Modularity from Monodromy

String Theory and Mathematics seminar - Kavli IPMU

Thorsten Schimannek

based on [1902.08215], T.S.

and [1910.01988], C. F. Cota, A. Klemm, T.S.

8.10.2019

Topological string partition function on elliptic/genus one fibered Calabi-Yau $X \to B$

dual via F-theory

[Klemm,Mayr,Vafa'96],...

Elliptic genera of strings from D3-branes wrapping curves in B

Exhibit ***modular properties!***

QUESTION: Can we understand the modularity directly within topological string theory on X?

(Goes back to **[Candelas,Font,Katz,Morrison'94]***!)*

Outline

1. Topological strings and branes on Calabi-Yau 3-folds

—*break*—

2. The quantum geometry of genus one fibered CY 3-folds

Part I:

Topological strings and topological branes on Calabi-Yau 3-folds

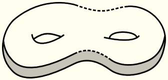
Topological string theory

- Consider string compactification: $\mathrm{X} = \mathcal{M}_{1,3} \times \underbrace{\mathrm{CY}_3}_{\boxed{\textit{N=(2,2) SCFT}}}$

  (eg. Type IIA/B strings)

- Worldsheet path integral over all embeddings

$$Z = \sum_{g=0}^{\infty} \int \mathcal{D}\phi \mathcal{D}\psi \, e^{-S_g}, \quad \phi :$$



$$\to \mathrm{X}$$

- *Supersymmetric path integral localizes on* $\delta_Q \psi = 0$

  (elsewhere fermions can be gauged away $\rightarrow \int \mathcal{D}\psi \, 1 = 0$)

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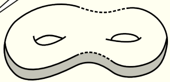

X

- *Supersymmetric path integral localizes on $\bar\partial\phi = 0$*

  (elsewhere fermions can be gauged away $\to \int \mathcal{D}\psi \, 1 = 0$)

Z is the partition function

ϕ are maps from worldsheet into target space X

ψ are fermionic fields on the worldsheet Σ

g is the worldsheet genus

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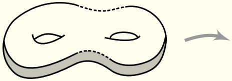

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Problem

- Variation of the action under SUSY transformation

$$\delta S = \int_\Sigma dz d\bar{z} \sqrt{h} \left(\nabla_\mu \epsilon^i G_i^\mu + \text{c.c} \right)$$

- No covariantly constant spinors on riemann surface $g \neq 1$
- Global supersymmetry is broken

But

- Sub-sectors of the theory localize!

Two different sub-sectors / restrictions / "twists"

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∇_μ : Covariant derivative w.r.t. worldsheet metric

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ϵ^i : Variational parameter, worldsheet spinor

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G_i^μ : Superpartner(s) of energy–momentum tensor

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Topological string theory

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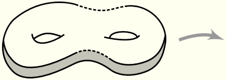
The closed B-model

B-twisted SUSY transformation

$$\delta_{Q_B}\psi \sim \epsilon_1 \partial\phi + \epsilon_2 \overline{\partial}\phi$$

Z localizes on constant maps *(easy to count, at least for $g < 2$)*

$\phi :$  $\bullet \in X$

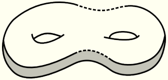
- Only depends on the *complex structure* of Calabi-Yau
- Does not receive worldsheet instanton corrections
- Genus 0 contributions $\sim$ variation of Hodge structure

The closed A-model

A-twisted SUSY transformation

$$\delta_{Q_A}\psi \sim \epsilon_1 \overline{\partial}\phi + \epsilon_2 \partial\overline{\phi}$$

localizes on **holomorphic maps** *(always hard to count)*

$\phi :$  $\xrightarrow{hol.}$ $C \subset X$

- Only depends on the *Kähler structure* of Calabi-Yau
- Receives w.s. instanton corrections → **stringy geometry**

The closed A-model

$Z_{\text{top.}}$ encodes highly non-trivial enumerative invariants!

Mathematically it can be defined as generating function of **Gromov-Witten invariants**, i.e. integrals over moduli stack of stable maps into CY (see e.g. book by **[Katz,Cox]**).
Those invariants are not integral!

Physically it encodes **multiplicities of BPS states** in 5d theory from M-theory on CY **[Gopakumar,Vafa'98]**.
Those are integral!

Mathematical definition only in special cases, e.g. via stable pair invariants.

$$\log(Z_{\text{top.}}) = \sum_{\beta \in H_2(M,\mathbb{Z})} \sum_{g=0}^{\infty} \sum_{m=1}^{\infty} \frac{n_\beta^g}{m} \left(2 \sin\left(\frac{m\lambda}{2} \right) \right)^{2g-2} q^{\beta m}$$

The closed A-model

n_β^g are the **Gopakumar-Vafa invariants**
$\sim$ weighted sum of BPS multiplicities

λ is the *topological string coupling*

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β is class of curve $C \subset X$

$q^\beta = \exp(2\pi i t^i \beta_i)$
t^i are *stringy Kähler parameters*

Sum over m due to *multi-coverings*

[Huang,Katz,Klemm'15]:

If Calabi-Yau 3-fold X is particular type of elliptic fibration, then $Z_{\text{top.}}$ can be expressed in terms of Jacobi forms.

(more on this later)

Interlude: **The stringy Kähler moduli space**

- String theories have anti-symmetric 2-form field $B_{\mu\nu}$
  $\rightarrow$ *combines into complexified Kähler class*

- Kähler cone is extended with cones of birational CY's and moduli cones of non-geometric worldsheet SCFTs
  (e.g. Landau-Ginzberg models or Hybrid phases)

- Points can be non-trivially identified via string dualities

In contrast, closed B-model only sees classical geometry!

Mirror symmetry

- There exist pairs of Calabi-Yau manifolds X, Y such that

$$\text{A-M{\scriptsize ODEL} on X} \quad \xleftrightarrow{\text{mirror symmetry}} \quad \text{B-M{\scriptsize ODEL} on Y}$$

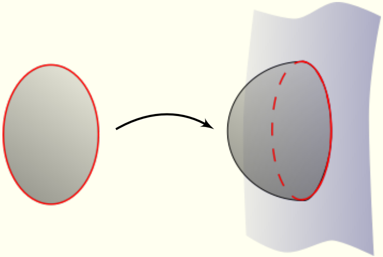
- From SCFT perspective "trivial" (choice of sign)
- Geometrically *highly non-trivial*

Loops in complex structure moduli space of B-model can be identified with very paths between dual points in stringy Kähler moduli space

What about open strings?

Open strings have boundaries that are mapped into branes.

$\phi :$



$\subset X$

Admissible topological branes depend on twist!

A brief reminder of topological B-branes

Naively, wrap holomorphic cycles in CY and carry gauge field

- IR equivalent to finite **complexes of vector bundles**

$$\mathcal{F}^\bullet = 0 \to V_1 \to \cdots \to V_n \to 0$$

$$(\cdots \to \text{brane} \to \text{anti-brane} \to \text{brane} \to \dots)$$

  Objects of *derived category of quasi-coherent sheaves $D^b(X)$*

  [Douglas'01]

- Central charge depends on Kähler class ω **[Iritani'09]**

$$Z(\mathcal{F}^\bullet) = \int_X e^\omega \, \Gamma(X) \, (\mathrm{ch}\, \mathcal{F}^\bullet)^\vee + (\text{instanton corr.})$$

  $\to$ For 2-branes just volume of support (no instantons!)

Homological mirror symmetry:

Monodromies in stringy Kähler moduli space
act as autoequivalences on $D^b(X)$! **[Kontsevich'96]**, **[Horja'99]**

Autoequivalences of $D^b(X)$ are always expressible as
Fourier-Mukai transformations **[Orlov'96]**

$$\Phi_{\mathcal{E}} : \mathcal{F}^\bullet \mapsto R\pi_{1*} \left(\mathcal{E} \otimes_L L\pi_2^* \mathcal{F}^\bullet \right) , \quad \mathcal{E} \in D^b(X \times X) \, .$$

The complex $\mathcal{E}$ is called the *Fourier-Mukai kernel.*

Physical intuition:

- Kernel $\mathcal{E}$ corresponds to *defect* between two non-linear sigma models with target space X
- FM-transformation $\approx$ *fusing defect with boundary*
  e.g. **[Brunner,Jockers,Roggenkamp'08]**

How does HKK relate to monodromies?

Idea: *Interpret Kähler moduli as central charges of 2-branes*

1. Identify actions on D-branes that generate modular group
2. Show that those actions arise from monodromies, i.e.
   they identify dual points in the stringy Kähler moduli space!
3. Use general automorphic properties of Z_{top}.

Part II:

The quantum geometry of genus one fibered Calabi-Yau 3-folds

Geometry of elliptic Calabi-Yau 3-folds

Consider *elliptically fibered* Calabi-Yau 3-fold $\pi : X \to B$

- Birationally equivalent to Weierstrass model $\pi' : \tilde{X} \to B$

$$\{y^2 = x^3 + fxz^4 + gz^6\} \subset \mathbb{P}_{231}(K_B^{-2} \oplus K_B^{-3} \oplus \mathcal{O})$$

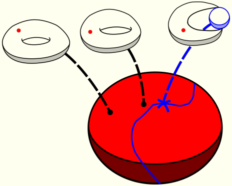
- Fiber of X degenerates over *discrimininant locus*

$$\{4f^3 + 27g^2 = 0\} \subset B$$

Degenerations classified by Kodaira

(not quite true for 3-folds, see **[Esole,Yau'11]**)

e.g. I_2 singularity

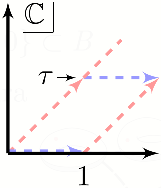


Theorem (Shioda-Tate-Wazir):

$$h^{1,1}(X) = h^{1,1}(B) + \#(\text{sections}) + \#(\text{fibral divisors})$$

Sections form group

- If fibration has section σ, construct fibers as $\mathbb{C}/(a\tau + b)$
- Identify $\sigma \cap$ *fiber* with $0 \in \mathbb{C}$



- Addition in $\mathbb{C}$ lifts to group law on sections:

$$\textit{Mordell-Weil group} \quad \mathbb{Z}^N \times \mathbb{Z}_M$$

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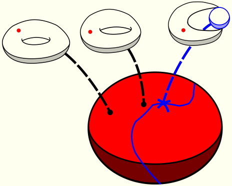

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Not all genus one fibrations are elliptic fibrations!

Some only have N**-sections**:

- An N-section intersects the generic fiber N times
- Points experience monodromy around loops in the base

Theorem (Shioda-Tate-Wazir-Braun-Morrison):

$$h^{1,1}(X) = h^{1,1}(B) + \#(N\text{-sections}) + \#(\text{fibral divisors})$$

By abuse of terminology:

"with N-sections" $\leftrightarrow$ there is no N'-section with $N' < N$

There are **four types of curves** on a genus 1 fibered Calabi-Yau:

1. Curves in the base
2. The generic fiber
3. Fibers of fibral divisors
4. Isolated components of reducible fibers

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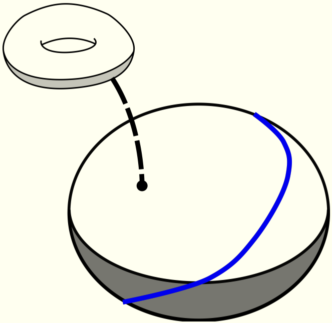

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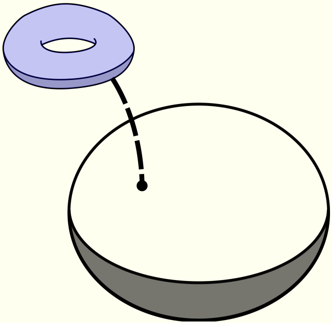

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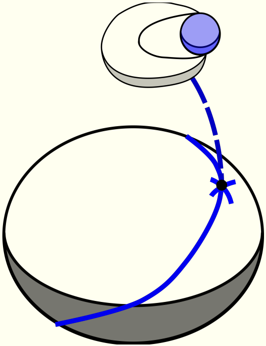

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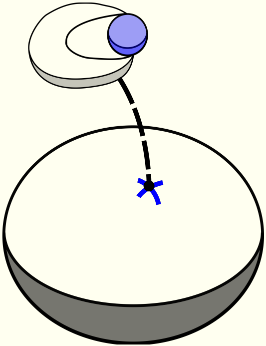

Geometry of elliptic Calabi-Yau 3-folds

30 second introduction to F-theory

(as an economic way to describe elliptic fibrations)

Fibral divisors

k free generators of Mordell-Weil group

$$G = \frac{G_{\text{non-Abelian}}}{\mathbb{Z}_n} \times U(1)^k \times \mathbb{Z}_m$$

n-torsional sections

No section but only m-sections

Topological strings and weak Jacobi forms

Conjecture: For trivial G *(i.e. no reducible fibers!)*
[Huang,Katz,Klemm'15]

$$Z_{\mathrm{top}}(\underline{t}, \lambda) = Z_0(\tau, \lambda) \left(1 + \sum_{\beta \in H_2(B,\mathbb{Z})} Z_\beta(\tau, \lambda) Q^\beta \right)$$

$$Z_{\beta > 0}(\tau, \lambda) = \frac{\phi_\beta(\tau, \lambda)}{\eta(\tau)^{12 \beta \cdot K_B} \prod_{l=1}^{b_2(B)} \prod_{s=1}^{\beta_l} \phi_{-2,1}(\tau, s\lambda)}$$

$\rightarrow$ *All-genus results for compact CY 3-folds!*

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t^j: (shifted) volumes of base curves

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$\phi_\beta, \phi_{-2,1}$: weak Jacobi forms

$\rightarrow$ *All-genus results for compact CY 3-folds!*

Topological strings and weak Jacobi forms

A **weak Jacobi form** $\phi(\tau, z)$
of weight k and index m

- Admits a Fourier expansion

$$\phi(\tau, z) = \sum_{n\geq 0}\sum_{r\in\mathbb{Z}} c(n,r)\, e^{2\pi i(n\tau + rz)}$$

- Satisfies *"Modular Transformation Law" (MTL)*

$$\phi\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \tau^k e^{\frac{2\pi i m z^2}{\tau}}\phi(\tau, z)$$

- MTL **implies** *"Elliptic Transformation Law" (ETL)*

$$\phi\left(\tau, z+\tau\right) = e^{-2\pi i m(\tau+2z)}\phi(\tau, z)$$

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Z_β has weight 0 and index $\frac{1}{2}\beta\cdot(\beta - c_1(B))$

$\rightarrow$ *All-genus results for compact CY 3-folds!*

gHKK conjecture:

1. For a "general" elliptic Calabi-Yau threefold, $Z_\beta(\tau, \lambda, \vec{m})$ is a *lattice Jacobi form*

$$Z_{\beta>0}(\tau, \lambda, \vec{m}) = \frac{\phi_\beta(\tau, \lambda, \vec{m})}{\eta^{12\beta\cdot K_B} \prod_{l=1}^{b_2(B)} \prod_{s=1}^{\beta_l} \phi_{-2,1}(\tau, s\lambda)}$$

2. The index matrix is encoded in the *anomaly polynomial* of the 6d F-theory effective action

3. Z_β is invariant under the action of the *affine Weyl groups*

[**DelZotto,Gu,Huang,Kashani-Poor,Klemm,Lockhart'17**], 2x[**Lee,Lerche,Weigand'18**]

(can be made more precise!)

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Volumes of fibral curves

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A **lattice Jacobi form** $\phi(\tau, z_1, ..., z_n)$

of weight k and index matrix M

- Admits a Fourier expansion and transforms as

$$\phi\left(-\frac{1}{\tau}, \frac{z_1}{\tau}, ..., \frac{z_n}{\tau}\right) = \tau^k e^{\frac{2\pi i M_{ij} z^i z^j}{\tau}} \phi(\tau, z_1, ..., z_n)$$

- Again, this implies elliptic transformation law

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(can be made more precise!)

Can we see this directly from topological strings?

- $Z_{\text{top.}}$ is wave function on moduli space of Calabi-Yau **[Witten'93]**
- Can we understand modular properties as consequence of monodromies in the stringy Kähler moduli space (SKM)?

Note: Invariance under $T: \tau \to \tau + 1$ trivial *(B-field shifts)*

(B-field shifts are also monodromies!)

Need to establish duality between τ *and* $-\frac{1}{\tau}$ *in SKM!*

Basically 2-fold T-duality, but what about the singular fibers?

What about genus one fibrations without sections?

Let's study monodromies!

Can we see this directly from topological strings?

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Basically 2-fold T-duality, but what about the singular fibers?

What about genus one fibrations without sections?

Let's study monodromies!

Warmup: Generic conifold monodromy

A-model: Monodromy around boundary of geometric cone where 6-brane becomes massless

B-model: Monodromy around principal component of the discriminant where base of SYZ fibration collapses

Topological B-branes:

- *0-brane transforms into 0-brane and anti-6-brane*
- FM-kernel is *ideal sheaf* $\mathcal{I}_\Delta$ *of diagonal in* $X \times X$

$$\mathcal{I}_\Delta \sim 0 \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0 \quad \text{in } D^b(X\times X)$$

Calculate induced action on brane charges from

$$\Phi_{\mathcal{I}_\Delta} : \mathcal{F}^\bullet \mapsto R\pi_{1*}\left(\mathcal{I}_\Delta \otimes_L L\pi_2^* \mathcal{F}^\bullet\right)$$

Grothendieck-Riemann-Roch implies that

$$\mathrm{ch}(f_* \mathcal{F}^\bullet) = f_* \left[\mathrm{ch}(\mathcal{F}^\bullet) \cdot f^* \mathrm{Td}(X)\right]$$

Therefore

$$\mathrm{ch}(\Phi_{\mathcal{I}_\Delta}(\mathcal{F}^\bullet)) = \pi_{1*}\left(\mathrm{ch}(\mathcal{I}_\Delta) \cdot \pi_2^* \left[\mathrm{ch}(\mathcal{F}^\bullet)\mathrm{Td}(X)\right]\right)$$

Using $\Phi_{\mathcal{O}_\Delta}(\mathcal{F}^\bullet) = \mathcal{F}^\bullet$ this leads to

$$\mathrm{ch}(\Phi_{\mathcal{I}_\Delta}(\mathcal{F}^\bullet)) = \mathrm{ch}(\mathcal{F}^\bullet) - \pi_{1*}\pi_2^* \left(\mathrm{ch}(\mathcal{F}^\bullet)\mathrm{Td}(X)\right)$$

Calculate induced action on brane charges from

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Grothendieck-Riemann-Roch implies that

$$\mathrm{ch}(f_* \mathcal{F}^\bullet) = f_* \left[\mathrm{ch}(\mathcal{F}^\bullet) \cdot f^* \mathrm{Td}(X)\right]$$

Therefore

"projects out everything that is not a point and transforms the point into X"

$$\mathrm{ch}(\Phi_{\mathcal{I}_\Delta}(\mathcal{F}^\bullet)) = \pi_{1*} \left(\mathrm{ch}(\mathcal{I}_\Delta) \cdot \pi_2^* [\mathrm{ch}(\mathcal{F}^\bullet) \mathrm{Td}(X)]\right)$$

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Using $\Phi_{\mathcal{O}_\Delta}(\mathcal{F}^\bullet) = \mathcal{F}^\bullet$ this leads to

$$\mathrm{ch}(\Phi_{\mathcal{I}_\Delta}(\mathcal{F}^\bullet)) = \mathrm{ch}(\mathcal{F}^\bullet) - \pi_{1*}\pi_2^*\left(\mathrm{ch}(\mathcal{F}^\bullet)\mathrm{Td}(X)\right)$$

For elliptic CY 3-folds without reducible fibers it is known that *ideal sheaf of relative diagonal* $\mathcal{I}_{\Delta_B}$ *acts as* $\tau \to \frac{\tau}{\tau+1}$
(when normalizing with 0-brane charge)

$\to$ **"fiberwise conifold monodromy"**

- Evaluation on brane charges uses *singular Riemann Roch*
  This requires suitable embedding into ambient space ("l.c.i. morphism") **[Baum,Fulton,MacPherson'75]**, **[Andreas,Curio,Ruipérez,Yau'00]**, reviewed in **[Andreas,Ruipérez'04]**

- $\mathcal{I}_{\Delta_B}$ auto-equivalence of $D^b(X)$ for *any genus one fibrations*
  [Ruipérez,López Martín,Sancho de Salas'06]

Idea:
- Use toric geometry, CY as complete intersection
- get l.c.i. morphism for free!
- evaluate action of $\mathcal{I}_{\Delta_B}$ on charges

Assume X is genus one fibered CY with N-sections that is a complete intersection in toric ambient space with compatible fibration. Then

$$\mathrm{ch}(\Phi_{\mathcal{I}_{\Delta_B}}(\mathcal{F}^\bullet)) = \mathrm{ch}(\mathcal{F}^\bullet) - \pi_{1*}\pi_2^*\left(\mathrm{ch}(\mathcal{F}^\bullet)\mathrm{Td}_{M/B}\right) \,,$$

and

$$\mathrm{Td}_{M/B} = 1 - \frac{1}{2}c_1(B) + \ldots \,.$$

Moreover, the **fiberwise conifold transformation** acts as

$$U: \left\{\begin{array}{rcl} \tau & \mapsto & \tau/(1+N\tau) \\ m_i & \mapsto & m_i/(1+N\tau)\,, \quad i=1,...,\mathrm{rk}(G) \\ Q_i & \mapsto & (-1)^{a_i}\exp\left(-\frac{N}{1+N\tau}\cdot\frac{1}{2}m^a m^b C^i_{ab} + \mathcal{O}(Q_i)\right)Q_i \end{array}\right.$$

[T.S.'19], **[Cota,Klemm,T.S.'19]**

Assume X is genus one fibered CY with N-sections that is a complete intersection in toric ambient space with compatible fibration. Then

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$\pi_{1,2}$ projections in $X \times_B X$

Mo … acts as

$$U: \begin{cases} \tau \mapsto \tau/(1+N\tau) \\ m_i \mapsto m_i/(1+N\tau)\,, \quad i = 1, ..., \mathrm{rk}(G) \\ Q_i \mapsto (-1)^{a_i}\exp\left(-\frac{N}{1+N\tau}\cdot\frac{1}{2}m^a m^b C^i_{ab} + \mathcal{O}(Q_i)\right) Q_i \end{cases}$$

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and

$$\mathrm{Td}_{M/B} = 1 - \frac{1}{2}c_1(B) + \ldots\,.$$

"Todd class of virtual relative tangent bundle"

Mo... acts as

$$U:\begin{cases} \tau & \mapsto \quad \tau/(1+N\tau) \\ m_i & \mapsto \quad m_i/(1+N\tau)\,, \qquad i=1,...,\mathrm{rk}(G) \\ Q_i & \mapsto \quad (-1)^{a_i}\exp\left(-\frac{N}{1+N\tau}\cdot\frac{1}{2}m^a m^b C^i_{ab}+\mathcal{O}(Q_i)\right)Q_i \end{cases}$$

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[T.S.'19], **[Cota,Klemm,T.S.'19]**

Assume X is genus one fibered CY with N-sections that is a complete intersection in toric ambient space with compatible fibration. Then

τ is $\frac{1}{N}$ times *volume of generic fiber*

For $N \leq 4$, U and T generate action of $\Gamma_1(N)$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,:\, a,d \equiv 1 \pmod{n}\,,\; c \equiv 0 \pmod{n} \right\}$$

Moreover, the **fiberwise conifold transformation** acts as

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[T.S.'19], **[Cota,Klemm,T.S.'19]**

Assume X is genus one fibered CY with N-sections that is a complete intersection in toric ambient space with compatible fibration. Then

$$\mathrm{ch}(\Phi_{T}(\mathcal{F}^\bullet)) = \mathrm{ch}(\mathcal{F}^\bullet) - \pi_1 \pi_2^*\left(\mathrm{ch}(\mathcal{F}^\bullet)\mathrm{Td}_{M/B}\right)\,,$$

m_i, $i=1,...,\mathrm{rk}(G)$ are *volumes of fibral curves*

They transform like *elliptic parameters*

$$\mathrm{Td}_{M/B} = 1 - \frac{1}{2}c_1(B) + \ldots\,.$$

Moreover, the **fiberwise conifold transformation** acts as

$$U: \begin{cases} \tau \mapsto \tau/(1+N\tau) \\ m_i \mapsto m_i/(1+N\tau)\,, \quad i=1,...,\mathrm{rk}(G) \\ Q_i \mapsto (-1)^{a_i}\exp\left(-\frac{N}{1+N\tau}\cdot\frac{1}{2}m^a m^b C^i_{ab} + \mathcal{O}(Q_i)\right)Q_i \end{cases}$$

[T.S.'19], **[Cota,Klemm,T.S.'19]**

Assume X is genus one fibered CY with N-sections that is a complete intersection in toric ambient space with compatible fibration. Then

$$Q_i = \exp(2\pi i \cdot t_i)\,, \quad i = 1, ..., h^{1,1}(B)$$

t_i are *shifted volumes of base curves*

$$t_i = \tilde{t}_i + \frac{\tilde{a}_i}{2N}\tau \text{ (for } \tilde{a}_i \text{ see paper)}$$

Q_i transforms like **Jacobi form** with index $-C^i_{ab}$!

Moreover, the **fiberwise conifold transformation** acts as

$$U: \begin{cases} \tau \mapsto \tau/(1+N\tau) \\ m_i \mapsto m_i/(1+N\tau)\,, \quad i = 1, ..., \mathrm{rk}(G) \\ Q_i \mapsto (-1)^{a_i} \exp\left(-\frac{N}{1+N\tau} \cdot \frac{1}{2} m^a m^b C^i_{ab} + \mathcal{O}(Q_i)\right) Q_i \end{cases}$$

[T.S.'19], **[Cota,Klemm,T.S.'19]**

Assume X is genus one fibered CY with N-sections that is a complete intersection in toric ambient space with compatible fibration. Then

$$\mathrm{ch}(\Phi \ldots (\mathcal{T}^\bullet)) \quad \mathrm{ch}(\mathcal{T}^\bullet) \quad = \quad \pi^*\left(\mathrm{ch}(\mathcal{T}^\bullet)\mathrm{Td} \ldots\right)$$

$$C_{ab}^{\beta} = -\tfrac{1}{N} \cdot \pi(D_a \cdot D_b) \cdot C_\beta, \quad C_\beta \in H_2(X)$$

$D_{a,b}$ are Shioda maps/fibral divisors

and

$$\mathrm{Td}_{M/B} = 1 - \frac{1}{2} c_1(B) + \ldots .$$

Moreover, the **fiberwise conifold transformation** acts as

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and

ggHKK $\Rightarrow$ $Q^\beta Z_\beta(\tau, \lambda, \vec{m})$ transforms as
Jacobi form of index 0 w.r.t. $\vec{m}$
under $\Gamma_1(N)$ action generated by U, T

Moreover, the **fiberwise conifold transformation** acts as

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[T.S.'19], **[Cota,Klemm,T.S.'19]**

But is U induced by a monodromy?

Should come from wall of geometric cone where fibre collapses

- Branes that wrap $\pi^{-1}(V)$ for some $V \subset B$ become massless
  this includes the 6-brane! **[Aspinwall,Horja,Karp'02]**

- Mirror $\subset$ principal component of discriminant

- It has to have tangency with intersection of other
  large complex structure divisors
  (otherwise get generic conifold monodromy)

Let us study example using the GLSM

[Herbst,Hori,Page'08], **[Hori,Romo'13]**, **[Knapp,Erkinger'17]**

GLSM branes
Matrix factorization
of superpotential

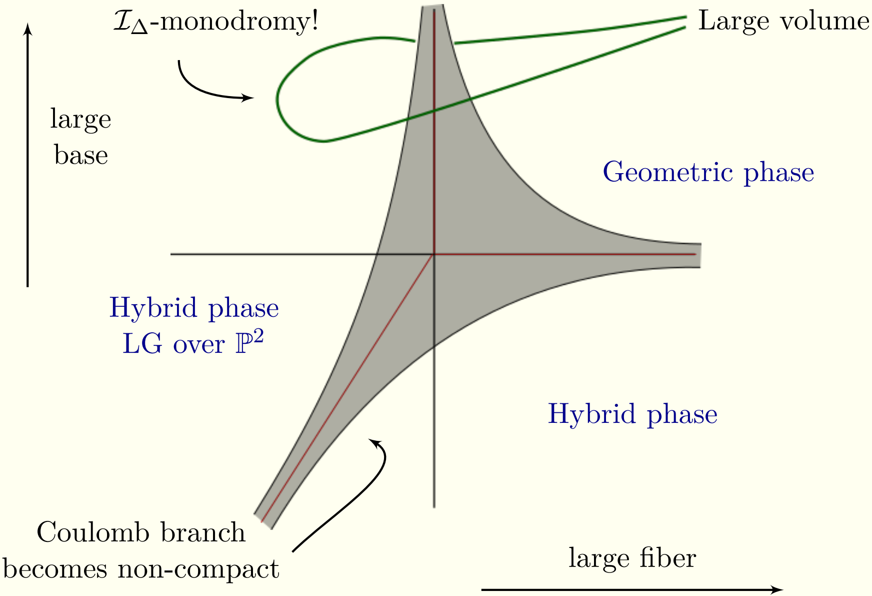
RG-flow

A-brane central charge
$Z(\Gamma) = \int_\Gamma \Omega$
depends on complex structure
Ω: (3,0)-form on W

$$\begin{array}{ccc} \text{B-branes on M} & \text{Mirror symmetry} & \text{A-branes on W} \\ \mathcal{F}^\bullet \in D^b(M) & \longleftrightarrow & \Gamma \in H_3(W) \end{array}$$

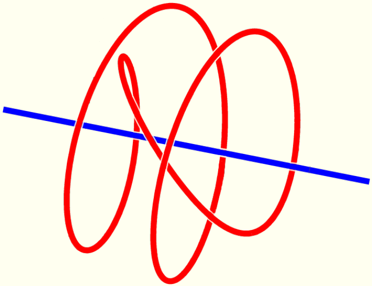
Fayet-Iliopoulos space of F_4 GLSM

(genus one fibration w/ 2-sections over $\mathbb{P}^2$)



How does this look in the B-model (open A-model)?

- Limit large base/small fiber coincides with *triple tangency* between discriminant and large base divisor
- Consider small **3-sphere around tangency** **[Aspinwall'01]**



This implies the relation

$$U = T_b^{-1} \cdot C \cdot T_b^{-1} \cdot C \cdot T_b^{-1} \cdot C \cdot T_b^{3} \,.$$

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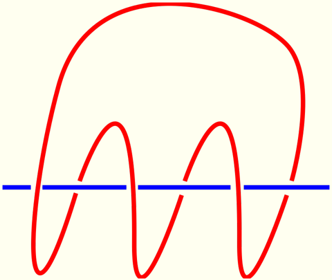


This implies the relation

$$U = T_b^{-1} \cdot C \cdot T_b^{-1} \cdot C \cdot T_b^{-1} \cdot C \cdot T_b^3 \,.$$

More generally:

Consider **any genus fibration** over $\mathbb{P}^2$.

Denote by C the generic conifold monodromy and by T_b the action of a shift of the "base" B-field by $t \to t+1$.

Then

$$U = T_b^{-1} \cdot C \cdot T_b^{-1} \cdot C \cdot T_b^{-1} \cdot C \cdot T_b^3 \, .$$

[Cota,Klemm,TS'19]

More generally:

Consider **any genus fibration** over $\mathbb{F}_n$, $n \in \mathbb{N}$.

Denote by C the generic conifold monodromy and by T_i the action of a shift of the B-field by $t_i \to t_i + 1$.

Then

$$U = \left(T_1^{-1} \cdot C \cdot T_2^{-1} \cdot C\right)^2 \cdot T_1^2 \cdot T_2^2 \,.$$

t_1: volume of $\mathbb{F}_n$ fiber, $\qquad t_2$: volume of $\mathbb{F}_n$ base

[Cota,Klemm,TS'19]

What properties does this imply for $Z_{\mathbf{top}}$?

mirror

- $Z_{\rm top}$ *wave function* on quantisation of phase space $H^3(Y)$ **[Witten'93]**
- Monodromies interpretable as *canonical transformations*

Monodromy matrix

$$M = \begin{array}{c} \Pi_{D6},\, \vec{\Pi}_{D4} \,|\, \vec{\Pi}_{D2},\, \Pi_{D0} \\ \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{pmatrix} \end{array} \begin{array}{l} \left.\begin{array}{l} \Pi_{D6}, \\ \vec{\Pi}_{D4} \end{array}\right\} \text{momenta} \\ \left.\begin{array}{l} \vec{\Pi}_{D2} \\ \Pi_{D0} \end{array}\right\} \text{positions} \end{array}$$

If $C = 0$, then $Z_{top}(\vec{t})$ is invariant under M!

[Aganagic,Bouchard,Klemm'06], **[Gunaydin,Neitzke,Pioline'06]**

Which monodromies don't mix momenta and positions?

- Large volume monodromies

$$T:\ \tau \mapsto \tau + 1\,, \quad M_a:\ m_a \mapsto m_a + 1\,, \quad T_i:\ t_i \mapsto t_i + 1$$

$\rightarrow$ *Fourier expansion*

- Generators of Weyl group **[Katz,Morrison,Plesser'96]**, **[Klemm,Mayr'96]**, **[Horja'99]**, **[Aspinwall'01]**, **[Szendroi,02]**

$\rightarrow$ *Weyl invariance*

- Furthermore, $E_a = M_a \cdot U^{-1} \cdot M_a^{-1} \cdot U$ acts as

$$E_a:\ \begin{cases} \tau & \mapsto & \tau \\ m_i & \mapsto & m_i\,, \quad i = 1, ..., \mathrm{rk}(G),\ i \neq a \\ m_a & \mapsto & m_a + N \cdot \tau\,, \\ Q_i & \mapsto & \exp\left(\frac{N^2 \cdot \tau}{2} C^i_{aa} + N C^i_{(ab)} m^b\right) Q_i \end{cases}.$$

$\rightarrow$ *Elliptic transformation law!*

Note: Elliptic fibrations with *torsional* sections have restricted $\Gamma_1(N)$ monodromy of fiber **complex structure** over base.

Conjecture:

[**Klevers,Pena,Oehlmann,Piragua,Reuter'14**], [**Oehlmann,Reuter,T.S.'16**]
T-dualizing fiber exchanges torsional sections and multi-sections

Fiber cplx structure monodromy $\leftrightarrow$ *Monodromy of Kähler class?*

How do we obtain the ansatz for the modular bootstrap on genus one fibrations with N-sections?

How do we obtain the ansatz for the modular bootstrap on genus one fibrations with N-sections?

By cheating!

Elliptic fibration
$G = \cdots \times U(1)$
Hypermultiplets w/ $q = N$

Higgsing
Restriction in $\mathcal{M}_{\text{Kähl}}$
$+$ deformation in $\mathcal{M}_{\text{cmplx.}}$

Genus one fibration
$G = \cdots \times \mathbb{Z}_N$
N-sections

Conjecturally
always exists

Higgsing in F-theory relates $Z_{\textbf{top.}}$s!

$$\tau \mapsto N \cdot \tau \,, \quad m \mapsto \tau$$

Can use relations among Jacobi/modular forms to obtain ansatz

Examples

Fibrations from fibers

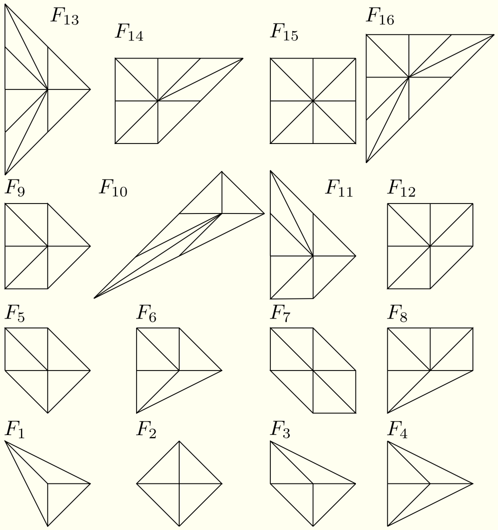
Engineering elliptic Calabi-Yau

see e.g. **[Cvetic,Klevers,Piragua'13]**, **[Klevers,Pena,Oehlmann,Piragua,Reuter'14]**

1. Obtain family of tori via Batyrev construction
2. Promote coefficients to sections of line bundles over base

- Properties of the fibration can be "tuned"

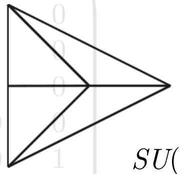
Fibrations from fibers



Example: F_4 **with base** $B = \mathbb{F}_1$

Toric data:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 \end{array}\right)$$



$SU(2) \times \mathbb{Z}_2$

Let us perform the modular bootstrap!

$$Z_{d_B,d_F} = \frac{\Delta_4^{2d_F+\frac{1}{2}d_B}}{\eta(2\tau)^{24d_F+12d_B}} \frac{\phi_{d_B,d_F}(\tau,\lambda,m)}{\prod_{k_1=1}^{d_F}\phi_{-2,1}(2\tau,k_1\lambda)\prod_{k_2=1}^{d_B}\phi_{-2,1}(2\tau,k_2\lambda)}$$

$$\begin{aligned}
\phi_F =& \frac{2}{9}\left(\Delta_4\right)^2\left[-8A^2g + AB\left(4g^2+h\right) + B^2g\left(18g^2-5h\right)\right] ,\\
\phi_B =& -2\sqrt{\Delta_4}g ,\\
\phi_{2B} =& \frac{\Delta_4}{288}\left[16A^2g^2 + 8ABg\left(h-2g^2\right) + B^2h\left(3h-11g^2\right)\right] ,\\
\phi_{B+F} =& \frac{\left(\Delta_4\right)^{\frac{5}{2}}}{216}\left[8A\left(8C^2g^2 - CDg\left(4g^2+h\right) - D^2g^2\left(18g^2-5h\right)\right)\right.\\
& B\left(-4C^2g\left(4g^2+33h\right) + 8CD\left(-4g^4+14g^2h+h^2\right)\right.\\
& \left.\left.-D^2g\left(4g^4-331g^2h+91h^2\right)\right)\right] ,
\end{aligned}$$

$$\begin{aligned}
A =& \phi_{0,1}(2\tau,\lambda) , \quad B = \phi_{-2,1}(2\tau,\lambda) ,\\
C =& \phi_{0,1}(2\tau,m) , \quad D = \phi_{-2,1}(2\tau,m) , \quad g = E_2^{(2)}(\tau) , \quad h = E_4(\tau) .
\end{aligned}$$

$$Z_{d_B,d_F} = \frac{\Delta_4^{2d_F+\frac{1}{2}d_B}}{\eta(2\tau)^{24d_F+12d_B}} \frac{\phi_{d_B,d_F}(\tau,\lambda,m)}{\prod_{k_1=1}^{d_F}\phi_{-2,1}(2\tau,k_1\lambda)\prod_{k_2=1}^{d_B}\phi_{-2,1}(2\tau,k_2\lambda)}$$

- Expressions for $Z_{\beta=n\cdot B}$ can be refined
  $\rightarrow$ *E-string w/ Wilson lines*

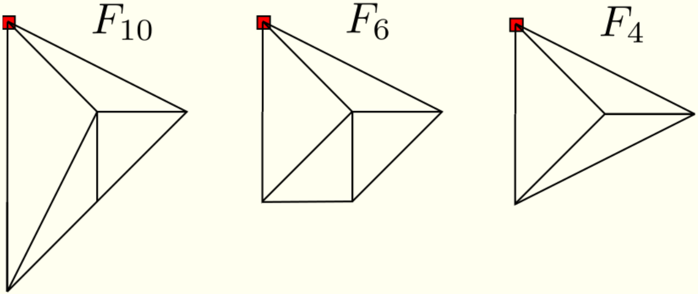
- Expressions for $Z_{\beta=n\cdot F}$ can be matched
  with one-loop calculation in heterotic strings on $(K3\times T^2)/\mathbb{Z}_2$

- $Z_{\beta=n\cdot B+m\cdot F}$ gives *non-perturbative prediction*!

$$A = \phi_{0,1}(2\tau,\lambda)\,,\quad B = \phi_{-2,1}(2\tau,\lambda)\,,$$

$$C = \phi_{0,1}(2\tau,m)\,,\quad D = \phi_{-2,1}(2\tau,m)\,,\quad g = E_2^{(2)}(\tau)\,,\quad h = E_4(\tau)\,.$$

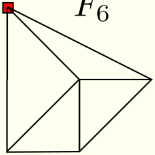
A Higgs chain: $F_{10} \to F_6 \to F_4$



$$SU(2) \xrightarrow[\text{higgsing}]{\text{adjoint}} U(1) \xrightarrow[\text{higgsing}]{\text{charge 2}} \mathbb{Z}_2$$

A Higgs chain: $F_{10} \to F_6 \to F_4$

F_6

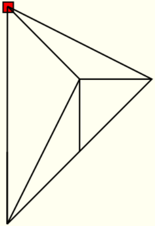


$U(1)$

restriction of
c.s. moduli

Z_{top} is equal!

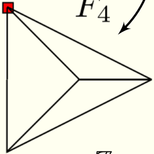
F_{10}



$SU(2)$

restriction of
Kähler moduli
$\tau \mapsto 2\tau, m \mapsto \tau$
$SL(2,\mathbb{Z}) \Rightarrow \Gamma_0(2)$

F_4



$\mathbb{Z}_2$

Bonus: Get X_{18} from F_6
via $m = 0$

A Higgs chain: $F_{10} \to F_6 \to F_4$

Relation between $F_{10} \to \mathbb{P}_2$ and $F_6 \to \mathbb{P}_2$

$F_{10} \to \mathbb{P}_2$

- Genus **g=10** curve of I_2 fibers
- $n = 72$ isolated fibral curves $\to$ fundamental matter
- 101 polynomial + **10** non-polynomial c.s. deformations

$F_6 \to \mathbb{P}_2$

- No fibral divisors but **two sections**
- **2g-2=18** charge two loci, $2n = 144$ charge one loci
- 111 polynomial + **0** non-polynomial c.s. deformations

Toric manifestation of discussion in **[Katz,Morrison,Plesser'96]**

Ansatz $F_4 \to \mathbb{P}^2$:

$$
\begin{aligned}
Z_d(\tau,\lambda) =& \frac{\Delta_4^{3d}}{\eta(2\tau)^{36d}} \frac{\phi_d(\tau,\lambda)}{\prod_{k=1}^{d} \phi_{-2,1}(2\tau,k\lambda)} \\
\phi_1(\tau,\lambda) =& 192\left[12\left(E_2^{(2)}\right)^2 + E_4\right]
\end{aligned}
$$

$$
E_2^{(N)}(\tau) = -\frac{1}{N-1}\partial_\tau \log\left(\frac{\eta(\tau)}{\eta(N\tau)}\right)\,, \quad \Delta_4(\tau) = \frac{1}{192}\left(E_4(\tau) - E_2^{(2)}(\tau)^2\right)
$$

$$
\begin{aligned}
\phi_2(\tau,\lambda) =& \frac{32}{9}A^4 \cdot \left(12g^2+h\right)^2 + A^3 B \frac{4}{27} g\left(1072g^4 - 7832g^2 h - 797h^2\right) \\
& - \frac{1}{54}A^2B^2 \cdot \left(4g^2 - h\right)\left(25504g^4 + 6924g^2 h + 227h^2\right) \\
& + AB^3 \cdot \frac{g\left(1425683g^6 + 7311527g^4 h - 733303g^2h^2 - 154563h^3\right)}{1728} \\
& + B^4 \cdot \frac{2550099g^8 - 20848992g^6 h + 2131870g^4h^2 + 885304g^2h^3 + 8887h^4}{6912}\,, \\
A =& \phi_{0,1}(2\tau,\lambda)\,, \quad B = \phi_{-2,1}(2\tau,\lambda)\,, \quad g = E_2^{(2)}(\tau)\,, \quad h = E_4(\tau)\,.
\end{aligned}
$$

How does this relate to Heterotic strings?

1. Heterotic on $K3 \times T^2$ dual to Type IIA on CY 3-fold **[Kachru,Vafa'95]**
2. Dualities of Heterotic strings on $K3 \times T^2$ contain $SL(2,\mathbb{Z})$
3. Realized as monodromies of dual Calabi-Yau **[Klemm,Lerche,Mayr'95]**
4. Dualities of Het. strings on $(K3 \times T^2)/\mathbb{Z}_N$ contain $\Gamma_1(N)$

$\rightarrow$ *Dual CY should be genus one fibration with N-sections*

For $N = 2$ we use modular bootstrap for genus one fibrations to perform all order checks against heterotic calculation by **[Chattopadhyaya,David'16]**

Summary

- Related absolute and relative conifold monodromy
  for *any* genus one fibration over $\mathbb{P}^2$, $\mathbb{F}_{n\in\mathbb{N}}$
- Derived elliptic transformation law for $Z_{\text{top.}}$
  w.r.t. geometric elliptic parameters
- Generalized modular bootstrap
  to genus one fibrations with N-sections
- Obtained new Type II duals to
  heterotic compactifications on $(K3 \times T^2)\mathbb{Z}_2$
- Using modular bootstrap, performed
  all order checks against heterotic one-loop amplitudes

Outlook

- What about genus one fibrations with N-sections, $N > 4$?
  More generators for $\Gamma_1(N)$ *and modularity of* $Z_{\beta=0}$ *puzzling*
  Potential source of swampland?
- BPS spectra of twisted compactifications of 6d SCFTs
  [Bhardwaj,Jefferson,Kim,Tarazi,Vafa'19]
- Find heterotic duals for $N > 2$

Thank you for your attention!