F-matrices in Cluster Algebra

Yasuaki Gyoda

Nagoya University

September, 12th, 2019

Quiver

Quiver: finite oriented graph

Example:

$$\begin{tikzcd}
& 1 \arrow[rr, bend left, "3"] & 2 \arrow[l] & 3 \arrow[l] & \\
4 \arrow[ur, "2"] & & & & 5 \arrow[ul]
\end{tikzcd}$$

- They do not have 1-loops and 2-cycles.

$$\begin{tikzcd}
i \arrow[loop left]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
j \arrow[r, bend left] & k \arrow[l, bend left]
\end{tikzcd}$$

Correspondence of Exchange Matrix and Quiver

B : skew-symmetric

$$B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \longleftrightarrow \quad 1 \longleftarrow 2 \longleftarrow 3$$

Quiver Mutations

Definition

Let Q be a quiver and j be a vertex of Q. The quiver mutation $\mu_j(Q)$ at j is the quiver obtained from Q as follows:

1) reverse all arrows incident with j.
2) for each subquiver $i \xrightarrow{b_{ij}} j \xrightarrow{b_{jk}} k$, add new arrow $i \xleftarrow{b_{ij}b_{jk}} k$.
3) remove all 2-cycles.

Example: mutation at 2

$$1 \longleftarrow 2 \longleftarrow 3 \quad \xmapsto{\mu_2} \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd}$$

Seed and Seed Mutation

Example:

$$
\begin{tikzcd}
1 & \color{red}{2} \arrow[l] & 3 \arrow[l]
\end{tikzcd}
\quad \xmapsto{\mu_2} \quad
\begin{tikzcd}
1 \arrow[r] & \color{red}{2} \arrow[r] & 3 \arrow[ll, bend right]
\end{tikzcd}
$$

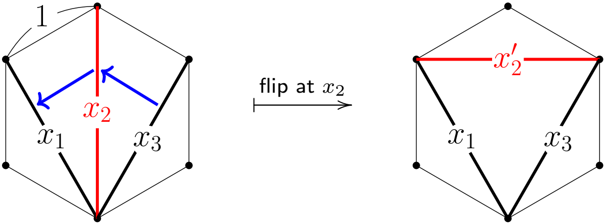
$$
(x_1, \color{red}{x_2}, x_3) \quad \xmapsto{\mu_2} \quad \left(x_1, \color{red}{\frac{x_1 + x_3}{x_2}}, x_3 \right)
$$

Seed and Seed Mutation

Example:

$$
\begin{tikzcd}
1 & 2 \arrow[l] & 3 \arrow[l] & {} \arrow[r, mapsto, "\mu_2"] & {} & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend right]
\end{tikzcd}
$$

$$
(x_1, {\color{red}x_2}, x_3) \overset{\mu_2}{\longmapsto} \left(x_1, {\color{red}\frac{x_1 + x_3}{x_2}}, x_3\right)
$$

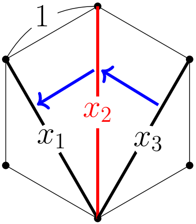


Seed and Seed Mutation

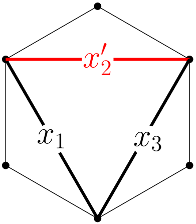
Example:

$$\begin{tikzcd} 1 & \color{red}{2} \arrow[l] & 3 \arrow[l] \end{tikzcd} \quad \xmapsto{\ \mu_2\ } \quad \begin{tikzcd} 1 \arrow[r] & \color{red}{2} \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd}$$

$$(x_1, {\color{red}x_2}, x_3) \quad \xmapsto{\ \mu_2\ } \quad \left(x_1, {\color{red}\frac{x_1 + x_3}{x_2}}, x_3\right)$$

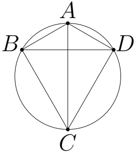


flip at x_2 $\longmapsto$



Ptolemy equation: $AC \times BD = AD \times BC + AB \times CD$

$$x_2' \cdot x_2 = 1 \cdot x_1 + 1 \cdot x_3$$

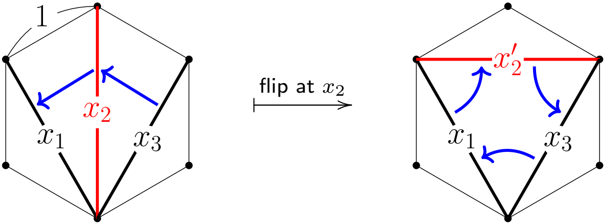


Seed and Seed Mutation

Example:

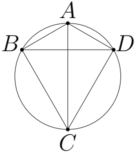
$$1 \longleftarrow \textcolor{red}{2} \longleftarrow 3 \quad \overset{\mu_2}{\longmapsto} \quad \begin{tikzcd} 1 \arrow[r] & \textcolor{red}{2} \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd}$$

$$(x_1, \textcolor{red}{x_2}, x_3) \quad \overset{\mu_2}{\longmapsto} \quad \left(x_1, \textcolor{red}{\frac{x_1 + x_3}{x_2}}, x_3\right)$$



Ptolemy equation: $AC \times BD = AD \times BC + AB \times CD$

$$x_2' \cdot x_2 = 1 \cdot x_1 + 1 \cdot x_3$$

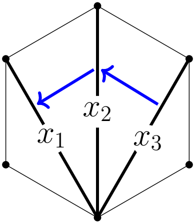


Quiver of Marked Surface Type

Marked surface : Riemann surface with boundary and marked point

Marked surface type

$$1 \longleftarrow 2 \longleftarrow 3 \qquad \longleftrightarrow$$



NOT marked surface type

$$1 \xleftarrow{3} 2 \qquad \longleftrightarrow \qquad \text{(No exists)}$$

Uniqueness of F-polynomials

Conjecture (G.-Yurikusa, 2019_+)

For any Q,

$$\{\mathbf{f}_{i;t}\}_{1\leq i\leq n} = \{\mathbf{f}_{i;s}\}_{1\leq i\leq n} \Rightarrow \{F_{i;t}^{B;t_0}(\mathbf{y})\}_{1\leq i\leq n} = \{F_{i;s}^{B;t_0}(\mathbf{y})\}_{1\leq i\leq n}.$$

Uniqueness of F-polynomials

Conjecture (G.-Yurikusa, 2019_+)

For any Q,

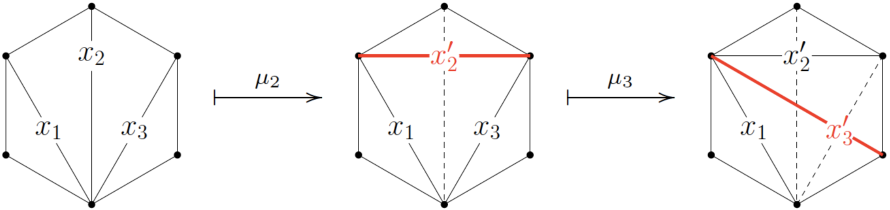
$$\{\mathbf{f}_{i;t}\}_{1\leq i\leq n}=\{\mathbf{f}_{i;s}\}_{1\leq i\leq n}\Rightarrow\{F^{B;t_0}_{i;t}(\mathbf{y})\}_{1\leq i\leq n}=\{F^{B;t_0}_{i;s}(\mathbf{y})\}_{1\leq i\leq n}.$$

We solve this conjecture partially:

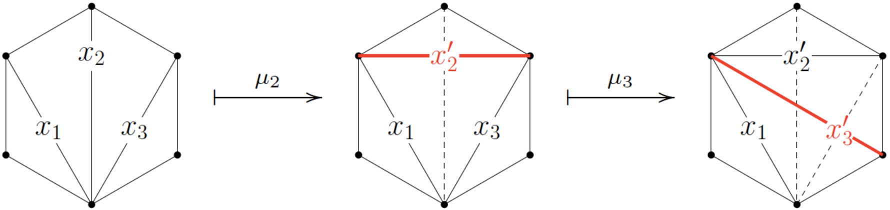
Theorem (G.-Yurikusa, 2019_+)

When the initial quiver Q is a quiver of marked surface type, the conjecture is correct.

Intersection Matrices and F-matrices

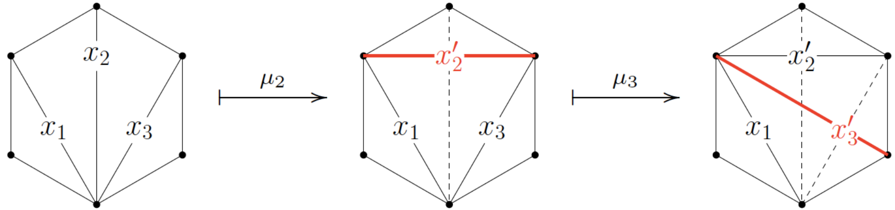


Intersection Matrices and F-matrices



Let's count the number of intersection points of the initial and a fliped triangulation.

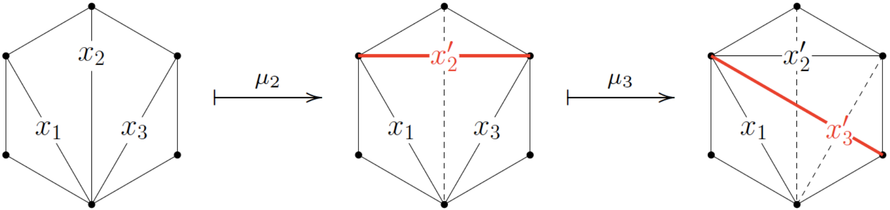
Intersection Matrices and F-matrices



Let's count the number of intersection points of the initial and a fliped triangulation.

x_1 intersects $x_1,\ x'_2,\ x'_3$ in $0, 0, 0$ point respectively.
x_2 intersects $x_1,\ x'_2,\ x'_3$ in $0, 1, 1$ point respectively.
x_3 intersects $x_1,\ x'_2,\ x'_3$ in $0, 0, 1$ point respectively.

Intersection Matrices and F-matrices



Let's count the number of intersection points of the initial and a fliped triangulation.

x_1 intersects $x_1,\ x'_2,\ x'_3$ in $0, 0, 0$ point respectively.
x_2 intersects $x_1,\ x'_2,\ x'_3$ in $0, 1, 1$ point respectively.
x_3 intersects $x_1,\ x'_2,\ x'_3$ in $0, 0, 1$ point respectively.

$$\mathrm{Int}_t^{B;t_0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem (Yurikusa, 2018_+)

We have the following equartion:

$$\mathsf{Int}_t^{B;t_0} = F_t^{B;t_0}$$

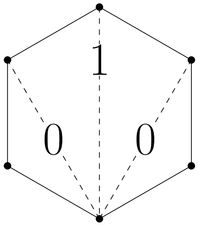
Sketch of Proof

Theorem (G.-Yurikusa, 2019_+)

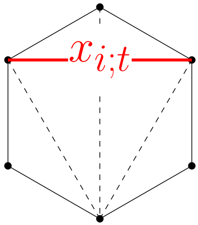
For any quiver Q of marked surface type,

$$\{\mathbf{f}_{i;t}\}_{1\le i\le n} = \{\mathbf{f}_{i;s}\}_{1\le i\le n} \Rightarrow \{F_{i;t}^{B;t_0}(\mathbf{y})\}_{1\le i\le n} = \{F_{i;s}^{B;t_0}(\mathbf{y})\}_{1\le i\le n}.$$

$$\mathbf{f}_{i;t} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xleftrightarrow{1:1}$$



almost $\xleftrightarrow{1:1}$



$$\xrightarrow{\text{Expansion formula}} x_{i;t} = \frac{y_2 + x_3}{x_2} \xrightarrow{x_i = 1} F_{i;t}^{B;t_0}(\mathbf{y}) = y_2 + 1$$