Gromov–Witten theory and integrable hierarchies

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Outline

1. Representation theory
   - Dirac see
   - The KP hierarchy

2. Gromov–Witten theory
   - Moduli spaces of curves
   - W-spin structures
   - Gromov–Witten Invariants

3. Summary of results
\[ V = \text{span}\left\{ v_j : j \in \mathbb{Z} \right\} \]

\[ F^{(0)} = \text{span} \left\{ v_{i_0} \wedge v_{i_1} \wedge \ldots \right\} \]

the vacuum state is \[ |0\rangle = v_0 \wedge v_{-1} \wedge \ldots \]

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Boson–Fermion isomorphism

- Representation of $\text{GL}_\infty$ on $F^{(0)}$:

\[ E_{ij} \mapsto (v_i \wedge \circ \text{contract } v_j) \]

- $F^{(0)} \cong \mathbb{C}[x_1, x_2, x_3, \ldots]$

- $[\Lambda_m, \Lambda_n] = m \delta_{m,-n}$ ($m \neq 0$) Heisenberg algebra relations

- $\Lambda_m \mapsto \frac{\partial}{\partial x_m}, \Lambda_{-m} \mapsto mx_m, \ m > 0.$

- $\Lambda_m \mapsto \sum_{i \in \mathbb{Z}} E_{i,i+m}$: (the normal ordering $: :$ means that we have to apply first the operation which annihilates the vacuum $|0\rangle$.)
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GW theory and integrable hierarchies
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GW theory and integrable hierarchies
The KP-hierarchy

- Decomposable vectors: $w_0 \wedge w_1 \wedge w_2 \wedge \ldots$, where $w_i = v_{-i}$ for $i \gg 0$.

- Plücker imbedding of the Grassmanian

  $\text{Gr} = \{ W \text{ subspace of } V \mid W \text{ projects isomorphically to } V_- \}$,

  where $V_-$ is the subspace spanned by $v_j, j < 0$.

- $\tau_W(x_1, x_2, x_3, \ldots)$ are called tau-functions of KP.

- One of the Plücker relations is the celebrated KP equation:

  \[
  (u_{xxx} + 12uu_x - u_x^3)_x + 3u_{x_2x_2} = 0,
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  where $x = x_1$ and $u = 2(\log \tau_W)_{xx}$. 
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GW theory and integrable hierarchies
\[ \langle \tau_{k_1}, \ldots, \tau_{k_n} \rangle_{g,n} = \sum_{\mathcal{M}_{g,n}} c_1(z_1)^{k_1} \cdots c_1(z_n)^{k_n} \in \mathbb{Q} \]

\[ k_1 + \ldots + k_n = 3g - 3 + n \]
**W-spin structures**

- $W(x_1, x_2, x_3)$ weighted-homogeneous polynomial with an isolated critical point at 0.

- Isolated singularities are classified by Dynkin diagrams. For example, the singularity corresponding to $D_N$ is: $W = x_1^{N-1} + x_1 x_2^2 + x_3^2$.

- A $W$-spin structure on a (nodal) Riemann surface is a choice of orbifold line bundles $L_1, L_2, L_3$ and isomorphisms

$$L_1^\otimes(N-1) \cong L_1 \otimes L_2^\otimes 2 \cong L_3^\otimes 2 \cong K_{\log},$$

where $K_{\log}$ is the canonical line bundle of the Riemann surface with logarithmic poles at marked and nodal points.
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\[ \{ \phi_1, \phi_2, \ldots, \phi_N \} \] basis of \( H^* (X; \mathbb{C}) \)

\[ \langle \tau_{k_1, a_1}, \ldots, \tau_{k_n, a_n} \rangle_{g,n} = \sum_d q^d \# \text{(degree-}d \text{ maps)} \]
We will be interested in formal power series

\[
\mathcal{D}_X = \exp \left( \sum \frac{\epsilon^{2g-2}}{n!} \langle \tau_{k_1}, a_1, \ldots, \tau_{k_n}, a_n \rangle g, n \ q_{k_1}^{a_1} \cdots q_{k_n}^{a_n} \right)
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in \( q_0, q_1, \ldots \), where \( q_k = (q_k^1, \ldots, q_k^N) \) are vector variables taking values in \( H^*(X) \), where \( N = \dim_{\mathbb{C}} X \).

**Question 1.** Is it true that the partial derivatives of \( \mathcal{D} \) satisfy quadratic equations similar to the differential equations of KP and is this system of equations an integrable hierarchy?
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**Question 1.** Is it true that the partial derivatives of \( D \) satisfy quadratic equations similar to the differential equations of KP and is this system of equations an integrable hierarchy?
A fundamental open question in Gromov–Witten theory is the Virasoro conjecture. It was formulated by a group of physicists: Egouchi–Hori–Xiong and S. Katz.

On the level of generating functions: $L_n D = 0$, $n \geq -1$ for some linear differential operators (in $q_0, q_1, \ldots$) which represent the vector fields $-\zeta^{n+1} \partial_\zeta$.

On the level of correlators the Virasoro conjecture says that the correlator

$$\langle \tau_{k,1}, \tau_{k,2}, a_2, \ldots, \tau_{k,n}, a_n \rangle_{g,n}$$

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Is it true that the correlator

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On the level of generating functions a positive answer to the above question would mean that there is an algebra of differential operators \( \mathcal{W} \) that contains Virasoro, such that \( D \) is a highest weigh vector.

**Question 2.** Does \( \mathcal{W} \) exist?
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Witten’s conjecture

- Witten conjectured and Kontsevich proved that $\mathcal{D}_{pt}$ is a tau-function of KdV, i.e., tau-function of KP independent of the even variables.

- The above fact allows us to compute all intersection numbers on $\overline{\mathcal{M}}_{g,n}$.

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The generalized Witten’s conjecture

- For any singularity Givental defined a total descendant potential – formal power series similar to $D_X$.
- Fan–Jarvis–Ruan proved that in the case of singularities of type A, D, and E, the total descendant potential of the singularity is a generating function for certain intersection numbers on the moduli space of $W$-spin curves.

**Theorem (A. Givental – T.M.)**

*The total descendant potential of a singularity of type A, D, or E is a tau-function for the Kac–Wakimoto hierarchies.*
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Theorem (B. Bakalov–T.M.)

The intersection numbers on the moduli space of $W$-spin curves, where $W$ is of type $A$, $D$, or $E$, satisfy $\mathcal{W}$-constraints similar to the ones described in Question 2.

Proof amounts to showing that the total descendant potential is a highest weight vector for certain vertex algebra $\mathcal{W}_\beta(g)$, with $\beta = 1$.

The $W$-spin intersection numbers are governed by a certain representation of the corresponding affine Lie algebra.
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Theorem

The total descendant potential of $\mathbb{C}P^1$ (both the equivariant and the non-equivariant) is a tau-function.

- The theorem is also known as the Toda conjecture (Eguchi and Young).
- It was proved by Getzler (non-equivariant case), Okounkov–Pandharipande (equivariant case), Dubrovin–Zhang (non-equivariant case), T.M. (both equivariant and non-equivariant case).

Theorem (T.M.–H.-H. Tseng)

The total descendant potential of $\mathbb{C}P^1_{k,m}$ (both the equivariant and the non-equivariant) is a tau-function.
GW theory of the projective line

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Summary

- Sympl. topology
- GW invariants
- Moduli spaces
- Mirror symmetry
- Complex structures
- Oscillating integrals
- Representations
- Integrable hierarchies

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