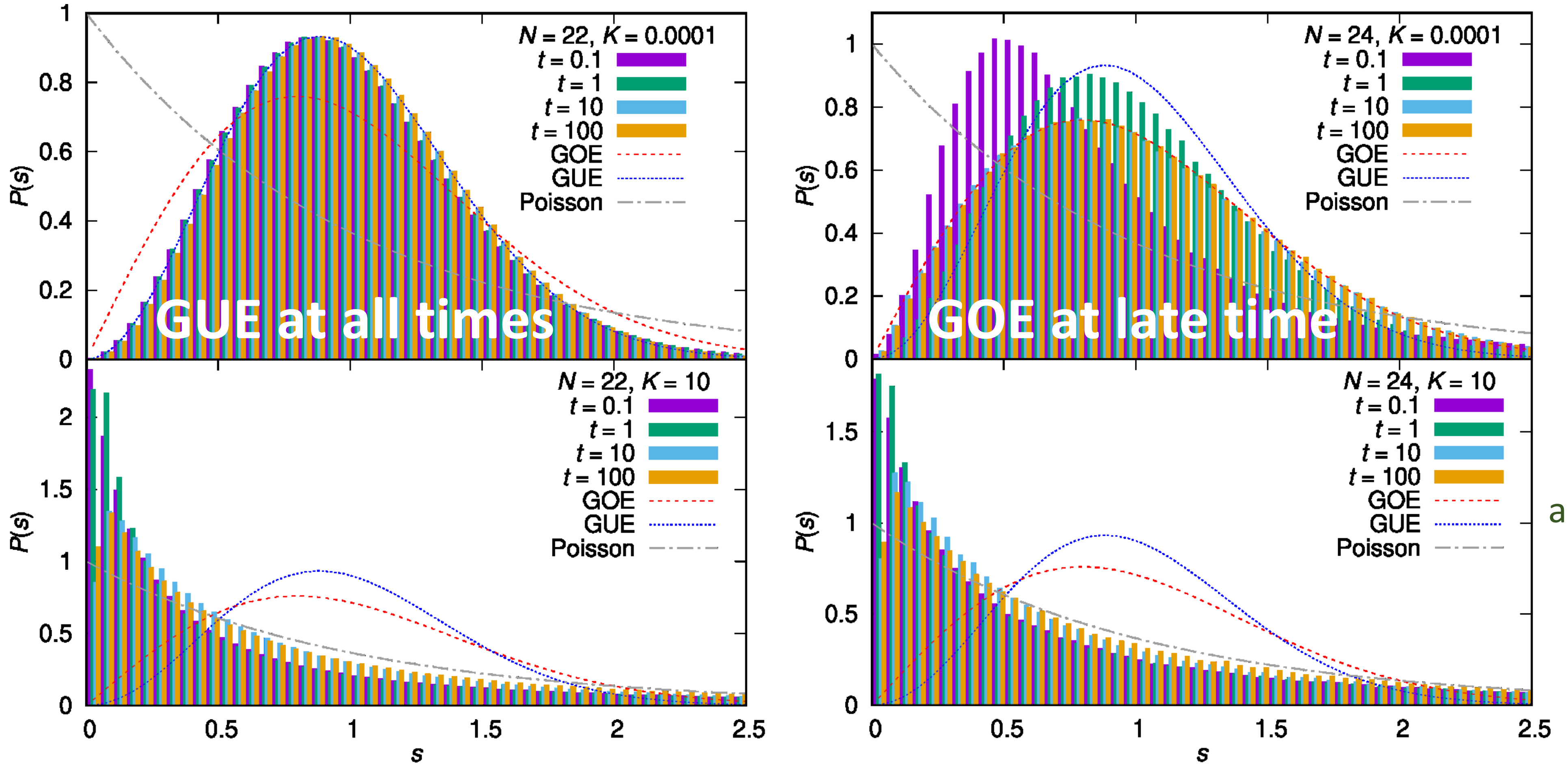
H. Gharibyan, M. Hanada, B. Swingle, and MT, arXiv:1902.11086

2. Singular value statistics of two-point functions

$$G_{ab}^{(\phi)} = \langle \phi | \hat{\chi}_a(t) \hat{\chi}_b(0) | \phi \rangle$$

SYK, larger $N/2$ exponents

$$\lambda_j = \log\left[\text{singular values of}\left(G_{ab}^{(\phi)}\right)\right]$$



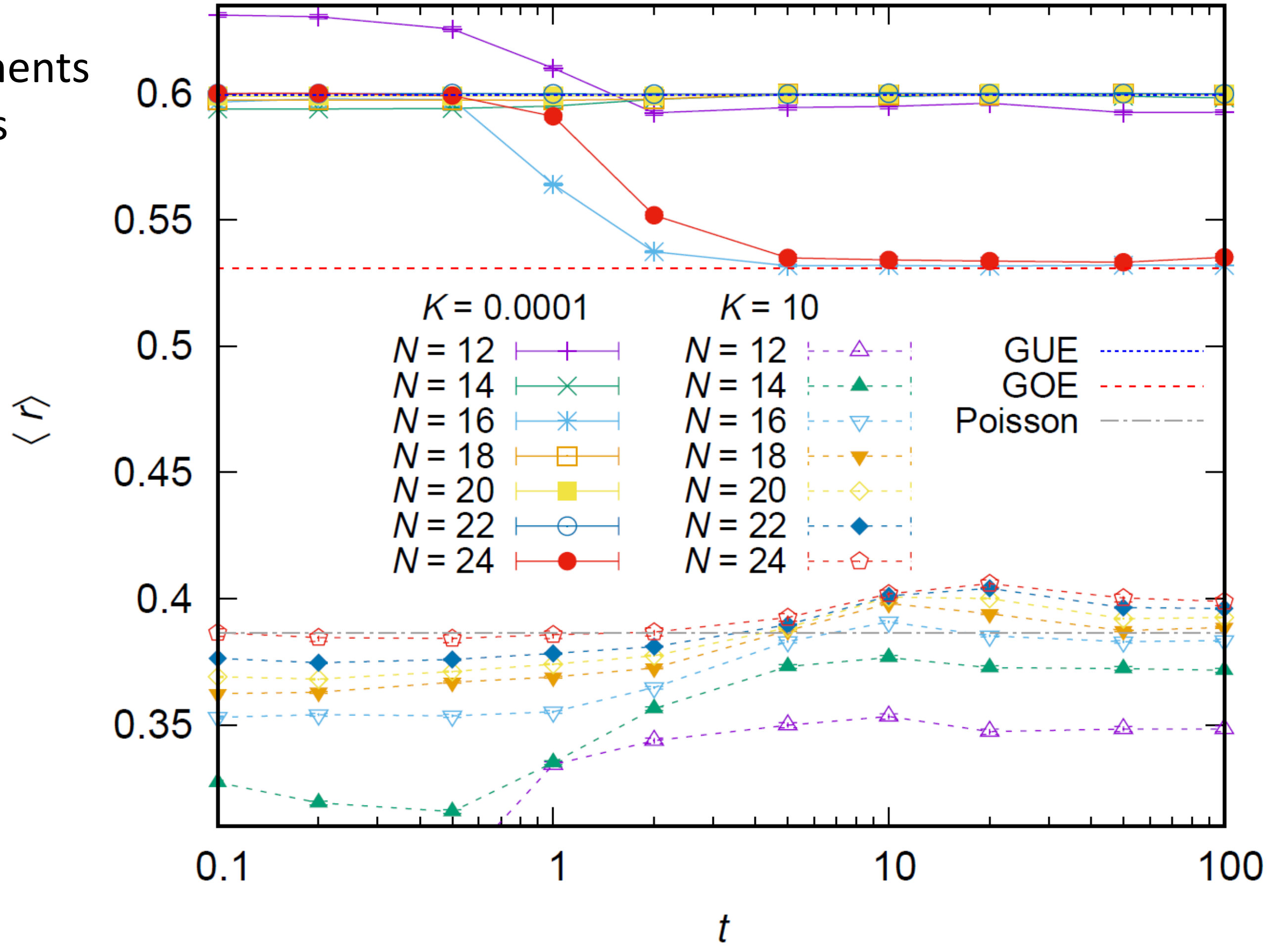
arXiv:**1902.11086**

$\langle r \rangle$: average of the adjacent gap ratio $\frac{\min(\lambda_{i+1}-\lambda_i\,,\lambda_{i+2}-\lambda_{i+1})}{\max(\lambda_{i+1}-\lambda_i\,,\lambda_{i+2}-\lambda_{i+1})}$

Uncorrelated (Poisson): $2\log 2 - 1 \approx 0.386$

Correlated: larger (GOE: 0.5307, GUE: 0.5996 etc.) [Atas *et al.*, PRL 2013]

SYK, larger *N*/2 exponents
ϕ: energy eigenstates



N mod 8 = 2, 4, 6: GUE

N mod 8 = 0: GOE
(the matrix is symmetric)

$$G_{ab}^{(\phi)} = \langle\phi|\hat{\chi}_a(t)\hat{\chi}_b(0)|\phi\rangle$$

$$\lambda_j = \log\left[\text{singular values of}\left(G_{ab}^{(\phi)}\right)\right]$$

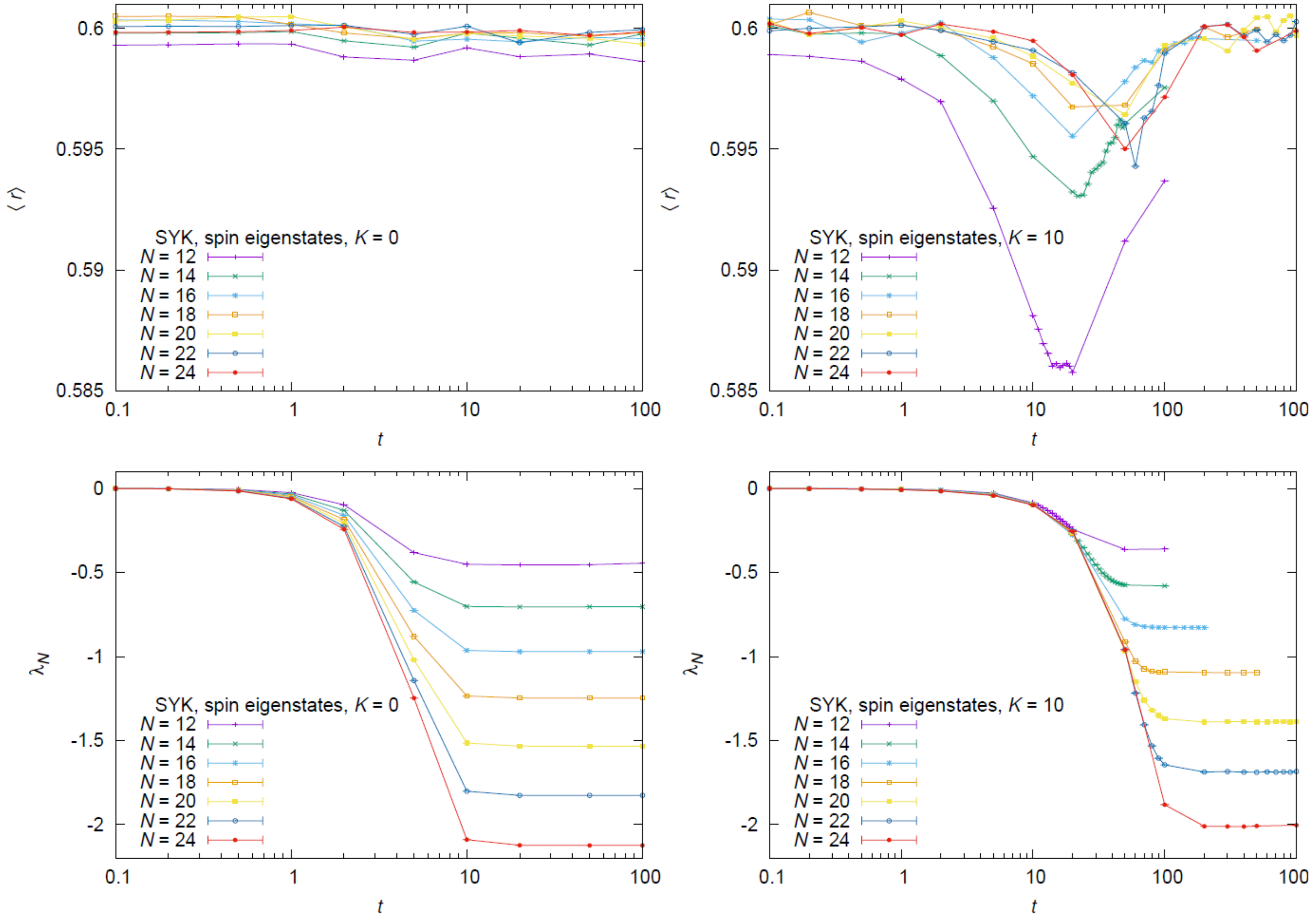
fixed-*i* unfolded

arXiv:**1902.11086**

At late time,
Random matrix behavior ⇔ chaotic (also for XXZ model + random field)

Random-matrix like for **complex fermion number eigenstates**, even for non-chaotic regime

Empty state in complex fermion description: state without long-range entanglement



The case of the random field XXZ model

$$\hat{H} = \sum_i^N \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1} + \sum_i^N h_i \widehat{S_i^z}$$

h_i: uniform distribution $[-W, W]$

Many-body localization transition at $W = W_c \sim 3.6$

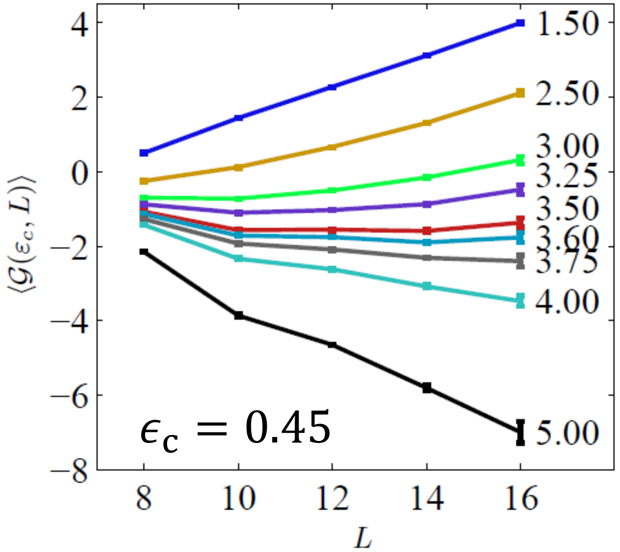
(though recently disputed; e.g. $W_c \geq 5$ proposed in E. V. H. Doggen et al., [1807.05051] using large systems with time-dependent variational principle & machine learning)

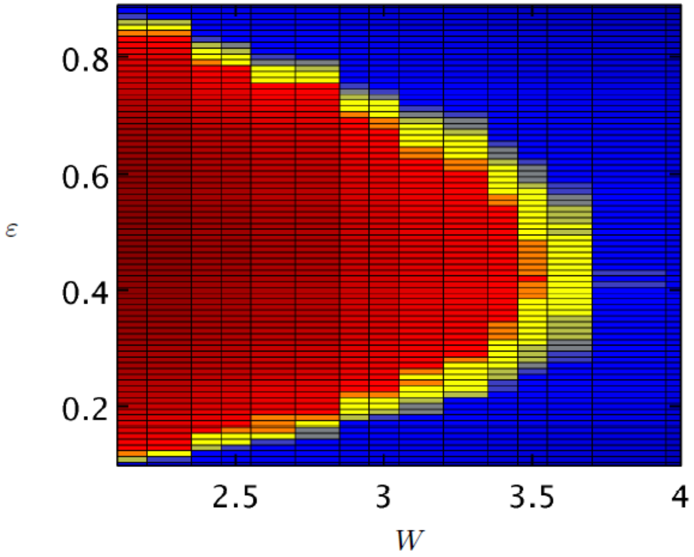
e.g. M. Serbyn, Z. Papic, and D. A. Abanin, Phys. Rev. X **5**, 041047 (2015) (arXiv:1507.01635)

Matrix element of local perturbation

$$\mathcal{G}(\varepsilon, L) = \ln \frac{|V_{n,n+1}|}{E'_{n+1} - E'_n}$$

Energy separation of neighboring energy eigenstates





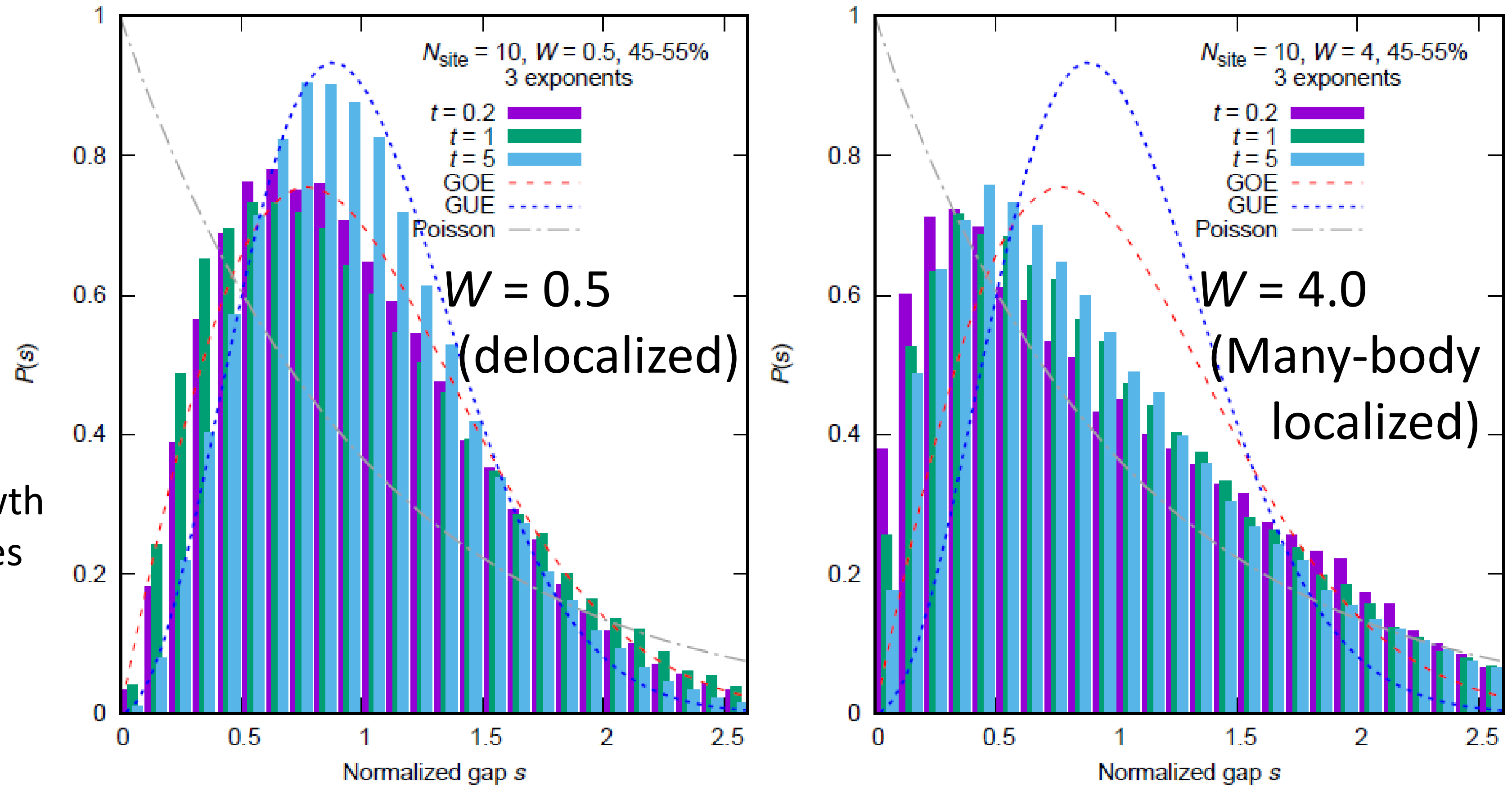
Spectral statistics of QLS for random field XXZ

arXiv:**1809.01671**

$$\hat{H} = \sum_{i}^{N} \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1} + \sum_{i}^{N} h_i \widehat{S_i^z}$$

h_i: uniform distribution $[-W, W]$

$$\hat{M}_{ab}(t) = \left[\widehat{S_a^+}(t), \widehat{S_b^-}(0)\right]$$



※ Exponential growth of the singular values is not observed, but the statistics approach GUE

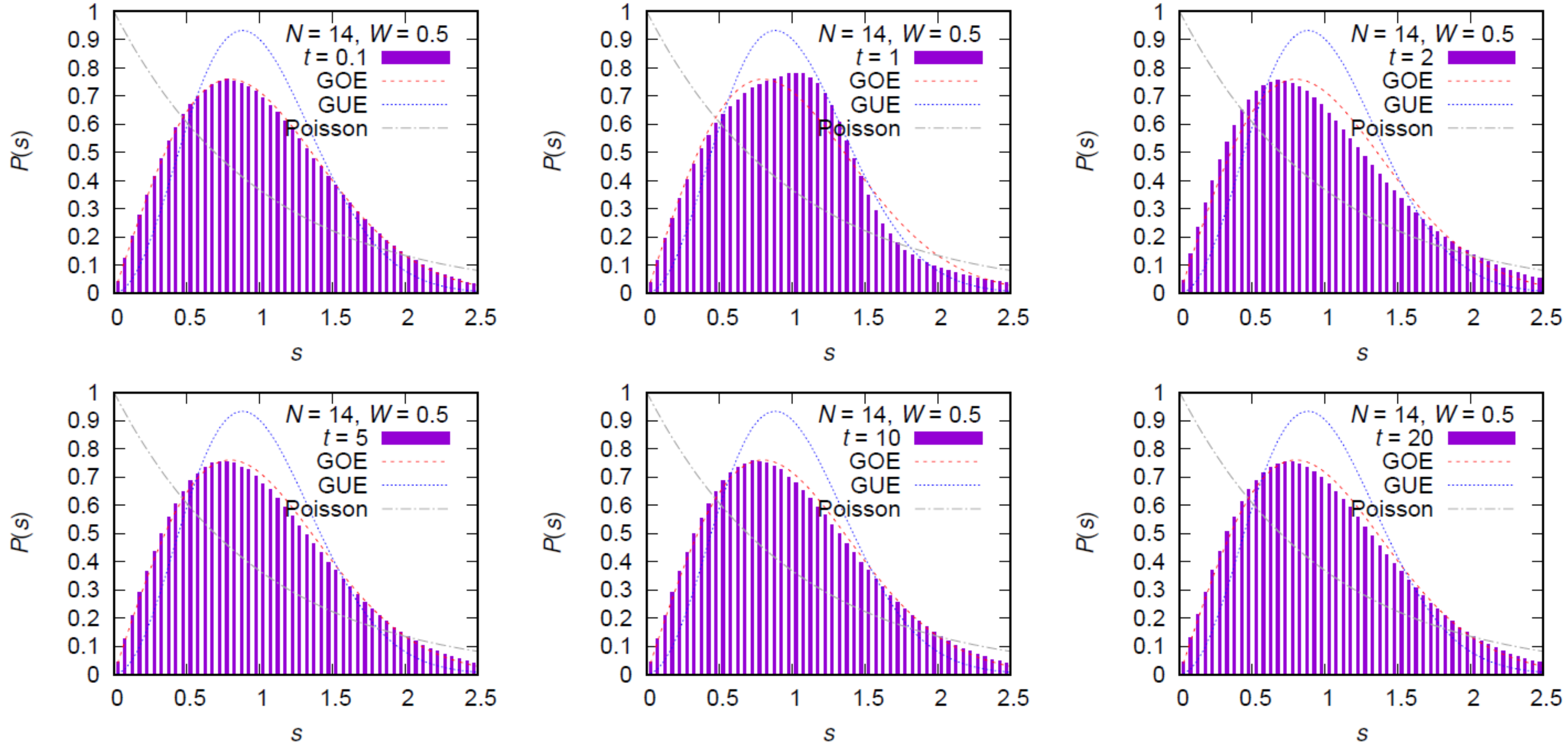
Quantum Lyapunov spectrum distinguishes chaotic and non-chaotic phases

Two-point function

$$\hat{H} = \sum_{i}^{N} \hat{S_i} \cdot \widehat{S_{i+1}} + \sum_{i}^{N} h_i \widehat{S_i^z} \qquad h_i \in [-W, W]$$

$$G_{ab}^{(\phi)} = \langle \phi | \widehat{\sigma^+}_a(t) \widehat{\sigma^-}_b(0) | \phi \rangle$$

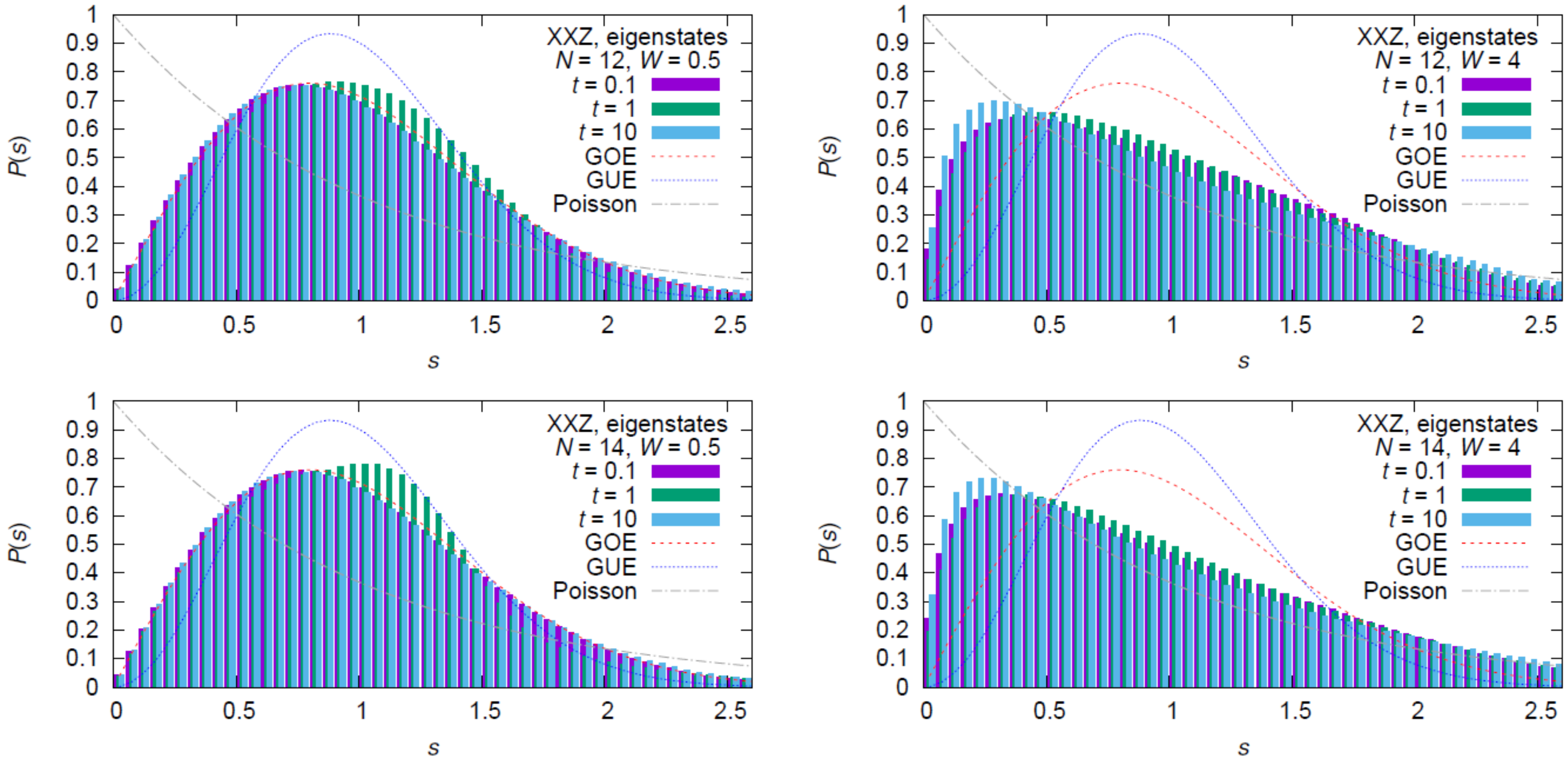
Energy eigenstates (not close to the spectral edges): GOE at short and long times for small *W*



Weak vs strong W

$$G_{ab}^{(\phi)} = \langle \phi | \widehat{\sigma^+}_a(t) \widehat{\sigma^-}_b(0) | \phi \rangle$$

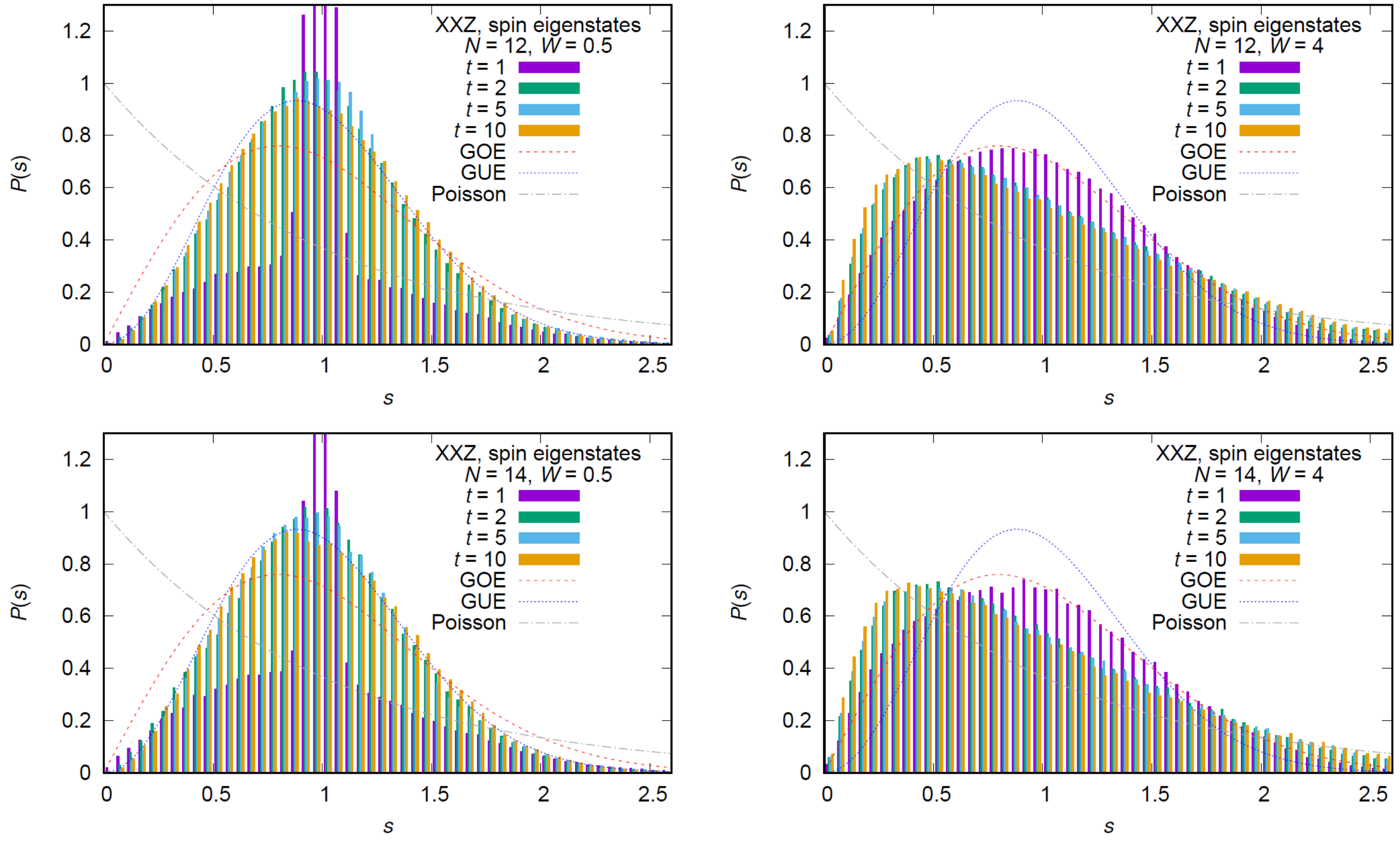
$$\widehat{H} = \sum_{i}^{N} \widehat{S_i} \cdot \widehat{S_{i+1}} + \sum_{i}^{N} h_i \widehat{S_i^z} \qquad h_i \in [-W, W]$$



Energy eigenstates GOE at short and long times for small *W*, close to Poisson at any time for large *W*

XXZ model: Spin eigenstates → GUE

$$G_{ab}^{(\phi)} = \langle\phi|\widehat{\sigma^+}_a(t)\widehat{\sigma^-}_b(0)|\phi\rangle$$



Singular value statistics of two-point correlation function

Model	Chaotic (small K / small W)	Not chaotic (large K / large W)
SYK_4 + SYK_2	Energy eig. ➔ GUE at late time except for $N \equiv 0 \pmod 8$: GOE Spin eig. ➔ GUE at any time	Energy eig. ➔ Poisson at any time Spin eig. ➔ off from GUE at some time
XXZ + random field	Energy eig. ➔ off from GOE at some time ($G_{ab}^{(\phi)} = \langle\phi\|\widehat{\sigma^+}_a(t)\widehat{\sigma^-}_b(0)\|\phi\rangle$ is symmetric) Spin eig. ➔ converges to GUE ($G_{ab}^{(\phi)} = \langle\phi\|\widehat{\sigma^+}_a(t)\widehat{\sigma^-}_b(0)\|\phi\rangle$ is not symmetric)	Energy eig. ➔ close to Poisson Spin eig. ➔ approaches Poisson from RMT-like

H. Gharibyan, M. Hanada, B. Swingle, and MT, arXiv:1902.11086

Outlook / related recent works

- Euclidean time; two-point correlations in classical dynamics; experiments?
  - In progress
- Time scale?
  - cf. "Onset of Random Matrix Behavior in Scrambling Systems"
    H. Gharibyan, M. Hanada, S. H. Shenker, and MT, JHEP07(2018)124 (1803.08050)
- Many-body localization (MBL) in other systems?
  - cf. MBL in a finite-range SYK model
    A. M. García-García and MT, PRB **99**, 054202 (2019) (1801.03204)
- Relation between randomness and chaos?
  - cf. SYK_2 model: "Randomness and chaos in qubit models"
    Pak Hang Chris Lau, Chen-Te Ma, Jeff Murugan, and MT, Phys. Lett. B in press (1812.04770)
- Holographic interpretation?
  - cf. "Effective Hopping in Holographic Bose and Fermi Hubbard Models"
    M. Fujita, R. Meyer, S. Pujari, and MT, JHEP01(2019)045 (1805.12584)

Summary

- Many-body quantum chaos: characterizations
- The Sachdev-Ye-Kitaev model
- Quantum Lyapunov spectrum defined from local operators: characterizes quantum chaos [1809.01671]
  - Random matrix behavior in chaotic systems
  - Lyapunov growth
  - Fastest entropy production in the SYK model?

$\hat{L}_{ab}(t) = \sum_{j=1}^{N} \hat{M}_{ja}(t)\hat{M}_{jb}(t)$ for

$$\hat{M}_{ab}(t) = \{\hat{\chi}_a(t), \hat{\chi}_b(0)\}$$

QLS: log(singular values of $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$)/(2t)

- Two-point correlation function: singular values exhibit random matrix behavior in chaotic cases [1902.11086]
  - Experiments should be possible with phase-sensitive measurements

$$G_{ab}^{(\phi)} = \langle\phi|\hat{\chi}_a(t)\hat{\chi}_b(0)|\phi\rangle$$

- Both characterizations of chaos demonstrated also for XXZ spin chain + random field