Halo-Independent *direct DM detection data analysis* **from convex hulls**

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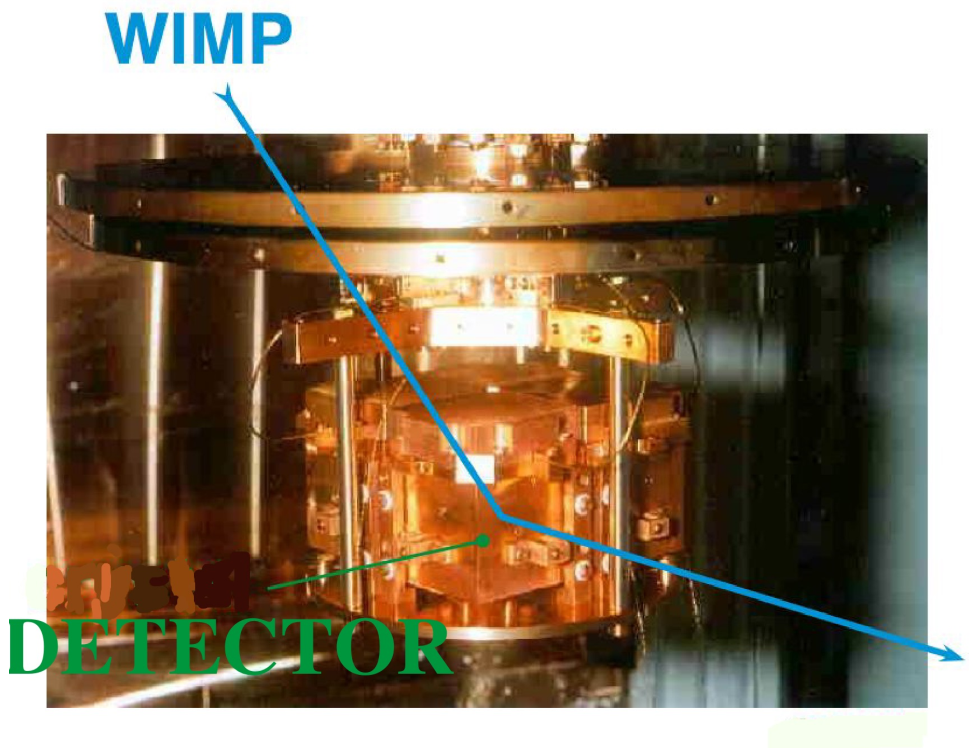
WIMP non-directional direct detection:

WIMP's interact coherently with nuclei in the detector, which recoil with energy E_R



Elements of the direct detection event rate

Event rate: events/(unit mass of detector)/(keV of recoil energy)/day

$$\frac{dR}{dE_R} = \sum_T \int_{v>v_{min}} N_T \times \frac{d\sigma_T}{dE_R} \times n v f(\vec{v},t) d^3 v$$

-E_R: nuclear recoil energy

- T: each target nuclide (elements and isotopes)

- $N_T = C_T/M_T$= Number of nuclides T in the detector = (mass fraction × Number of nuclides T per unit target mass);

- v_{min} min WIMP speed to impart E_R to the target T, $v_{min}(E_R) = \frac{1}{\sqrt{2M_T E_R}}\left|\frac{M_T E_R}{\mu_T} + \delta\right|$

- $\mu_T = mM_T/(m + M_T)$, reduced mass

-$\rho = nm$, $f(\vec{v},t)$: local DM density and $\vec{v}$ distribution depend on halo model.

The recoil rate dR/dE_R is not directly accessible to experiments, they observe only a proxy $E^{'}$ for the recoil energy E_R with $E^{'}$-dependent energy resolutions/efficiencies.

Observed event rate:

$$\frac{dR}{dE^{'}} = \varepsilon(E^{'}) \int_0^{\infty} dE_R \sum_T G_T(E_R, E^{'}) \frac{dR_T}{dE_R}$$

- $E^{'}$: detected energy (in keVee or number of PE), C_T: mass fraction in target nuclide T;
- $\varepsilon(E^{'})$: **counting efficiency or cut acceptance;** $G_T(E_R, E^{'})$: **energy response function**

$$\frac{dR_T}{dE_R} = \frac{C_T}{M_T} \int_{v>v_{min}} \frac{d\sigma_T}{dE_R} \times \frac{\rho}{m} v f(\vec{v}, t) d^3v$$

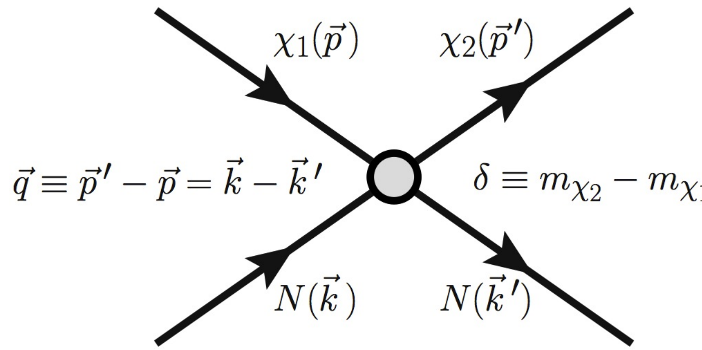
Elements of the rate: Each with its own uncertainties

[Event Rate] = [Detector Response] x [Cross Section] x [Halo Model]

Elements of the Event Rate

[Event Rate] = [Detector Response] x [Cross Section] x [Halo Model]

How does the DM particle couple to the nuclei?



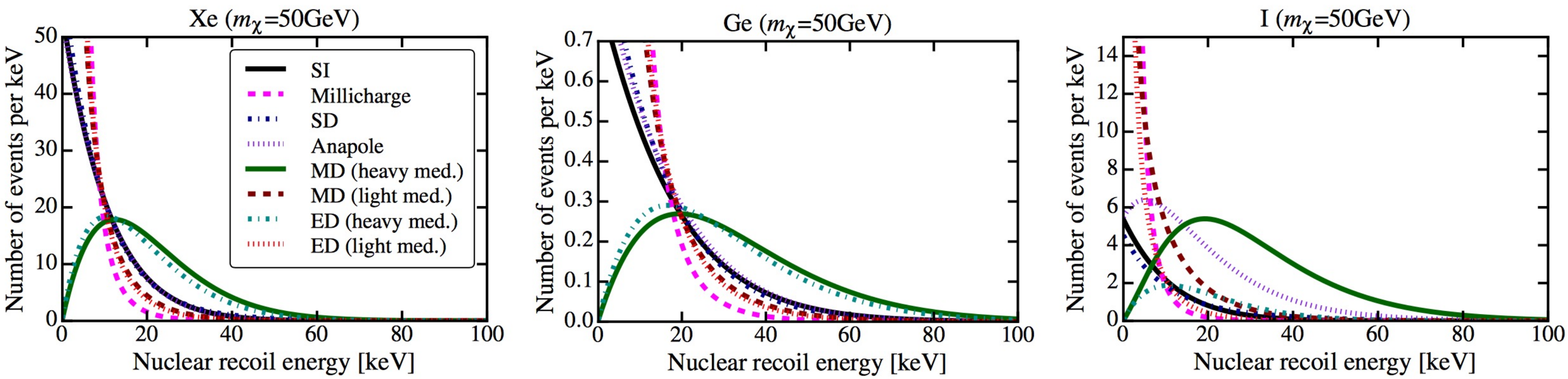
- Starting with fundamental interactions, DM particles couple to quarks/gluons, then pass from quarks/gluons to protons and neutrons, then to nuclei
- besides the DM mass m, this is the only input of Particle Physics

Cross sections can be very different: e.g. SI and Magnetic Dipole

$$\frac{d\sigma_T^{SI}}{dE_R} = \sigma_{ref}^{SI}\frac{|\vec{q}_{ref}|^4}{M^4}\frac{m_T}{2\mu_N^2 v^2}\left[A_T\right]^2 F_{SI,T}^2 \qquad |\vec{q}| \ \textit{independent}$$

$$\frac{d\sigma_T^{MD}}{dE_R} = \sigma_{ref}^{MD}\frac{|\vec{q}_{ref}|^2}{M^4}\frac{m_T^2}{4v^2\mu_N^2}\left[Z_T^2\left(4v^2|\vec{q}|^2 - |\vec{q}|^4\left\{\frac{1}{\mu_T^2} - \frac{1}{m_\chi^2}\right\}\right) F_{E,T}^2 + 2\frac{|\vec{q}|^4}{m_N^2}\frac{\lambda_T^2}{\lambda_N^2}\left(\frac{J_T+1}{3J_T}\right) F_{M,T}\right]$$

Rates can be very different than for SI Fig. from Gluscevic et al. 1506.04454

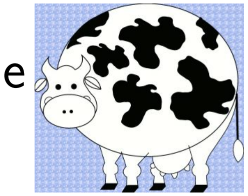


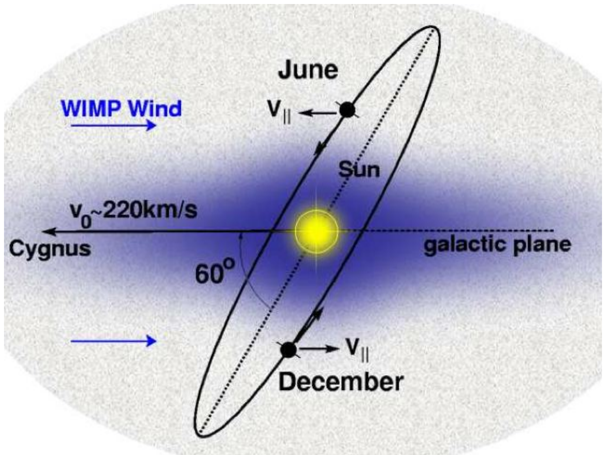
Elements of the Event Rate in Direct DM detection

[Event Rate] = [Detector Response] x [Cross Section] x [Halo Model]

How many DM particles are passing through the detector and with which velocity distribution?

The usually assumed Standard Halo Model is a good first approximation but not expected to be correct. Uncertainty in measurements of key parameters, and Earth could be within a DM clump, or streams, and maybe a dark disk and there are debris flows, triaxiality....

Standard Halo Model (SHM) The  of halo models

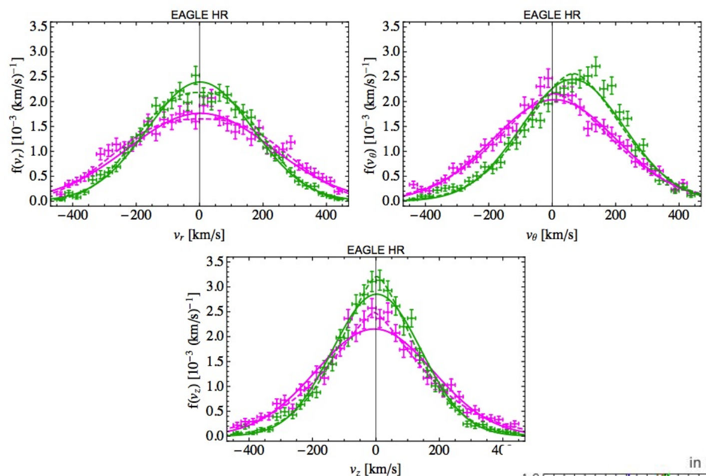


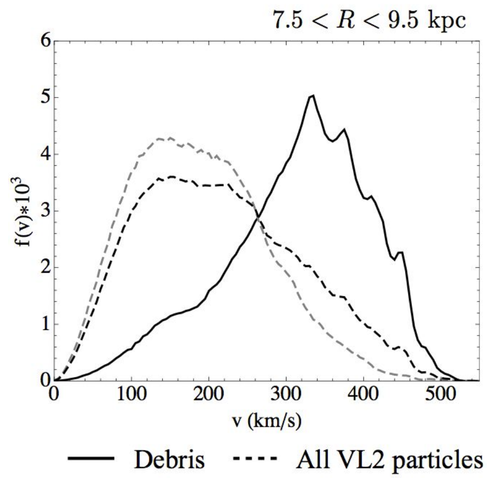
- $\rho_{SHM} = 0.3^{+0.2}_{-0.1}$ GeV/cm^3 local Dark Matter density
- $f(v,t)$: Maxwellian $\vec{v}$ distribution at rest with the Galaxy, $v_\odot \simeq$220km/s, $v_{esc} \simeq$500-650km/s

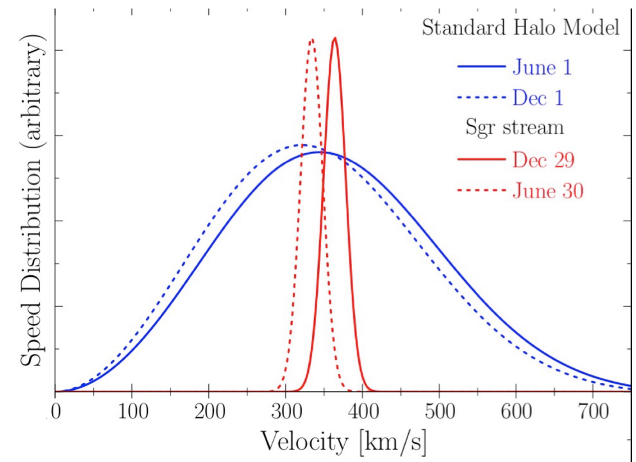
Expected annual modulation due to the max Galactic Earth velocity in late May-early June, min. in Dec. (unless gravitational focussing from the Sun is important) (Drukier, Freese, Spergel 1986)

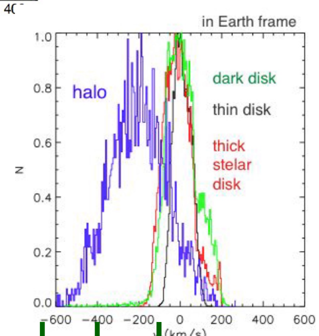
...but triaxiality, debris flows, streams, dark disk... "DM particles" in simulations have $> 10^3$ M$_\odot$....

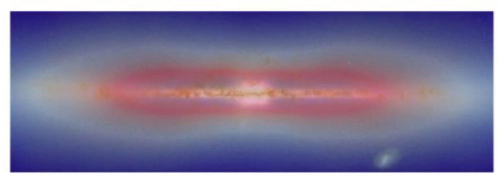
Distributions of radial, azimuthal, and vertical velocity components:











Avoid using a dark halo model to analyze Direct Detection data?
Lots of work done since 2010 on "Halo Independent" methods...

"Halo-Independent": Recall the event rate:

For a WIMP-nucleus contact differential cross section (for momentum transfer and velocity-independent interaction operators) e.g. for Spin Independent interactions

$$\frac{d\sigma_T}{dE_R} = \frac{\sigma_T(E_R)\ M_T}{2\mu_T^2 v^2} \qquad \sigma_T(E_R) \sim \sigma_{ref}$$

$$\frac{dR}{dE_R} = \sum_T \frac{\sigma_T(E_R)}{2m\mu_T^2}\ \rho\eta(v_{min}, t), \qquad \eta(v_{min}, t) = \int_{v>v_{min}} \frac{f(\vec{v}, t)}{v} d^3v = \int_{v_{min}} \frac{F(v, t)}{v} dv$$

- ρ, $f(\vec{v}, t)$: local DM density, Earth's frame $\vec{v}$ distribution depend on halo model

"Halo-Dependent": Given $\rho\eta(v_{min})$ plots in (m, σ_{ref}) plane (usual)

"Halo-Independent": Given m, $d\sigma_T/dE_R$ plots in $(v_{min}, \tilde{\eta}(v_{min}))$ plane,

$$\tilde{\eta}(v_{min}, t) = \frac{\sigma_{ref}}{m} \rho\eta(v_{min}, t)$$

contains all halo dependence in ANY experiment!

Fox, Liu, Weiner 1011.1915; Frandsen et al 1111.0292; Gondolo-Gelmini 1202.6359...

for ANY interaction, energy resolutions, efficiencies...

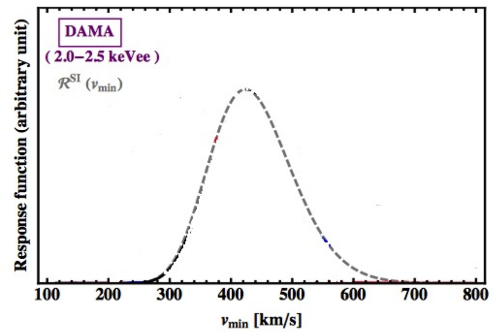
Gondolo-Gelmini 1202.6359; Del Nobile, Gelmini, Gondolo and Huh, 1306.5273

We write the predicted observable rate for any cross section as

$$R_{[E_1',E_2']} = \int_0^\infty dv_{min}\ \mathscr{R}_{[E_1',E_2']}(v_{min})\, \tilde{\eta}(v_{min},t)$$

$\mathscr{R}_{[E_1',E_2']}$ **: experiment and interaction dependent response function**

(non zero only for an interval in v_{min} given a measured energy interval $[E_1', E_2']$)



Proof: leave v integration for last and integrate by parts:

$$R_{[E_1',E_2']} = C\int_0^\infty dv\ \mathscr{H}_{[E_1',E_2']}(v)\ F(v,t) = C\int_0^\infty dv_{min}\ \mathscr{R}_{[E_1',E_2']}(v_{min})\, \eta(v_{min},t)$$

Early halo Independent analysis

Gondolo-Gelmini 1202.6359, Del Nobile, Gelmini, Gondolo and Huh, 1306.5273

Rate measurements: just translated into weighted averages of the $\tilde{\eta}$ function:

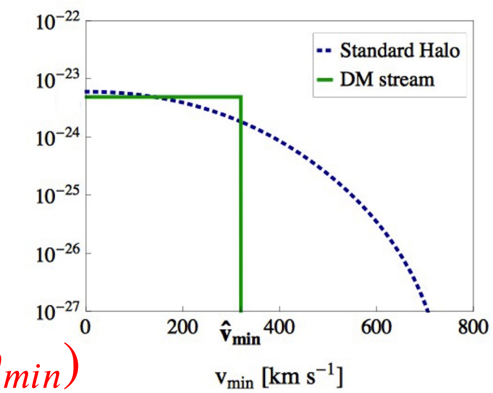
$$R^{measured}_{[E_1',E_2']} = \widetilde{<\eta_{[E_1',E_2']}>} \int_0^\infty dv_{min}\ \mathscr{R}_{[E_1',E_2']}(v_{min})$$

$\widetilde{<\eta_{[E_1',E_2']}>}$: weighted average of $\tilde{\eta}$ with weight $\mathscr{R}_{[E_1',E_2']}(v_{min})$

For both the average value over a year $\widetilde{\eta}_0$ and the annual modulation amplitude $\widetilde{\eta}_1$:

$\tilde{\eta}(v_{min}, t) = \widetilde{\eta}_0(v_{min}) + \widetilde{\eta}_1(v_{min})\ cos[\omega(t - t_0)]$; $R(t) = R_0 + R_1\ cos[\omega(t - t_0)]$

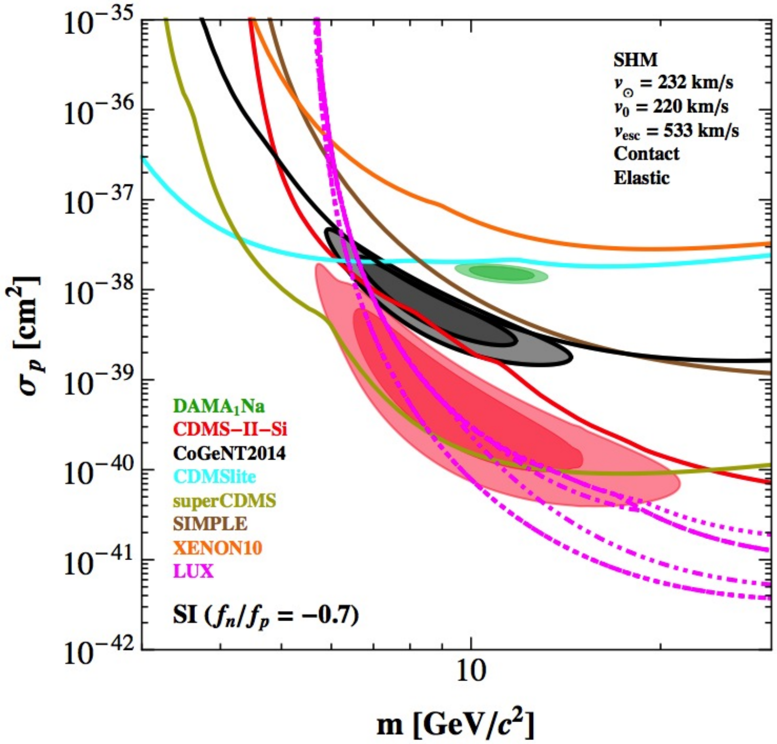
Upper limits: $\tilde{\eta}$ is a non decreasing function of v_{min}: the smallest possible halo passing by $(\hat{v}, \widetilde{\eta}_0)$ is $\tilde{\eta}(v_{min}) = \widetilde{\eta}_0 \Theta(\hat{v} - v_{min})$. Thus, compute the rate with this downward step $\tilde{\eta}$ function and ask for this rate to be at most equal to the measured limit for $\widetilde{\eta}_0 = \widetilde{\eta}_0^{\,lim}$.

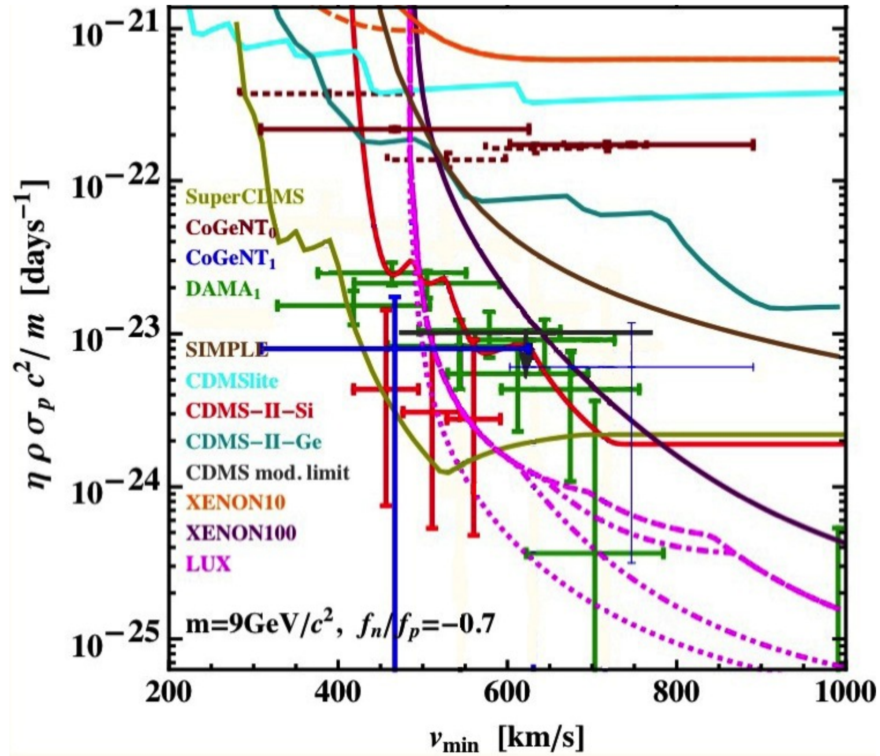


$$R^{limit}_{[E_1',E_2']} = \widetilde{\eta}_0^{\,limit}(\hat{v}) \int_0^{\hat{v}} dv_{min}\, \mathscr{R}_{[E_1',E_2']}(v_{min})$$

Halo Dependent vs Independent comparison for elastic SI IV

Del Nobile, Gelmini, Gondolo, Huh 1304.6183, 1311.4247, 1405.5582





LEFT: Part of the 90%CL CDMS-II-Si region survives all 90%CL limits.

RIGHT: $m = 9$GeV. CDMS-II-Si rate small for CoGeNT/DAMA mod. CoGeNT annual mod. compatible with zero at $\simeq 1\sigma$, with best fit phase of DAMA- Comparison of crosses and limits???

Likelihood based "Halo Independent" data analysis:

1- **Infer properties of the dark halo from Direct Detection data**, e.g. for the coefficients of the harmonic expansion of $\tilde{\eta}(v_{min}, t)$ (mostly its time average).

2- **Determine the compatibility of different data by comparing their inferred halo characteristics**, e.g. for the time-averaged $\tilde{\eta}(v_{min}, t)$:
– putative measurements translate into regions in the $(v_{min}, \tilde{\eta}(v_{min}))$ plane,
– upper limits into upper limits on $\tilde{\eta}(v_{min})$

Main Problem: Likelihood methods are good for parameter estimation, **but here we want to estimate a function,** $\tilde{\eta}$ or the local WIMP speed distribution F which the predicted rates depend on $\left(C = \frac{\sigma_{ref}\rho}{m}\right.$ is a constant$\left.\right)$

$$R_{[E_1',E_2']} = C \int_0^\infty dv_{min}\ \mathscr{R}_{[E_1',E_2']}(v_{min})\, \eta(v_{min}, t) = C \int_0^\infty dv\ \mathscr{H}_{[E_1',E_2']}(v)\ F(v, t)$$

In 2014-15 solved the problem only for unbinned data (Extended Likelihood)

All likelihoods depend on the dark halo velocity distribution only through the predicted rates:

For experiments $\beta = 1, 2, \ldots, n_{unbinned}$ with unbinned data, one can use an extended likelihood,

$$\mathscr{L}_\beta[\tilde{\eta}] = e^{-\nu_\beta[\tilde{\eta}]} \prod_{j=1}^{N_{O\beta}} (MT)_\beta \left(\frac{dR_\beta}{dE'}[\tilde{\eta}] + \frac{dR_{\beta b}}{dE'} \right)\Bigg|_{E'=E'_j},$$

ν_β = total number of expected events, $\nu_\beta[\tilde{\eta}] = (MT)_\beta (R_\beta[\tilde{\eta}])_{total}$.

For experiments $\alpha = 1, \ldots, n_{binned}$, with binned data α, usually a Poisson or a Gaussian likelihood

$$\mathscr{L}_\alpha[\tilde{\eta}] = \prod_{j=1}^{N_{bin-\alpha}} \frac{(\nu_{\alpha j}[\tilde{\eta}] + b_{\alpha j})^{n_{\alpha j}}}{n_{\alpha j}!} exp[-(\nu_{\alpha j}[\tilde{\eta}] + b_{\alpha j})],$$

$$\mathscr{L}_\alpha[\tilde{\eta}] = \prod_{j=1}^{N_{bin-\alpha}} \frac{1}{\sigma_{\alpha j}\sqrt{2\pi}} exp[-(\nu_{\alpha j}[\tilde{\eta}] + b_{\alpha j} - n_{\alpha j})^2/\sigma_{\alpha j}^2],$$

$\nu_{\alpha j}[\tilde{\eta}]$ = number of events predicted in the bin j of exp. α, $\nu_{\alpha j}[\tilde{\eta}] = (MT)_\alpha R_{\alpha j}[\tilde{\eta}]$,

$R_{\alpha j}$ = integrated predicted rate in the same bin, $(MT)_\alpha$ = exposure of experiment α.

(In calculations we minimize $L = -2ln\mathscr{L}$ instead of maximizing $\mathscr{L}$)

Halo-Independent analysis as of 2015

Regions for putative DM (time averaged) rate measurements: With unbinned data (e.g. three CDMS-II-Si events), using at least one extended likelihood, we found (Fox, Kahn and McCullough 1403.6830; Gelmini, Georgescu, Gondolo and Huh 1507.03902; Gelmini, Huh and Witte 1607.02445)

1 - a unique piecewise constant best fit $\tilde{\eta}(v_{min})$ with a number of downward steps $\leq$ number of data points, by extending to functionals the Karush-Kuhn-Tucker (KKT) maximization conditions (Fox, Kahn and McCullough 1403.6830), and defined

2 -a statistically meaningful two-sided point-wise band at a chosen CL. (Gelmini, Georgescu, Gondolo and Huh, 1507.03902)

Karush-Kuhn-Tucker (KKT) conditions are first order necessary conditions for an extremization problem with inequality conditions. They generalize the method of Lagrange multipliers, which allows only equality constraints.

Minimize $f(x)$ subject to $g_i(x) \leq 0$. Then one minimizes

$$L(x, \mu_i) = f(x) + \sum_i \mu_i g_i(x)$$

where μ_i are KKT multipliers and $\mu_i \geq 0$

Then, if x^* is a local optimum,

$\vec{\nabla} f(x^*) = \sum_i \mu_i \vec{\nabla} g_i(x^*)$ i.e. and $\mu_i g_i(x^*) = 0$ for all i are necessary conditions.

We extended this procedure for functions to functionals.

Karush-Kuhn-Tucker (KKT) conditions to minimize $L[\tilde{\eta}] = -2\ \mathscr{L}[\tilde{\eta}]$

Gelmini, Georgescu, Gondolo and Huh, 1507.03902, where $q(v_{min})$ is the KKT multiplier for the condition that $\tilde{\eta}(v_{min})$ is non-increasing (III):

$$\text{(I)} \quad q(v_{min}) = \int_{v_\delta}^{v_{min}} dv \, \frac{\delta L}{\delta \tilde{\eta}(v)}$$

$$\text{(II)} \quad q(v_{min}) \geq 0$$

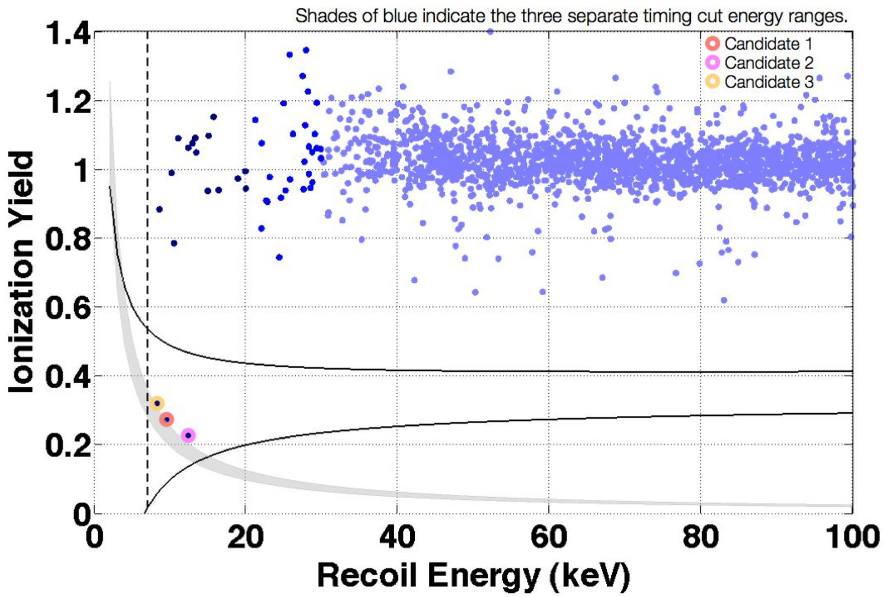
$$\text{(III)} \quad \forall \varepsilon > 0, \quad \tilde{\eta}(v_{min} + \varepsilon) \leq \tilde{\eta}(v_{min})$$

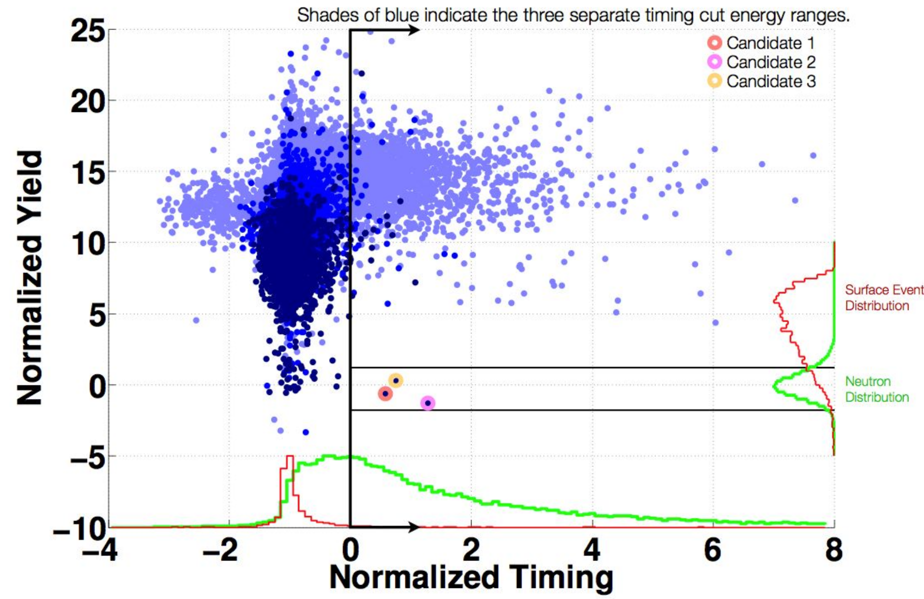
$$\text{(IV)} \quad q(v_{min}) \, {}_{\varepsilon \to +0} \frac{\tilde{\eta}(v_{min} + \varepsilon) - \tilde{\eta}(v_{min})}{\varepsilon} = 0 \, .$$

In 1507.03902 we proved that for an extended likelihood $q(v_{min})$ has only at most N_O (number of events observed) isolated zeroes, thus the $\tilde{\eta}(v_{min})$ that minimizes L is **piece-wise constant with at most N_O downward steps.**

CDMS three candidate events in Si (April14, 2013)

140.23 kg-day from July 2007 to Sep. 2008 in 8 Si detectors, expected background events < 0.7 $(0.41^{+0.20+0.28}_{-0.08-0.24})$, 5.4% probability of being known backgrounds

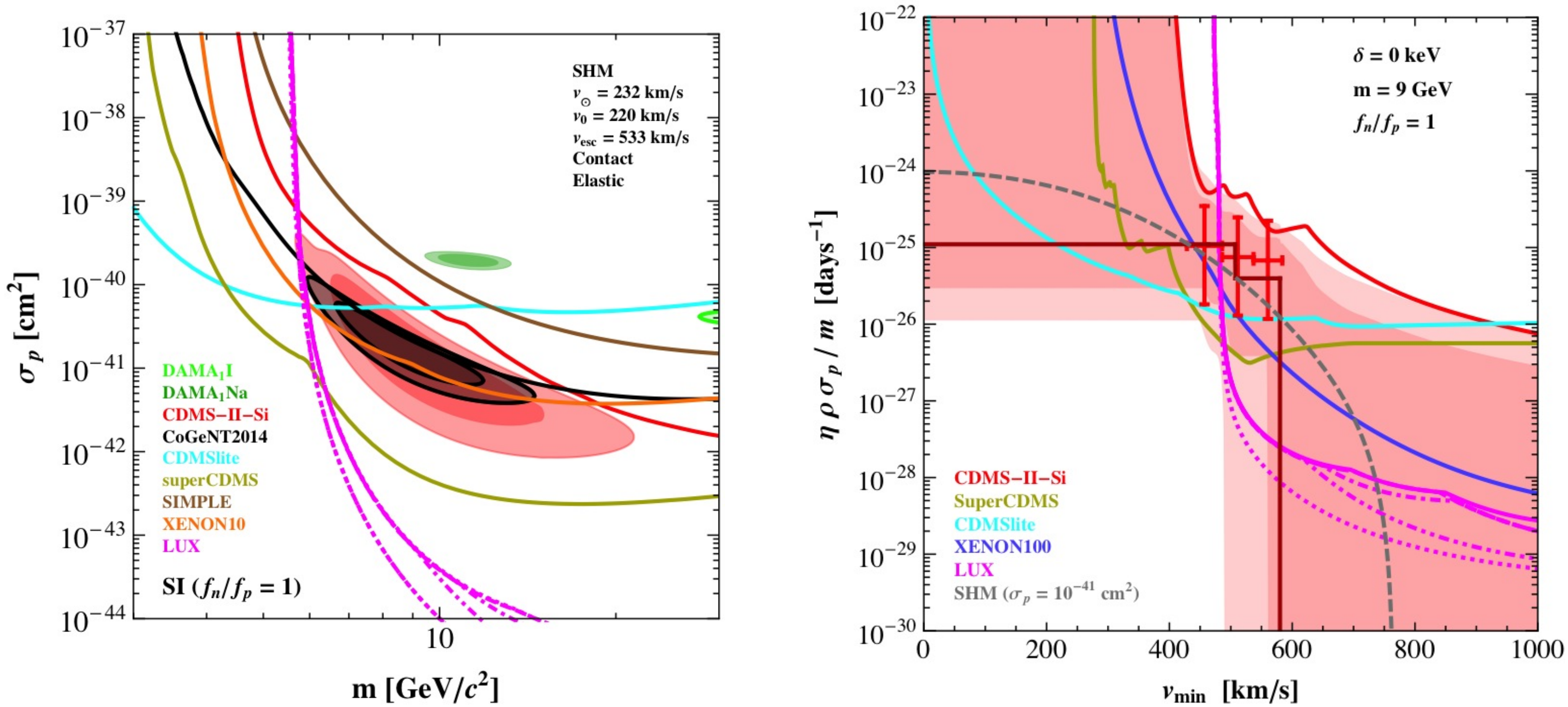




Extendent likelihood Halo Independent method: forbidden model

Fox, Kahn and McCullough 1403.6830; Gelmini, Georgescu, Gondolo and Huh, 1507.03902

LEFT: halo dependent Figs. from Del Nobile, Gelmini, Gondolo, Huh 1405.5582 RIGHT: halo independent

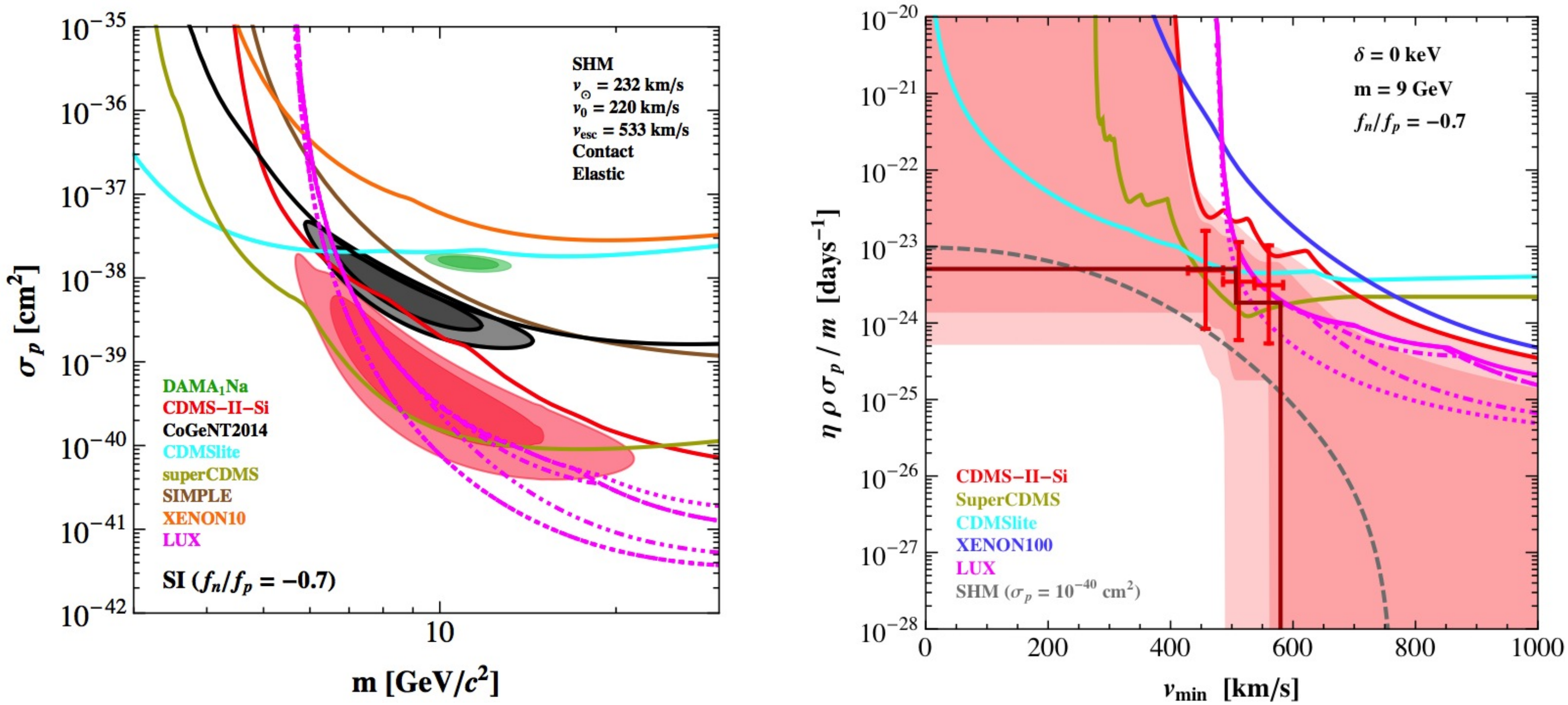


90%CL bounds and the 68% and 90%CL regions (Left) and confidence confidence bands (Right) for CDMS-II-Si, $m = 9$ GeV elastic SI and $f_n/f_p = 1$. No continuous part of the bands allowed

Extendent likelihood Halo Independent method: allowed model

Fox, Kahn and McCullough 1403.6830; Gelmini, Georgescu, Gondolo and Huh, 1507.03902

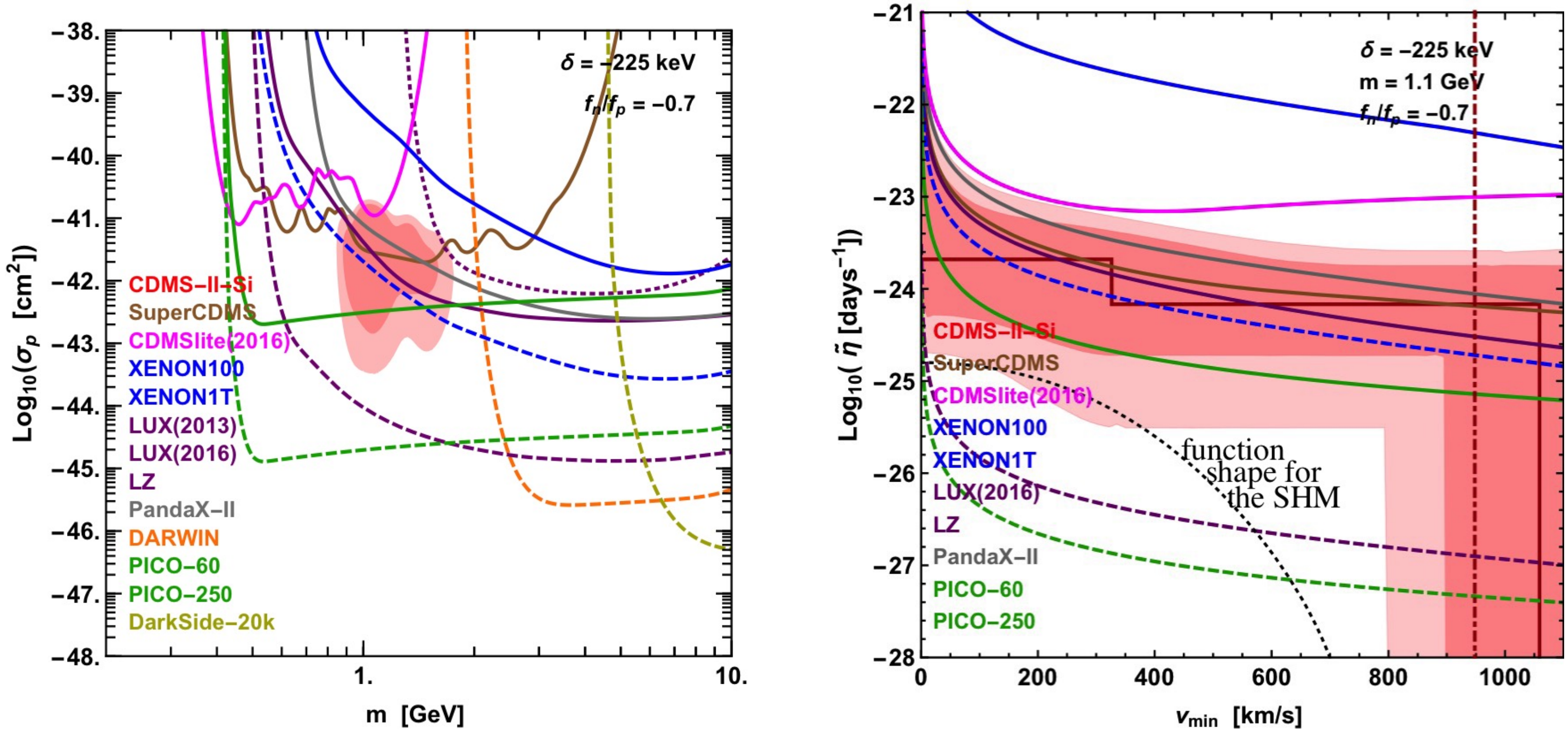
LEFT: halo dependent Figs. from Gelmini, Georgescu, Huh 1404.7484 RIGHT: halo independent



90%CL bounds and the 68% and 90%CL regions and confidence bands for CDMS-II-Si, $m = 9$ GeV elastic SI f_n/f_p=-0.7. A continuous part of the bands (so any $\tilde{\eta}$ contained in it) is allowed

Halo-Dependent and Independent analyses CDMS-II-Si data

inelastic exothermic DM with SI IV coupling, $\delta = -225$ keV Witte, Gelmini 1703.06892



LEFT: assuming the SHM RIGHT: Halo independent, m=1.1GeV

Can be ruled out by an LZ or PICO-250 like experiment (not XENON1T)

We later extended this result Gelmini, Huh and Witte, 1607.02445 to a global likelihood consisting of at least one extended likelihood (requires unbinned data) and an arbitrary number of Gaussian or Poisson likelihoods (which use binned data):

- $\tilde{\eta}(v_{min})$ that minimizes L is piece-wise constant with at most N downward steps (N$=$ number of data entries);

- this best fit $\tilde{\eta}(v_{min})$ is unique (L does not have a degenerate minimum);

- we showed how to construct a two sided pointwise confidence band at any desired confidence level,

We wanted to extend these results to any likelihood (e.g. with just binned data, and to modulation data) and suspected that the $\tilde{\eta}(v_{min})$ is always piecewise constant, but we found that $q(v_{min})$ may have extended zeroes and thus we could not use KKT (IV) to prove the shape of $\tilde{\eta}(v_{min})$.

Now comes the use of the convex hull...

Let us first concentrate on time-averaged rates (so no t dependence)

A deeper understanding of Halo-Independent methods for all Likelihoods Gelmini, Huh and Witte 1707.07019

Why a piecewise constant best fit $\tilde{\eta}(v_{min})$ with the number of downward steps $\leq$ the number of data points????

Without making any assumption on $F(v)$, theorems in convex geometry (Caratheodory, Fenchel-Eggleston) provide the answer: for d (time-averaged) rate data points the DM speed distribution $F(v)$, normalized to 1, is

$$F(v) = \sum_{n=1}^{d} F_n \, \delta(v - v_n) \quad with \quad \sum_{n=1}^{d} F_n = 1$$

Now we have at most 2d parameters F_n, v_n to estimate using the Likelihood **and the integral** $\tilde{\eta}(v_{min}) = C \int_{v_{min}}^{\infty} dv \; F(v)/v$ **is the sum of at most d step functions** $\Theta(v_n - v_{min})$

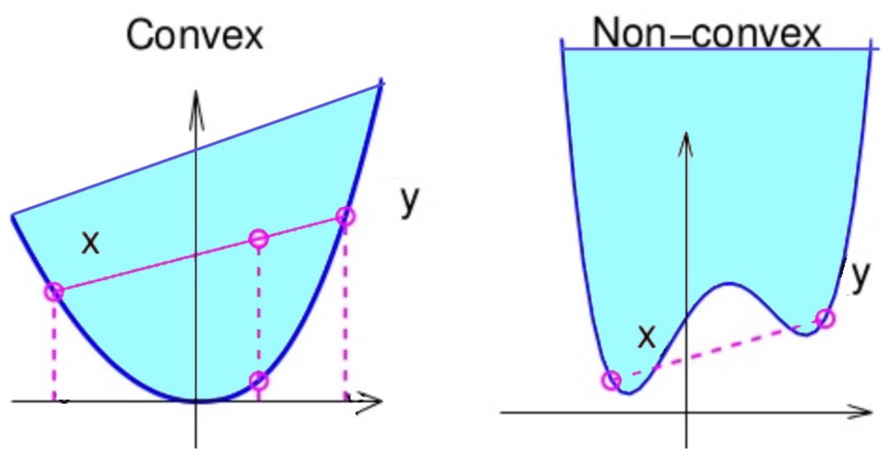
(A refinement I will not return to: because the rates are non negative, the maximum number of steps is actually $d - 1$)

Convexity has to do with the maximization/minimization problem, i.e. optimization.

A "convex set" of points is such that it includes all the points in a straight line connecting any two points in the set. Each point in a straight line between $\vec{x}$ and $\vec{y}$ is given by

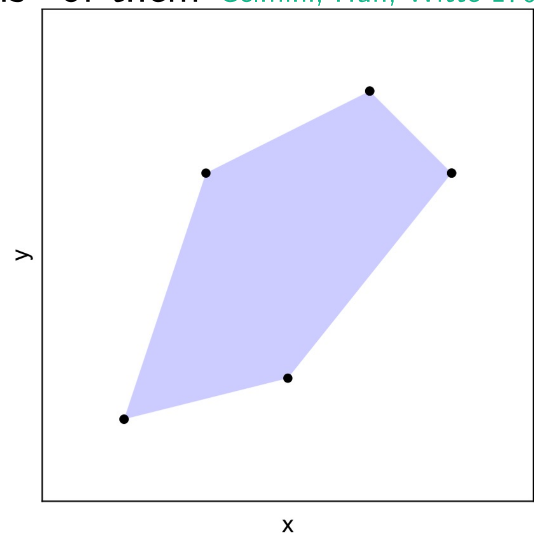
$$\vec{r} = a\vec{x} + b\vec{y}$$

where a, b are real non-negative numbers and $a + b = 1$. This is a "convex combination" of $\vec{x}$ and $\vec{y}$.



Convex hull of “generating” vectors of d-dimensions

Contains all “convex combinations” of them Gelmini, Huh, Witte 1707.07019

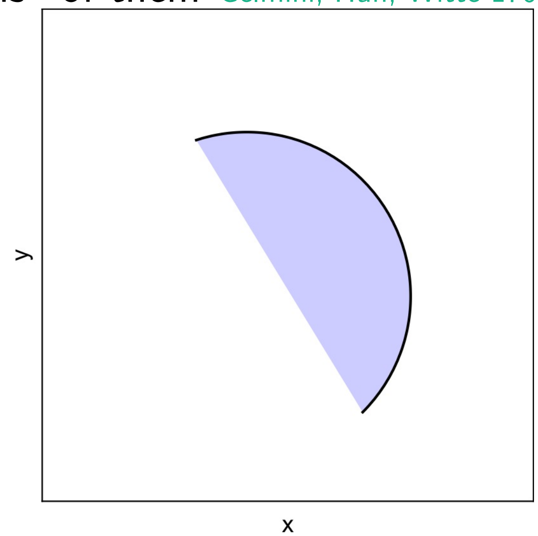


Discrete set of generating vectors $\vec{r}_i$: any vector in the hull is a **“convex combination”** of them i.e. a linear combination $\vec{r} = \sum_{i=1}^{n} a_i \vec{r}_i$ **with coefficients a_i real non-negative and $\sum_{i=1}^{n} a_i = 1$**

Notice: any $\vec{r}$ in the hull can be written as combination of at most d+1 (here 3) generating vectors. (Caratheodory Theorem)

Convex hull of "generating" vectors of d-dimensions

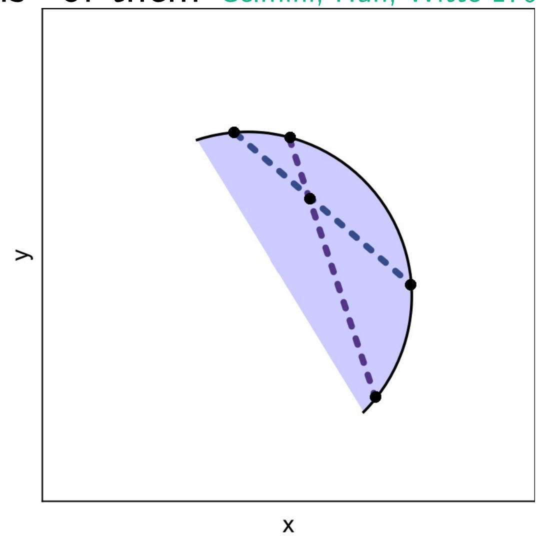
Contains all "convex combinations" of them Gelmini, Huh, Witte 1707.07019



Connected set of generating vectors $\vec{r}(\lambda)$ (with a real parameterλ): any vector in the hull is defined by $\vec{r} = \int \vec{r}(\lambda)\ a(\lambda)\ d\lambda$ with $\int a(\lambda)\ d\lambda = 1$ (Choquet theorem)

Convex hull of "generating" vectors, of d-dimensions

Contains all "convex combinations" of them Gelmini, Huh, Witte 1707.07019



Connected set of generating vectors $\vec{r}(\lambda)$ (with a real parameter λ): any vector in the hull is defined by $\vec{r} = \int \vec{r}(\lambda)\ a(\lambda)\ d\lambda$ with $\int a(\lambda)\ d\lambda = 1$

Fenchel-Eggleston: **any $\vec{r}$ in the hull can be written as combination of at most d (here 2) generating vectors**, i.e. $\vec{r} = \sum_{n=1}^{d} \vec{r}(\lambda_n) a(\lambda_n)$

Same as saying: $a(\lambda) = \sum_{n=1}^{d} a(\lambda_n)\delta(\lambda - \lambda_n)$

Predicted Rates Let us write each rate (here integrated- but same for differential) using the DM speed distribution, $F(v,t)$ ($C = \sigma_{ref}\rho/m$, constant)

$$R_{[E_1',E_2']}(t) = C\int_0^\infty dv\ \mathscr{H}_{[E_1',E_2']}(v)\ F(v,t)$$

by integration by parts can be written as before

$$R_{[E_1',E_2']}(t) = \int_0^\infty dv_{min}\ \mathscr{R}_{[E_1',E_2']}(v_{min})\ \tilde{\eta}(v_{min},t)$$

with

$$\tilde{\eta}(v_{min},t) = C\int_{v_{min}}^\infty dv\ \frac{F(v,t)}{v}, \qquad \mathscr{R}_{[E_1',E_2']}(v) = \frac{\partial}{\partial v}\left[v\mathscr{H}_{[E_1',E_2']}(v)\right]$$

and consider just average rates for simplicity (so no t dependence)

With d data points, d predicted rates enter in the Likelihood. Consider these rates as the d components of a rate vector $\vec{R}$. Each component has its kernel $\mathscr{H}$, which we put in a kernel vector $\overrightarrow{\mathscr{H}}$, then, using the speed distribution $F(v)$

$$\vec{R} = C \int_0^\infty \overrightarrow{\mathscr{H}}(v)\ F(v)\ dv$$

$F(v) \equiv v^2 \int d\Omega_v f(\vec{v})$ is normalized to 1: $\int_0^\infty dv\ F(v) = 1$. This means that **all possible predicted rate vectors $\vec{R}$ form the convex hull of the originating connected kernel vectors $\overrightarrow{\mathscr{H}}(v)$ (a set with one real parameter v)**.

Fenchel-Eggleston says that all vectors in the hull can be written as a convex combination of at most d kernel vectors $\vec{R} = \sum_{n=1}^{d} \overrightarrow{\mathscr{H}}(v_n)\ F(v_n)$
which amounts to writing our result,

$$F(v) = \sum_{n=1}^{d} F_n\ \delta(v - v_n), \quad with \quad F_n = F(v_n)$$

Confidence and degeneracy bands for any Likelihood

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In this way we can always find a best fit piecewise constant $\widetilde{\eta_{BF}}$ **but it may or may not be unique!**

With at least one extended likelihood it is unique, and we can define a confidence band at any CL.

With Gaussian and Poisson distributions only, it may or may not be unique. If it is not, we can find a degeneracy band within which all $\tilde{\eta}$ functions maximize equally the Likelihood.

Confidence and degeneracy bands

Define a two-sided pointwise band as the region filled by all possible $\tilde{\eta}$ functions satisfying (here L is the -2 log-likelihood)

$$\Delta L[\tilde{\eta}] \equiv L[\tilde{\eta}] - L_{\text{min}} \leq \Delta L^*$$

with ΔL^* corresponding to a desired CL (we need to know the prob. distribution).

In practice: find the subset of $\tilde{\eta}$ functions which minimize $L[\tilde{\eta}]$ subject to the constraint of passing by a particular point $(v^*, \widetilde{\eta}^*)$ i.e. $\tilde{\eta}(v^*) = \widetilde{\eta}^*$.
If $L^c_{\text{min}}(v^*, \widetilde{\eta}^*)$ is the minimum of $L[\tilde{\eta}]$ subject to the constraint,

$$\Delta L^c_{\text{min}}(v^*, \widetilde{\eta}^*) = L^c_{\text{min}}(v^*, \widetilde{\eta}^*) - L_{\text{min}} \leq \Delta L^*$$

If we find $\Delta L^* \neq 0$, we define a **confidence band.**
If we find solutions with $\Delta L^* = 0$ this means that the best fit eta function is not unique and define instead a **degeneracy band.**

Confidence and degeneracy bands

If there is no degeneracy band, i.e. the best fit $\tilde{\eta}$ function is unique, Wilks theorem can be applied. We prove this by discretizing in v_{min}.

Take K discrete values in $v_{\rm min}$: $(v^0_{min}, \dots, v^{K-1}_{min})$. The constraint to pass by the point $(v^*, \widetilde{\eta}^*)$ is now $v^k_{min} \le v^* < v^{k+1}_{min}$ and $\widetilde{\eta}^* = \widetilde{\eta}_k$ for some k, $0 \le k \le K-1$. $\Delta L^c_{\rm min}(v^*, \widetilde{\eta}^*)$ is replaced by the function $\Delta L^{k,c}_{\rm min}(\widetilde{\eta}^*)$ with the index k corresponding to v^*,

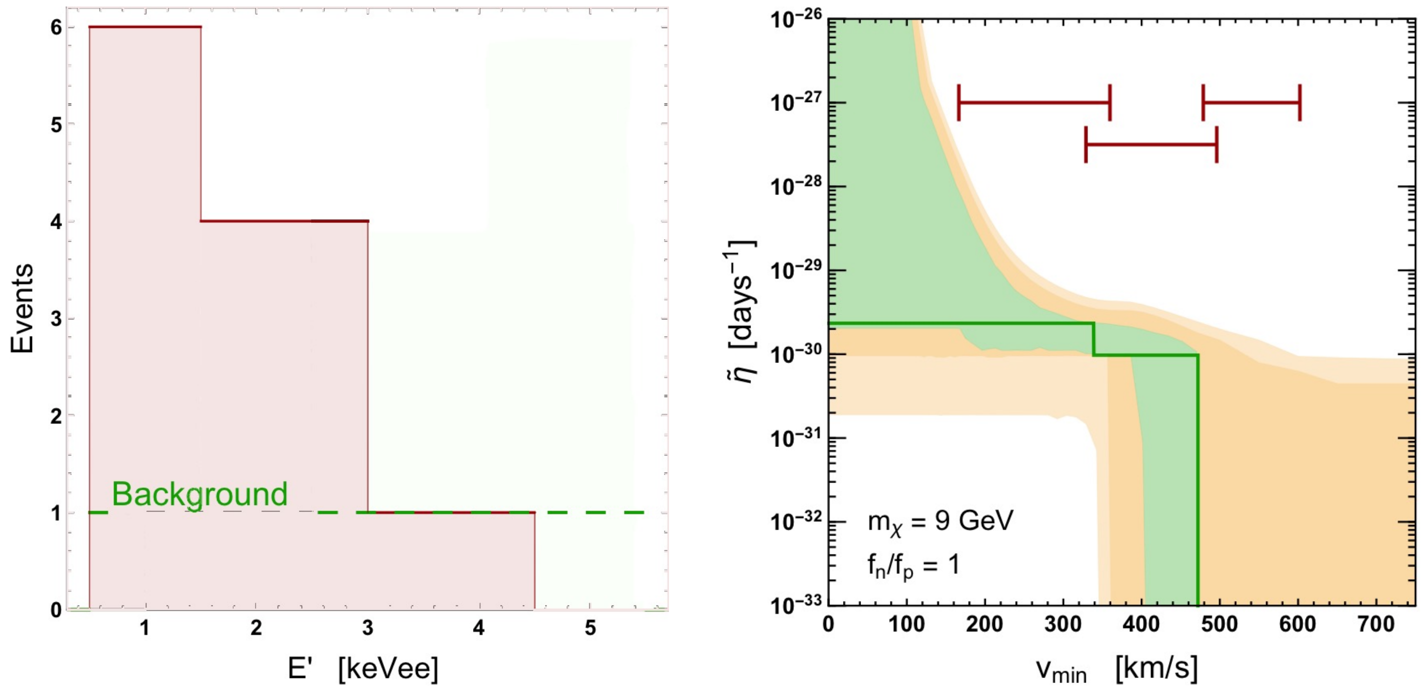
$$\Delta L^{k,c}_{\rm min}(\widetilde{\eta}^*) = -2 \left[\frac{\mathscr{L}(\hat{\hat{\eta}}_0, \dots, \hat{\hat{\eta}}_{k-1}, \widetilde{\eta}_k = \widetilde{\eta}^*, \hat{\hat{\eta}}_{k+1}, \dots, \hat{\hat{\eta}}_{K-1})}{\mathscr{L}(\hat{\eta}_0, \dots, \hat{\eta}_k, \dots, \hat{\eta}_{K-1})} \right]$$

$\hat{\hat{\eta}}_i$ are the $\widetilde{\eta}_i$ which maximize $\mathscr{L}$ subject to the constraint $\widetilde{\eta}_k = \widetilde{\eta}^*$, $\hat{\eta}_i$ maximize $\mathscr{L}$ without it

$\Delta L^{k,c}_{\rm min}(\widetilde{\eta}^*)$ now defines the -2 of the profile likelihood ratio with one parameter $(\widetilde{\eta}_k)$, thus by Wilks' theorem the distribution of $\Delta L^{k,c}_{\rm min}(\widetilde{\eta}^*)$ approaches the chi-square distribution with one degree of freedom in the limit where the data sample is very large. This is not changed when we take the continuum limit $K \to \infty$. If so, $\Delta L^* = 1.0\ (2.7)$ correspond to the confidence intervals $\tilde{\eta}$ at $68\ (90)\%$ CL for each v_{min} value.

Fake binned data Best fit not unique- degeneracy band (green)

$\Delta L = 1.0$ (darker yellow) and $\Delta L = 2.7$ (lighter yellow) Gelmini, Huh, Witte 1707.07019



Fake binned data Best fit unique- Confidence bands,

68%CL ($\Delta L = 1.0$) darker purple, and 90%CL ($\Delta L = 2.7$) (lighter purple)

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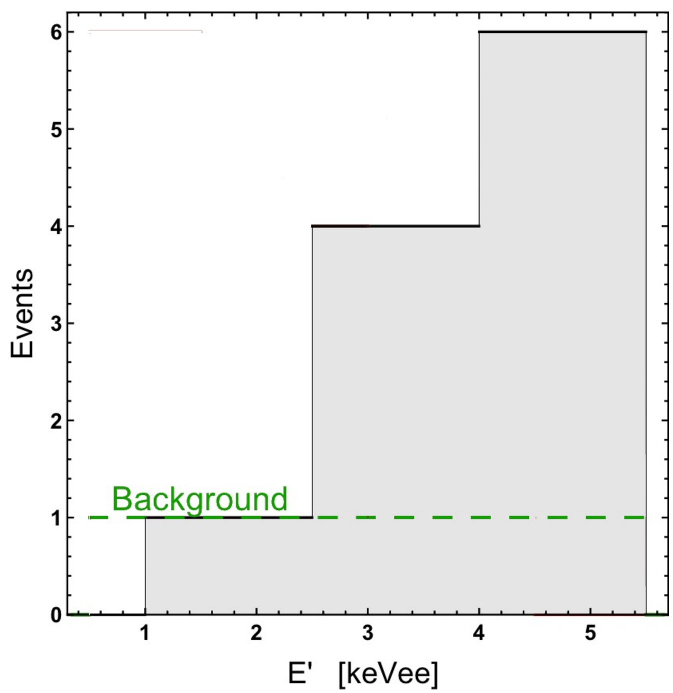

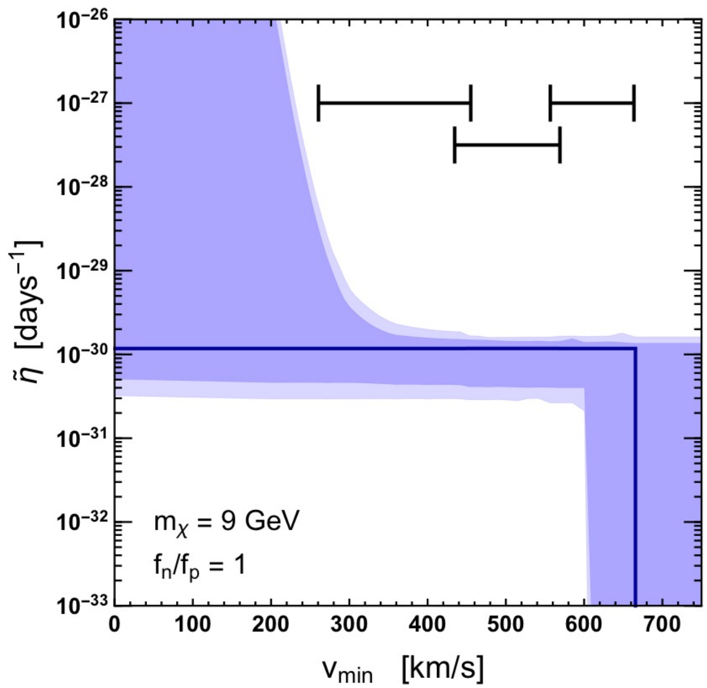

Formalism for modulation amplitudes Gelmini, Huh, Witte 1707.07019

actually for all coefficients of a harmonic expansion of the rate.

The theorems we use require the DM velocity distribution to be time independent: In the Galactic frame $f^{gal}(\vec{u})$ does NOT depend on t, the (detector and particle candidate dependent) kernel which defines the rate DOES depend on t

$$R_{[E_1',E_2']}(t) = \int d^3u\ \mathscr{H}^{gal}_{[E_1',E_2']}(\vec{u},t)\ f^{gal}(\vec{u})$$

Expand $R(t)$ and $\mathscr{H}^{gal}(\vec{u},t)$ as a harmonic series in time with coefficients R^a and $\mathscr{H}^{gal-a}(\vec{u})$, $a = 0, 1....$ For d data points on R^a **Fenchel-Eggleston gives**

$$f^{gal}(\vec{u}) = \sum_{h=1}^{d} f_h^{gal-a}\ \delta^{(3)}(\vec{u} - \vec{u}_h^a)$$

We maximize the likelihood now with at most 4d parameters.

This is a sum of streams with negligible velocity dispersion.

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actually for all coefficients of a harmonic expansion of the rate.

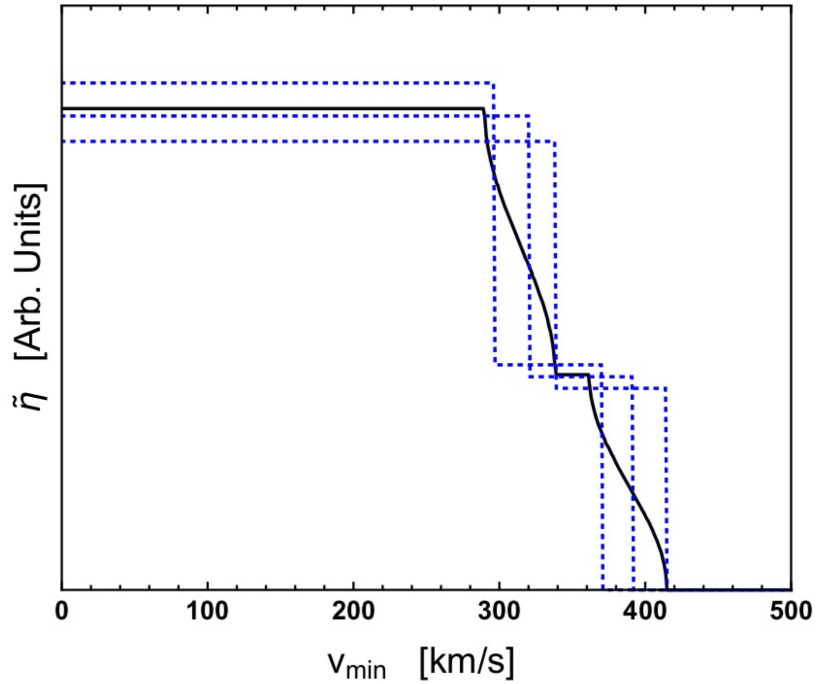
Using a Galilean transformation to Earth's frame $\vec{u} = \vec{v}_\odot + \vec{v}_\oplus(t) + \vec{v}$ we compute the **BEST FIT** $\tilde{\eta}(v_{min}, t)$,

$$\begin{aligned}\widetilde{\eta_{BF}}(v_{min}, t) &\equiv C \int_{|\vec{v}| \geq v_{min}} d^3v \, \frac{f^{gal}(\vec{v}_\odot + \vec{v}_\oplus(t) + \vec{v})}{v} \\ &= \sum_{h=1}^{\mathcal{N}} \frac{C \;\, f_h^{gal-a}}{|\vec{u}_h^a - \vec{v}_\odot - \vec{v}_\oplus(t)|} \Theta(|\vec{u}_h^a - \vec{v}_\odot - \vec{v}_\oplus(t)| - v_{min}) \,,\end{aligned}$$

which is piecewise constant only for fixed time t, and find its time-averaged $\widetilde{\eta_{BF}}$ (which is NOT piecewise constant) and construct a band around it.

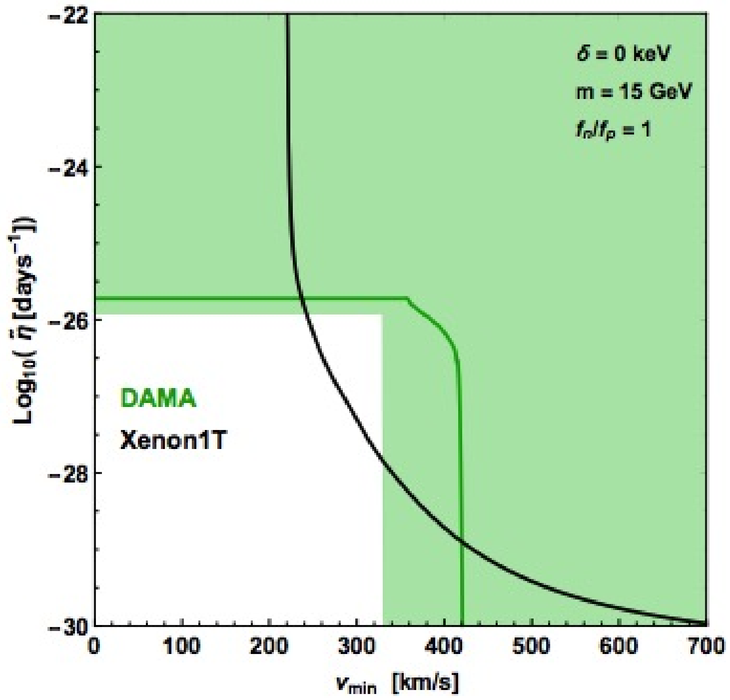
Example of $\widetilde{\eta_{BF}}(v_{min}, t)$ and its time average

at constant t it is piecewise constant, but its time average is not



Example: two-bin 2013 DAMA modulation data, 2-6 and 6-14 eVee (preliminary) $\Delta L = 2.7$ band

$$S_{ik} \quad = \quad S_{0,k} + S_{m,k} \cos\omega(t_i - t_0) + Z_{m,k} \sin\omega(t_i - t_0)$$



Outlook

The Halo Independent method to compare data of different direct DM searches is independent of the usual comparison in the m, σ plane which must be done assuming a particular halo model.

It answers two questions under the assumption of a DM particle model:

1- What can we say about the local dark halo with (non-directional) DD data ?

2- How DD data from different experiments can be compared without assuming any model for the local dark halo?

Finding that putative signals and limits yield compatible inferred halo features when assuming a particular DM particle model and not others, would constitute a clear indication that the signals come from DM and which model is right.

The method in now on a firm mathematical ground but still not mature to be widely implemented. I believe that if there are DD signal, the method will be adopted as complementary to the usual halo dependent.

Sure, what we most need are dark matter signals...

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EXTRA SLIDES

Event rate R in direct detection:

$dR = N_T \times \sigma \times$ {flux of projectiles with speed v}

N_T= number of targets

σ= interaction cross section Thus $N_T \times \sigma$= total area presented by targets to the projectiles

{Flux of projectiles with speed v} = $\{v\ dn(v)\}$= $\{[v\ (dt$ area$)\ dn(v)]/$ (area $dt)\}$ = number of projectiles with speed v reaching the detector per unit time per unit area

$$dn(v) = n f(\vec{v}, t) d^3 v$$

with the velocity distribution $f(\vec{v}, t)$ (and speed distribution $F(v,t) \equiv v^2 \int d\Omega_v f(\vec{v}, t)$) normalized to 1:

$$\int f(\vec{v}, t) d^3 v = \int F(v, t) dv 1$$

n is the total number density = number of projectiles per unit volume

In general a convex combination of a set of K vectors $\overrightarrow{r^{(j)}}$, is a linear combination

$$\vec{R} = \sum_{j=1}^{K} a_j \overrightarrow{r^{(j)}}$$

with real non-negative coefficients a_j which sum to 1, i.e. $\sum_{j=1}^{K} a_j = 1$.
The **"convex hull" of a set of vectors** $\overrightarrow{r^{(j)}}$ (called "generating vectors") is the set of all convex combinations of these vectors, i.e. here the set of all $\vec{R}$.
If the dimension of the vectors $\vec{R}$ is n, the dimension d of the hull is $d \leq n$.

Caratheodory theorem: any $\vec{R}$ in the convex hull of dim. d of a generating set of vectors belongs to the convex hull of at most $(d+1)$ of the generating vectors, ie. **any $\vec{R}$ can be written as the convex combination of at most $(d+1)$ generating vectors** (not necessarily fixed).

The Caratheodory theorem applies to the convex hull of any set of generating vectors. However, if the generating vectors are a **connected set**, the Fenchel-Eggleston theorem reduces the max. number of necessary generating vectors to be **at most d.**

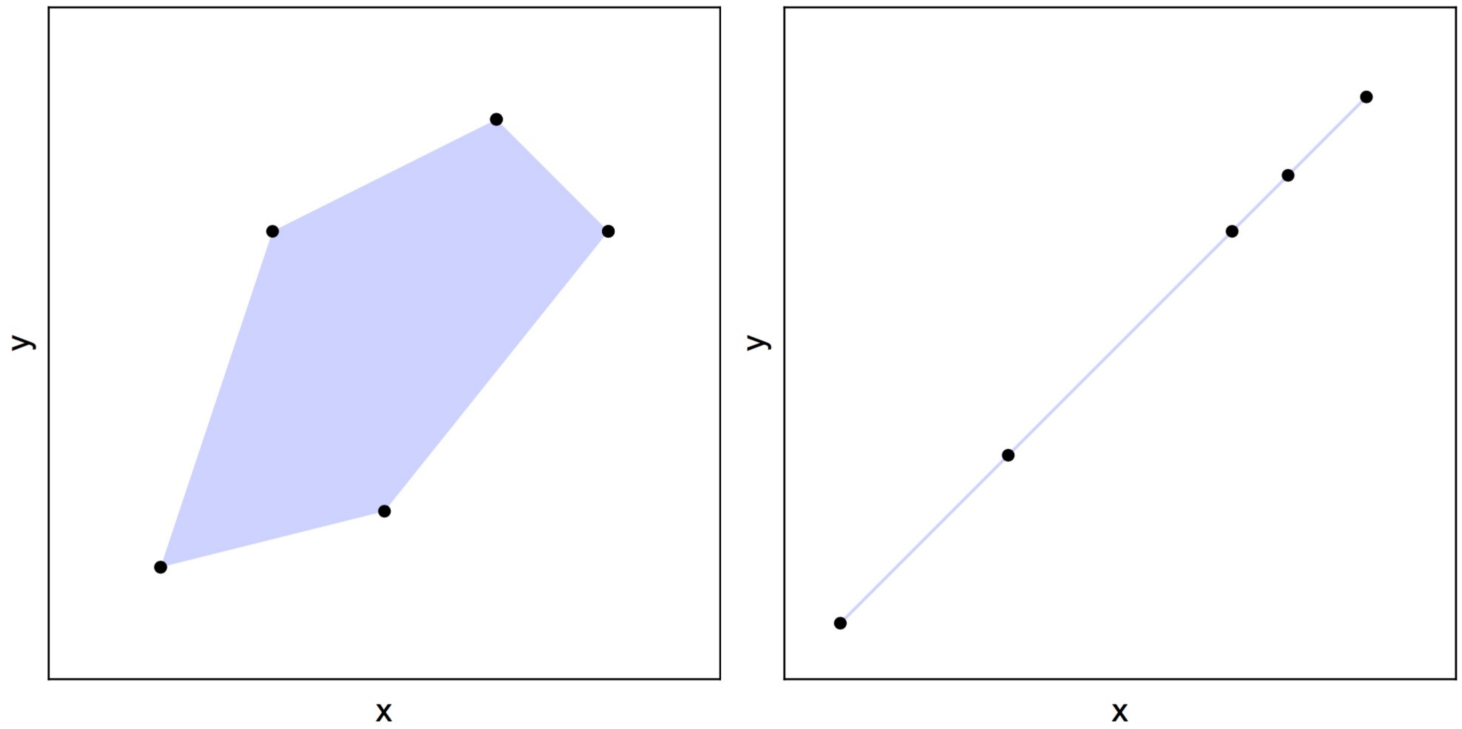

Figure 6. 6.a (left) The light blue region is the convex hull of the five points indicated with black dots. Its dimension is $d = 2$. The Caratheodory theorem says that any point in the hull can be written as a convex combination of at most $d+1 = 3$ of the generating points (which is obvious from the figure). 6.b (right) The convex hull is now the line joining the five generating points in the figure. Its dimension is $d = 1$. The Caratheodory theorem says that any one of the points in the line can be written as the convex combination of at most $d+1 = 2$ points (which is obvious from the figure).

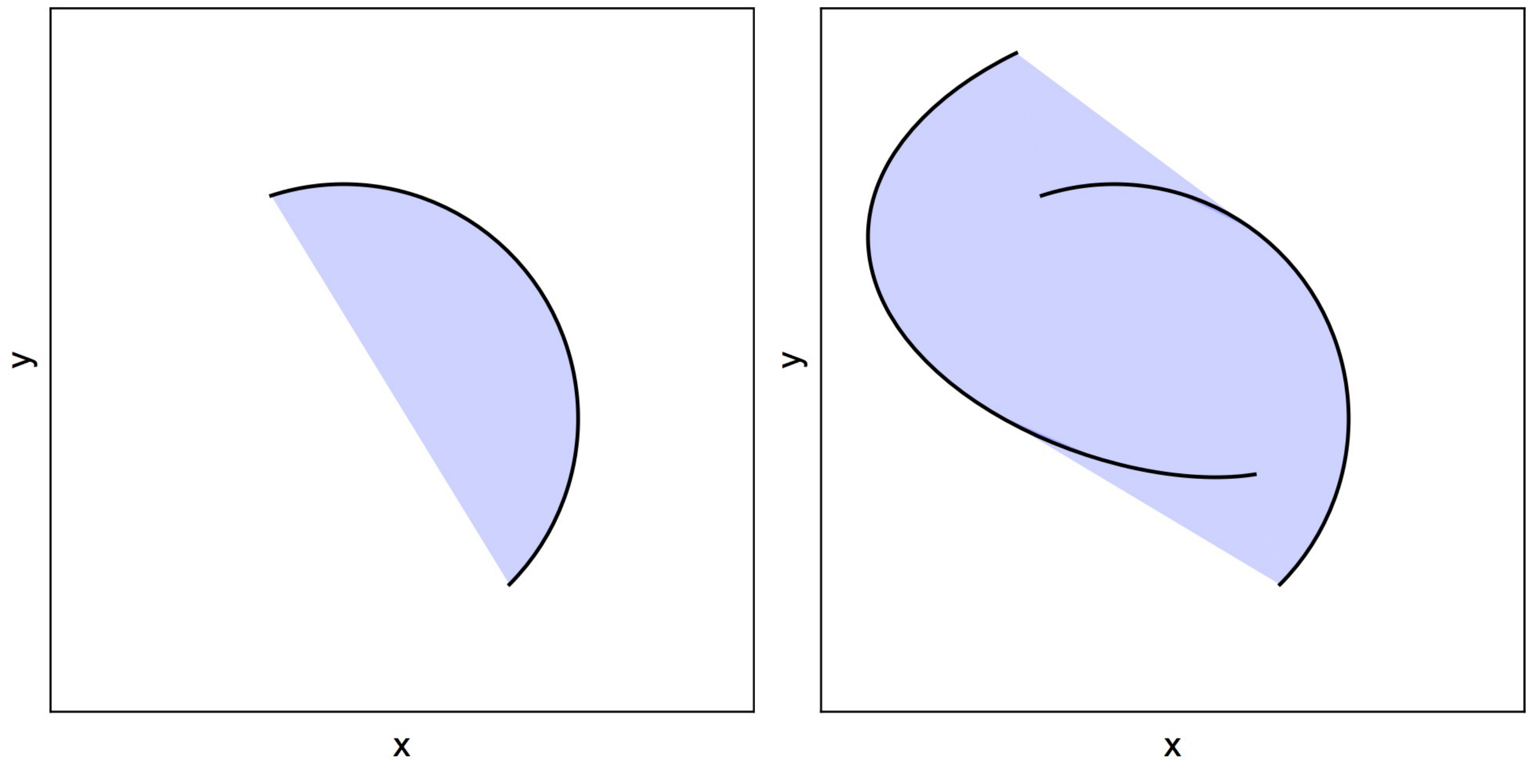

Figure 7. In 7.a (left) there is one line of generating points and in 7.b(right) there are two lines of generating points. In both cases the hulls (light blue regions) have dimension $d = 2$ and any point in each of them can be given as the convex combination of at most $d = 2$ points of the generating lines. Notice that the theorem applies to a maximum of $d = 2$ connected generating set of points (i.e. a maximum of two generating lines in this example).

If the generating vectors are a **connected set** the index j becomes a continuous real positive variable, call it v (Choquet theorem). The generating set is $\vec{R}(v)$ and the most general convex combination of the generating set, instead of $\vec{R} = \sum_{j=1}^{K} \lambda_j \overrightarrow{r^{(j)}}$ with $\sum_{j=1}^{K} \lambda_j = 1$, becomes

$$\vec{R} = \int_0^\infty dv \ \vec{R}(v)\lambda(v)$$

with $\int_0^\infty \lambda(v) dv = 1$.

The set of all $\vec{R}$ constitute the convex hull the generating set $\vec{R}(v)$.

And the Fenchel-Eggleston theorem say that any $\vec{R}$ can be given as

$$\vec{R} = \sum_{h=1}^{d} \vec{R}(v_h)\lambda_h$$

with $\sum_{h=1}^{d} \lambda_h = 1$, where $d \leq n$ and n is the dimension of the vectors $\vec{R}$.

Formalism for modulation amplitudes actually for all coefficients of a harmonic expansion of the rate.

Main idea: go to Galactic frame so the DM velocity distribution $f^{gal}(\vec{u})$ does NOT depend on t, it is the detector (thus the response function) which does. Change LAB equation

$$R_i(t) = \int d^3v\ \mathscr{H}_i(\vec{v}) f(\vec{v}, t) \quad with \quad \mathscr{H}_i(\vec{v}) = \frac{\mathscr{C}\mathscr{H}_i(\vec{v})}{v}$$

to the Galactic rest frame:

$$R_i(t) = \int d^3u\ \mathscr{H}_i^{gal}(\vec{u}, t)\ f^{gal}(\vec{u})$$

Writing $\vec{R}(t)$ and $\overrightarrow{\mathscr{H}}^{gal}(\vec{u}, t)$ as a harmonic series in time with coefficients $\overrightarrow{R}^a$ and $\overrightarrow{\mathscr{H}}^{gal-a}(\vec{u})$, $a = 0, 1...$ and all $\overrightarrow{R}^a$ are the convex hull of $\overrightarrow{\mathscr{H}}^{gal-a}(\vec{u})$ (which now have 3 parameters instead of 1).

Formalism for modulation amplitudes actually for all coefficients of a harmonic expansion of the rate. Thus Fenchel-Eggleston implies

$$\overrightarrow{R}^a = \sum_{h=1}^{d} \overrightarrow{\mathscr{H}}^{gal-a}(\overrightarrow{u_h})\ f_h^{gal-a}, \qquad \sum_{h=1}^{d} f_h^{gal-a} = 1$$

$d \leq N$, where N is the number of components of the vectors $\overrightarrow{\mathscr{H}}^{gal}(\vec{u}, t)$.
Equivalently, the Galactic velocity distribution can be written as a sum of at most d delta functions in Galactic velocity $\vec{u}$, namely streams with zero velocity dispersion,

$$f^{gal}(\vec{u}) = \sum_{h=1}^{d} f_h^{gal-a}\ \delta^{(3)}(\vec{u} - \overrightarrow{u_h^a})$$

Using a simple Galilean transformation, the best fit $\tilde{\eta}(v_{min}, t)$ at fixes t is again piecewise constant, and we could construct a confidence band at any fixed t. However, may be more useful to find the time average $\tilde{\eta}$ function (NOT piecewise constant) and construct a band around it.