Reconstruction and the search for the exotic in Conformal Field Theory

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Overview

- conformal field theory
- subfactors
- conformal nets of factors
- Haagerup subfactor and its double
- reconstruction of a CFT from a modular tensor category

CFT - the search for the exotic

Verlinde ring as representation theory of loop group at level k

modules or representations of

- vertex operator algebra
- conformal nets of von Neumann algebras

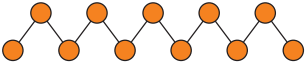
$Rep(G)$ or $Rep(LG)_k$ is a category

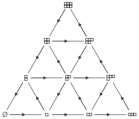
tensor $\lambda \otimes \mu$ trivial object id

morphism $\lambda \otimes \mu \simeq \mu \otimes \lambda$

tensor category $\mathcal{N} = Rep(\Phi)$ where $\Phi =$???

WZW loop group

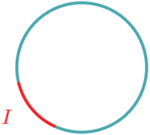




- $\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I'} SU(n))'$
  $\lambda = 0 : \qquad N = N \qquad\qquad \lambda N \subset N$ Wassermann

λ endomorphism, N type III_1 factor: $\lambda\mu = \sum_\nu N^\nu_{\lambda\mu}\nu$

representations of conformal net of factors $N(I)$ $\qquad I \subset S^1$



braided system of endomorphisms $\qquad Rep(N) \qquad \pi = \pi_0 \circ \lambda$

$Bimod_\lambda \quad a.x.b = ax\lambda(b)$

$$\lambda\mu = \mathrm{Ad}u(\lambda, \mu)\mu\lambda, \qquad u(\lambda, \mu) =$$

$$\chi_\lambda = \mathrm{trace}_{\mathcal{H}_\lambda}\, q^{L_0 - c/24} \qquad q = e^{2\pi i\tau}$$

2D Ising

$$Z = \Sigma_\sigma exp(-\beta H(\sigma)) = \Sigma\ \Pi\ \text{Boltzmann weights} = trace T^N$$

$$T \quad = \quad V^{1/2}\, W\ V^{1/2} = e^{-\mathcal{H}}$$

$$[\exp \beta J \sigma \sigma'] = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} \sim expK^* \sigma_z$$

$$\sinh 2K\ \sinh 2K^* = 1$$

$$V = exp\, K \Sigma \sigma_j^x \sigma_{j+1}^x \qquad W = exp\, K^* \Sigma \sigma_j^z$$

$$\sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho : \sigma_j^x \sigma_{j+1}^x \leftrightarrow \sigma_j^z$$

$$\rho\, \sigma_j^z = \sigma_j^x \sigma_{j+1}^x \qquad \rho\, \sigma_j^x = \sigma_1^z \sigma_2^z \cdots \sigma_j^z$$

Ising sectors

$M_2 \otimes M_2 \otimes \cdots \subset$ Cuntz algebra $\mathcal{O}_2$

$\mathcal{O}_2 = C^*(s_+, s_-); s_+s_+^* + s_-s_-^* = 1, s_\pm^* s_\pm = 1$

$M_2 =$ span of $s_+s_+^*, s_-s_+^*, s_+s_-^*, s_-s_-^*$

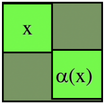
$\rho : \sigma^x = s_+s_+^* - s_-s_-^* \to \sigma^z = s_+s_-^* + s_-s_+^*.$

$s_+s_+^* \longrightarrow \Sigma_{h,k} s_h s_k^*/2$

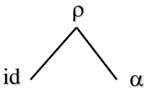
ρ on $\mathcal{O}_2$ by $s_+ \to \pm(\Sigma_h s_h/\sqrt{2})$

$\rho(s_g) = \pm Ad(U(g))(\Sigma_h s_h/\sqrt{2})$

$\rho^2(x) = s_+ x s_+^* + s_- \alpha(x) s_-^* \qquad \alpha : s_+ \leftrightarrow s_-$



$\rho^2 = id + \alpha$



$\rho^2 = 1 + \alpha + 0.\rho \qquad \mathbb{Z}_2$

Ising conformal field theory

Dynkin diagram A_3, $\lambda \in \{\bullet, +, -\}$
Fermions $g_a : a \in \mathbb{N} - 1/2 \quad or \quad \mathbb{N}$
$L_0 = \Sigma_{r\in\mathbb{N}-1/2} r g_r^* g_r \to \chi_\pm$ $\qquad$ $L_0 = \Sigma_{n\in\mathbb{N}} n g_n^* g_n \to \chi_\bullet$

$$\chi_+ \pm \chi_- = q^{-1/48}\Pi_{n\in\mathbb{N}}(1 \pm q^{n-1/2})$$

$$\chi_\bullet = q^{1/24}\Pi_{n\in\mathbb{N}}(1 + q^n)$$

$$\chi = \mathrm{trace}\, q^{L_0 - c/24} \quad q = e^{2\pi i\tau}$$

$$\tau \to \tau + 1 \qquad S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$\tau \to -1/\tau \qquad T = \mathrm{diag}(e^{-\pi i/24}, e^{-\pi i/12}, e^{-\pi i 23/24})$$

MTC from subfactors

index $= 4$

$$\begin{array}{ccc} R & \subset & R \otimes M_2 \\ \cup & & \cup \\ R^G & \subset & (R \otimes M_2)^G \end{array}$$

$G \subset SU(2)$ affine ADE classification + cohom. obstruction

index < 4 ADE classification

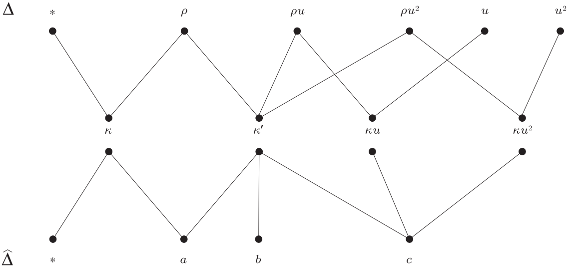
Beyond 4: Haagerup subfactor at index $(5 + \sqrt{13})/2$

quantum double of finite group, Haagerup subfactor etc

$\mathcal{X}$ on N : $A \subset N \otimes N^{opp}$, $\iota\bar{\iota} = \Sigma_{\nu \in \mathcal{X}} \nu \otimes \nu^{opp}$

Ocneanu, Longo-Rehren, Izumi, Popa

Principal graphs of the Haagerup $(5+\sqrt{13})/2$ subfactor



$$\alpha^3 = 1\,, \quad \rho\alpha = \alpha^2\rho\,, \quad \rho^2 = 1 + \rho + \rho\alpha + \rho\alpha^{\mathbf{2}}$$

$$d_\lambda = [M, \lambda M]^{1/2} \qquad d_\rho^2 = 1 + 3d_\rho; \quad d_\rho = (3+\sqrt{1}3)/2$$

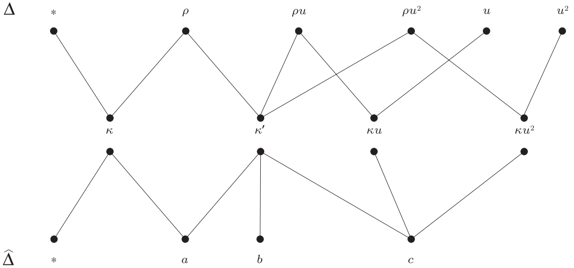
generalised Haagerup $\qquad \rho^2 = 1 + \sum_{g\in G} \rho g\,, \quad g\rho = \rho g^{-1}$

Izumi: $\qquad \mathbb{Z}_3, \mathbb{Z}_5 \quad |G|^2 + 4 = 13, 29$

E-Gannon: $\qquad \mathbb{Z}_7, \mathbb{Z}_{\mathbf{9}}, \mathbb{Z}_{\mathbf{11}}, \mathbb{Z}_{13}, \mathbb{Z}_{15}, \mathbb{Z}_{17}, \mathbb{Z}_{\mathbf{19}}$

$|G|^2 + 4 \qquad 53, \mathbf{85}, \mathbf{125}, 173, 229, 293, \mathbf{365}$

Principal graphs of the Haagerup $(5+\sqrt{13})/2$ subfactor



$$\alpha^3 = 1\,, \quad \rho\alpha = \alpha^2\rho\,, \quad \rho^2 = 1 + \rho + \rho\alpha + \rho\alpha^{\mathbf{2}}$$

$$d_\lambda = [M, \lambda M]^{1/2} \qquad d_\rho^2 = 1 + 3d_\rho; \quad d_\rho = (3+\sqrt{13})/2$$

near group $\rho^2 = \sum_{g\in G} g + n|G|\rho$ $n = 0, 1, 2, \cdots$

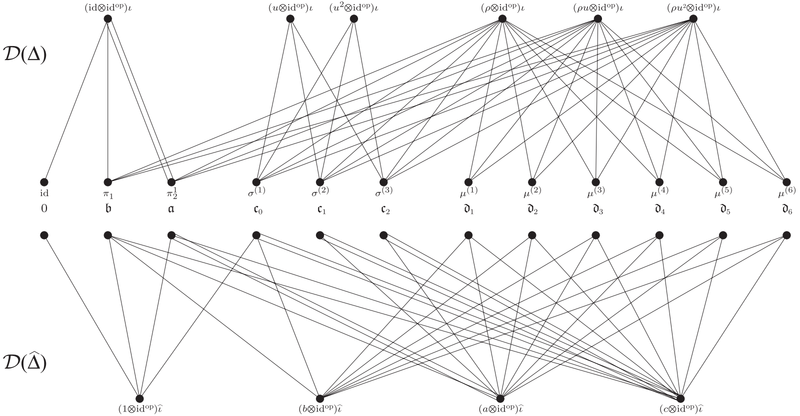
$\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2$ $2, 2, 1$ Izumi

$\mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{10}, \mathbb{Z}_{11}, \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_{13}$

3, 4, 2, 8, 4, 2, 1, 4, 4, 4, 2, 4

E-Gannon

Dual principal graphs for doubles of Δ and $\widehat{\Delta}$



Modular data for Haagerup $\mathcal{D}$Hg

$$S = \frac{1}{3}\begin{pmatrix} x & 1-x & 1 & 1 & 1 & 1 & y & y & y & y & y & y \\ 1-x & x & 1 & 1 & 1 & 1 & -y & -y & -y & -y & -y & -y \\ 1 & 1 & 2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & -y & 0 & 0 & 0 & 0 & c(1) & c(2) & c(3) & c(4) & c(5) & c(6) \\ y & -y & 0 & 0 & 0 & 0 & c(2) & c(4) & c(6) & c(5) & c(3) & c(1) \\ y & -y & 0 & 0 & 0 & 0 & c(3) & c(6) & c(4) & c(1) & c(2) & c(5) \\ y & -y & 0 & 0 & 0 & 0 & c(4) & c(5) & c(1) & c(3) & c(6) & c(2) \\ y & -y & 0 & 0 & 0 & 0 & c(5) & c(3) & c(2) & c(6) & c(1) & c(4) \\ y & -y & 0 & 0 & 0 & 0 & c(6) & c(1) & c(5) & c(2) & c(4) & c(3) \end{pmatrix}$$

$$T = \mathrm{diag}(1, 1, 1, 1, \xi_3, \overline{\xi_3}, \xi_{13}^{6}, \xi_{13}^{-2}, \xi_{13}^{2}, \xi_{13}^{5}, \xi_{13}^{-6}, \xi_{13}^{-5})$$

$$x = (13 - 3\sqrt{13})/26 \qquad y = 3/\sqrt{13} \qquad c(j) = -2y\,\cos(2\pi j/13) \qquad \xi = e^{2\pi i/13}$$

$$S_{jj'} = (-2y/3)\cos(2\pi j j'/13)$$

E-Gannon

$$N_{i,j}^{k} = \sum_{l} \frac{S_{i,l}}{S_{0,l}} S_{j,l} S_{k,l}^{*}\,, \qquad S_{i,j} = \overline{T}_{i,i}\overline{T}_{j,j}T_{0,0}\sum_{k} T_{k,k}S_{k,0}N_{i,j}^{k}\,.$$

Modular data for $SO(13)_2$

$$S = \frac{1}{3}\begin{pmatrix}
y/2 & y/2 & 3/2 & 3/2 & y & y & y & y & y & y \\
y/2 & y/2 & -3/2 & -3/2 & y & y & y & y & y & y \\
3/2 & -3/2 & 3/2 & -3/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/2 & -3/2 & -3/2 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & y & 0 & 0 & -c(1) & -c(2) & -c(3) & -c(4) & -c(5) & -c(6) \\
y & y & 0 & 0 & -c(2) & -c(4) & -c(6) & -c(5) & -c(3) & -c(1) \\
y & y & 0 & 0 & -c(3) & -c(6) & -c(4) & -c(1) & -c(2) & -c(5) \\
y & y & 0 & 0 & -c(4) & -c(5) & -c(1) & -c(3) & -c(6) & -c(2) \\
y & y & 0 & 0 & -c(5) & -c(3) & -c(2) & -c(6) & -c(1) & -c(4) \\
y & y & 0 & 0 & -c(6) & -c(1) & -c(5) & -c(2) & -c(4) & -c(3)
\end{pmatrix}$$

$$T = \mathrm{diag}(-1, -1; -i, i; -\xi_{13}^{6l^2})$$

$$y = 3/\sqrt{13} \qquad c(j) = -2y\cos(2\pi j/13) \qquad \xi = e^{2\pi i/13}$$

$$S_{jj'} = (+2y/3)\cos(2\pi jj'/13)$$

Characters for $\mathcal{D}$Hg, $c = 8$, $\gamma = 0, 1$

$$\begin{pmatrix} ch_0(\tau) \\ ch_{\mathfrak{b}}(\tau) \\ ch_{\mathfrak{a}}(\tau) = ch_{\mathfrak{c}_0}(\tau) \\ ch_{\mathfrak{c}_1}(\tau) \\ ch_{\mathfrak{c}_2}(\tau) \\ ch_{\mathfrak{d}_1}(\tau) \\ ch_{\mathfrak{d}_2}(\tau) \\ ch_{\mathfrak{d}_3}(\tau) \\ ch_{\mathfrak{d}_4}(\tau) \\ ch_{\mathfrak{d}_5}(\tau) \\ ch_{\mathfrak{d}_6}(\tau) \end{pmatrix} = \begin{pmatrix} q^{2/3}\left(q^{-1} + (6+13\gamma) + (120+78\gamma)q + (956+351\gamma)q^2 + (6010+1235\gamma)q^3 + \cdots\right) \\ q^{2/3}\left((80-13\gamma) + (1250-78\gamma)q + (10630-351\gamma)q^2 + (65042-1235\gamma)q^3 + \cdots\right) \\ q^{2/3}\left(81 + 1377q + 11583q^2 + 71037q^3 + \cdots\right) \\ 3 + 243q + 2916q^2 + 21870q^3 + \cdots \\ q^{1/3}\left(27 + 594q + 5967q^2 + 39852q^3 + \cdots\right) \\ q^{5/39}\left((7-\gamma) + (292-6\gamma)q + (3204-43\gamma)q^2 + (23010-146\gamma)q^3 + \cdots\right) \\ q^{20/39}\left((42+16\gamma) + (777+121\gamma)q + (7147+547\gamma)q^2 + (45367+2000\gamma)q^3 + \cdots\right) \\ q^{32/39}\left(\gamma q^{-1} + (11\gamma+119) + (73\gamma+1623)q + (300\gamma+12996)q^2 + (76429+1063\gamma)q^3 + \cdots\right) \\ q^{2/39}\left((5-3\gamma) + (229-50\gamma)q + (2738-252\gamma)q^2 + (19942-1032\gamma)q^3 + \cdots\right) \\ q^{8/39}\left((13-5\gamma) + (347-37\gamma)q + (3804-212\gamma)q^2 + (26390-794\gamma)q^3 + \cdots\right) \\ q^{11/39}\left((14+7\gamma) + (441+61\gamma)q + (4445+303\gamma)q^2 + (30329+1167\gamma)q^3 + \cdots\right) \end{pmatrix}$$

Potts model and orbifold $H(\sigma) = -\sum_{i,j\ n \cdot n} J\delta(\sigma_i, \sigma_j)$

$$W_i W_{i+1} = e^{2\pi i/Q} W_{i+1} W_i\,, \quad W_i W_j = W_j W_i\,, |i-j| > 1 \text{ in } M_{Q^\infty}$$

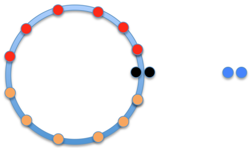
$$e_i = \text{Spectral}(W_i, 1) \qquad e_i e_{i\pm1} e_i = e_i/Q$$

$$V = \exp L \sum e_{2i+1}\,, \quad W = \exp L^* \sum e_{2i} \quad (e^L - 1)(e^{L^*} - 1) = Q$$

$$\mu \text{ on } \mathcal{O}_Q \qquad \mu^2 = \textstyle\sum_g \alpha_g \qquad g \in \mathbb{Z}_Q$$

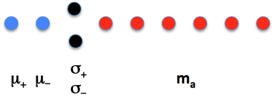
Orbifold $g \leftrightarrow -g$; take $\times\mathbb{Z}_2$

$$\longrightarrow \hat{\mathbb{Z}}_2 = \sigma_\pm \qquad \mu_\pm \qquad m_g = \alpha_g \oplus \alpha_{-g}$$

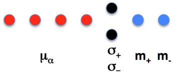


$$m_a m_b = m_{a+b} + m_{a-b} \qquad m_0 = \sigma_+ + \sigma_-$$
$$\mu_\tau \mu_{\tau'} = \sigma_{\tau+\tau'} + \textstyle\sum m_a \qquad a \sim -a \neq 0$$
$$\mu_\tau m_a = \mu_+ + \mu_- \qquad \sigma_\pm m_a = m_a$$

fusion rules of double of Haagerup



$\mathbb{Z}_{13}$

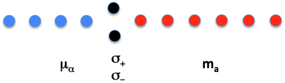


$\mathbb{Z}_3 \times \mathbb{Z}_3$

$$\sigma_\pm \simeq \mathbb{Z}_2 \quad m_a = e^{ia} + e^{-ia} \ : \ m_a m_b = m_{a+b} + m_{a-b}$$

$$\mu_\tau \mu_{\tau'} = \sigma_{\tau+\tau'} + \sum m_a \qquad a \sim -a \neq 0$$

$$\mu_\tau m_a = \mu_+ + \mu_- \qquad \sigma_\pm m_a = m_a$$



$$\sigma_+ = \text{identity}$$

$$\sigma_-^2 = \sigma_+ + \sigma_- + \sum \mu_\alpha + \sum m_a = R$$

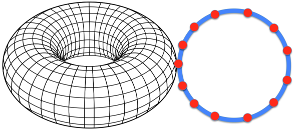
$$\mu_\alpha m_b = R - \sigma_+ = R_-$$

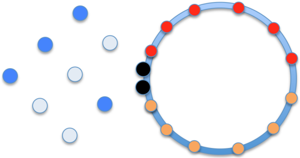
$$\mu_\alpha \mu_\beta = R_- + \mu_{\alpha+\beta} + \mu_{\alpha-\beta} \qquad \mu_0 = \sigma_+ + \sigma_-$$

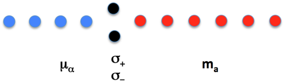
$$m_a m_b = R_- - m_{a+b} - m_{a-b} \qquad m_0 = -\sigma_+ + \sigma_-$$

$$\sigma_- \mu_\alpha = R_- + \mu_\alpha \qquad \sigma_- m_a = R_- - m_a$$

Haagerup bundles







Groups and Orbifolds

- for $\omega \in H^3(G)$, $\exists$ a conformal net with $Rep(\mathcal{A}) = Rep\,\mathcal{D}^\omega(G)$
- If $\mathrm{Rep}(\mathcal{A}) \simeq \mathcal{D}^\omega(G)$, then $\mathcal{A} \simeq \mathcal{V}^G$ for a holomorphic net $\mathcal{V}$
- $\mathcal{V}$ holomorphic conformal net, $G \subset S_k$
  $\mathrm{Rep}\ (\mathcal{V}^{k\otimes})^G = \mathrm{Rep}\mathcal{D}^\omega(G)$ - with $\omega^3 = 1$

E-Gannon

input : classification of group actions Jones 1978

$$\cdots \to H^2(K/N) \to H^2(K) \to \Lambda \to H^3(K/N) \to H^3(K) \to \cdots$$

- for G finite abelian odd, $\exists$ conformal net $Rep(\mathcal{A}) = TY(G)^{\mathbb{Z}_2}$
- for G finite abelian, $\exists$ conformal net with $Rep(\mathcal{B}) = \mathcal{D}TY(G)$

Bischoff, E-Gannon