Gromov–Witten theory and integrable hierarchies

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Outline

Dirac see

- $V = \mathrm{span}\{v_j \ : \ j \in \mathbb{Z}\}$
- $F^{(0)} = \mathrm{span}\ \{v_{i_0} \wedge v_{i_1} \wedge \dots\}$
- the vacuum state is $|0\rangle = v_0 \wedge v_{-1} \wedge \dots$
- every wedge monomial in $F^{(0)}$ differs from the vacuum only in finitely many places
- It helps to think of v_j as a particle of energy j and charge -1

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Boson–Fermion isomorphism

- Representation of GL_∞ on $F^{(0)}$:

$$E_{ij} \mapsto (v_i \wedge) \circ (\mathrm{contract} v_j)$$

- $F^{(0)} \cong \mathbb{C}[x_1, x_2, x_3, \ldots]$
- $[\Lambda_m, \Lambda_n] = m\delta_{m,-n}$ $(m \neq 0)$ Heisenberg algebra relations
- $\Lambda_m \mapsto \frac{\partial}{\partial x_m}$, $\Lambda_{-m} \mapsto m x_m$, $m > 0$.
- $\Lambda_m \mapsto \sum_{i \in \mathbb{Z}} : E_{i,i+m} :$ (the normal ordering $:\ :$ means that we have to apply first the operation which annihilates the vacuum $|0\rangle$.)

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The KP-hierarchy

- Decomposable vectors: $w_0 \wedge w_1 \wedge w_2 \wedge \ldots$, where $w_i = v_{-i}$ for $i >> 0$.
- Plücker imbedding of the Grassmanian

$$\mathrm{Gr} = \{W \text{ subspace of } V \,|\, W \text{ projects isomorphically to } V_-\},$$

where V_- is the subspace spanned by $v_j, j < 0$.

- $\tau_W(x_1, x_2, x_3, \ldots)$ are called tau-functions of KP.
- One of the Plücker relations is the celebrated KP equation:

$$(u_{xxx} + 12uu_x - u_{x_3})'_x + 3u_{x_2x_2} = 0,$$

where $x = x_1$ and $u = 2(\log \tau_W)_{xx}$.

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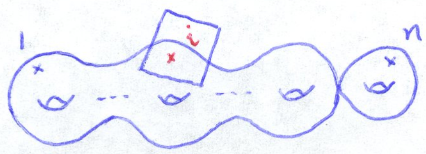
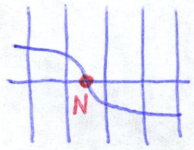
Moduli spaces of curves



$$\mathcal{L}_i \downarrow \overline{\mathcal{M}}_{g,n}$$



$$L \downarrow M \hookleftarrow N$$

$$c_1(L) \cap N$$

$$\langle \tau_{k_1}, \dots, \tau_{k_n} \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_1)^{k_1} \cdots c_1(\mathcal{L}_n)^{k_n} \in \mathbb{Q}$$

$$k_1 + \dots + k_n = 3g - 3 + n$$

W-spin structures

- $W(x_1, x_2, x_3)$ weighted-homogeneous polynomial with an isolated critical point at 0.
- Isolated singularities are classified by Dynkin diagrams. For example the singularity corresponding to D_N is: $W = x_1^{N-1} + x_1 x_2^2 + x_3^2$.
- A W-spin structure on a (nodal) Riemann surface is a choice of orbifold line bundles L_1, L_2, L_3 and isomorphisms

$$L_1^{\otimes(N-1)} \cong L_1 \otimes L_2^{\otimes 2} \cong L_3^{\otimes 2} \cong K_{\log},$$

where $K_{\log}$ is the canonical line bundle of the Riemann surface with logarithmic poles at marked and nodal points.

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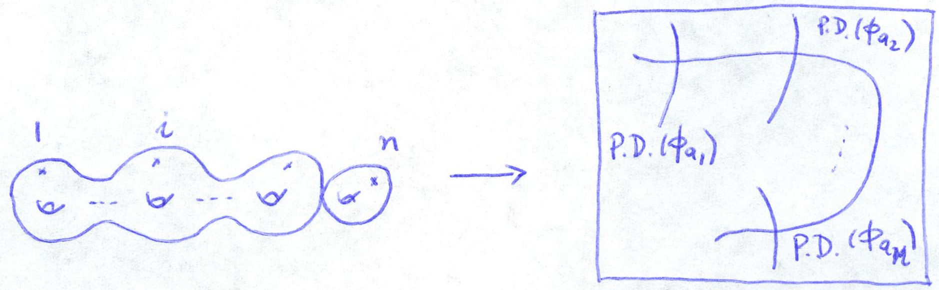
GW invariants

$\{\phi_1, \phi_2, \dots, \phi_N\}$ basis of $H^*(X; \mathbb{C})$



$$\langle \tau_{k_1,a_1}, \dots, \tau_{k_n,a_n} \rangle_{g,n} = \sum_d Q^d \, \# \left(\text{degree-}d \text{ maps as above} \right)$$

Total descendant potential

- We will be interested in formal power series

$$\mathcal{D}_X = \exp\Big(\sum \frac{\epsilon^{2g-2}}{n!}\langle \tau_{k_1,a_1},\dots,\tau_{k_n,a_n}\rangle_{g,n}\, q_{k_1}^{a_1}\dots q_{k_n}^{a_n}\Big)$$

in $q_0, q_1, \ldots$, where $q_k = (q_k^1, \ldots, q_k^N)$ are vector variables taking values in $H^*(X)$, where $N = \dim_{\mathbb{C}} X$.

- **Question 1.** Is it true that the partial derivatives of $\mathcal{D}$ satisfy quadratic equations similar to the differential equations of KP and is this system of equations an integrable hierarchy?

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Virasoro constraints

- A fundamental open question in Gromov–Witten theory is the Virasoro conjecture. It was formulated by a group of physicists: Egouchi–Hori–Xiong and S. Katz.
- On the level of generating functions: $L_n\mathcal{D} = 0$, $n \geq -1$ for some linear differential operators (in $q_0, q_1, \ldots$) which represent the vector fields $-\zeta^{n+1}\partial_\zeta$.
- On the level of correlators the Virasoro conjecture says that the correlator

$$\langle \tau_{k,1}, \tau_{k_2,a_2}, \cdots, \tau_{k_n,a_n} \rangle_{g,n}$$

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Witten's conjecture

- Witten conjectured and Kontsevich proved that $\mathcal{D}_{\mathrm{pt}}$ is a tau-function of KdV, i.e., tau-function of KP independent of the even variables
- The above fact allows us to compute all intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
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The generalized Witten's conjecture

- For any singularity Givental defined a total descendant potential – formal power series similar to $\mathcal{D}_X$.
- Fan–Jarvis–Ruan proved that in the case of singularities of type A, D, and E, the total descendant potential of the singularity is a generating function for certain intersection numbers on the moduli space of W-spin curves.

Theorem (A. Givental – T.M.)

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W-spin curves and representation theory

Theorem (B. Bakalov–T.M.)

The intersection numbers on the moduli space of W-spin curves, where W is of type A, D, or E, satisfy $\mathcal{W}$-constraints similar to the ones described in Question 2.

- Proof amounts to showing that the total descendant potential is a highest weight vector for certain vertex algebra $\mathcal{W}_\beta(\mathfrak{g})$, with $\beta = 1$.
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GW theory of the projective line

Theorem

The total descendant potential of $\mathbb{C}P^1$ (both the equivariant and the non-equivariant) is a tau-function.

- The theorem is also known as the Toda conjecture (Egouchi and Young).
- It was proved by Getzler (non-equivariant case), Okounkov–Pandharipande (equivariant case), Dubrovin–Zhang (non-equivariant case), T.M. (both equivariant and non-equivariant case).

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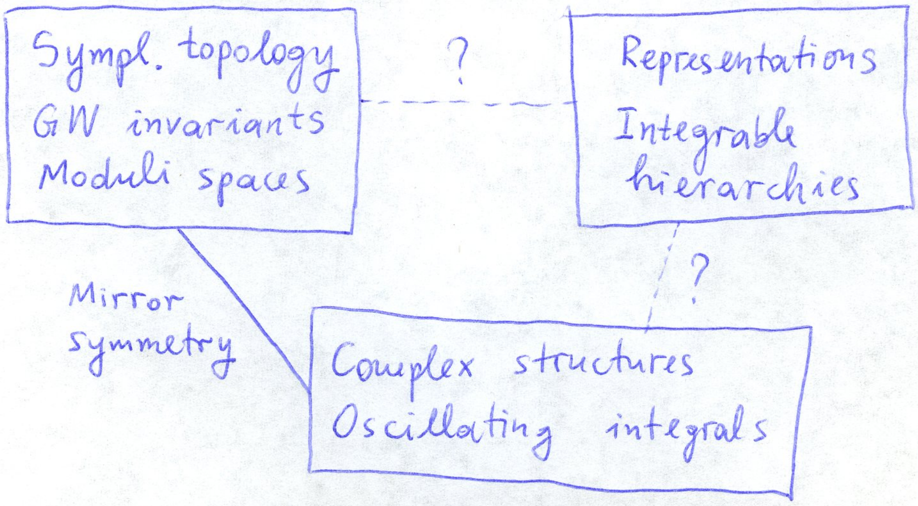
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