A physicist-friendly reformulation of the Atiyah-Patodi-Singer index

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OSAKA UNIVERSITY
Live Locally, Grow Globally

F, Onogi, Yamaguchi

PRD96(2017) no.12, 125004 [arXiv:1710.03379]

F, Furuta (U. Tokyo), Matsuo (Nagoya U.),

Onogi, Yamaguchi, Yamashita (U.Tokyo->RIMS, Kyoto U.),

arXiv:1910.01987

F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi,

arXiv:1910.09675

Supported by Kobo-Kenkyu budget 18H04484 in

JSPS Grant-in-Aid for Scientific Research on Innovative Areas "Discrete Geometric Analysis for Materials Design" Grant Number 17H06460

My topic today

In 2017, we proposed “A physicist-friendly reformulation of the Atiyah-Patodi-Singer index”

F, Onogi, Yamaguchi PRD96(2017) no.12, 125004 [arXiv:1710.03379]

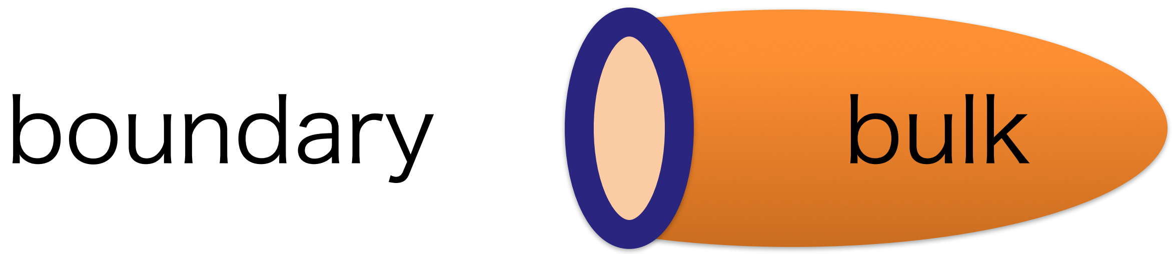
Recently, we invited 3 mathematicians and succeeded in a mathematical proof.

F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita, arXiv:1910.01987

(Lattice version

F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, 1910.09675)

Atiyah-Patodi-Singer (APS) index theorem [1975]



curvature

$$Ind(D_{\rm APS}) = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\,\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

* Here we (mainly) consider 4-dimensional flat Euclidean space with boundary at $x_4=0$.

Topological insulator

gapped material in the bulk but conductor on boundary (edge).

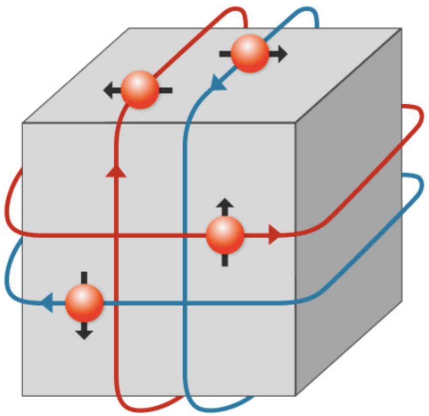
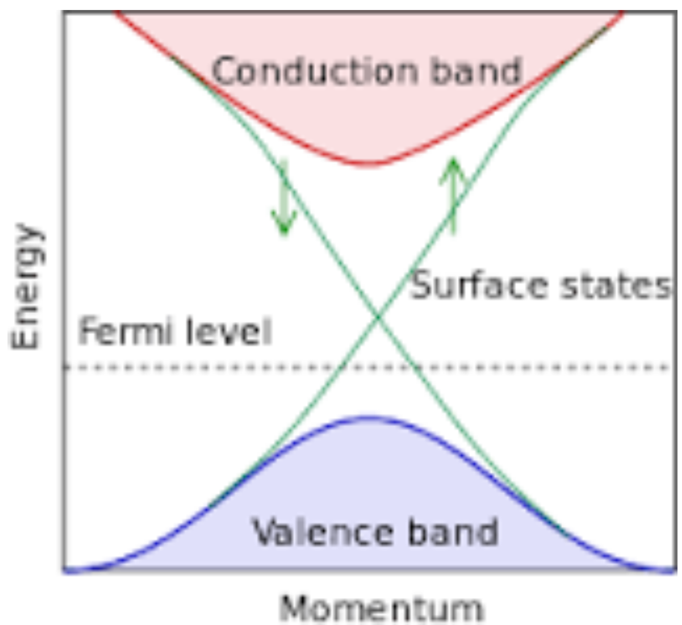


Figure from Wikipedia



2005 predicted by Kane et al.

2007 discovered [Koenig et al.].

APS index in topological insulator

Witten 2015 : APS index is a key to understand bulk-edge correspondence in symmetry protected topological insulator:

fermion path integrals

$$Z_{\rm edge} \propto \exp(-i\pi\eta(iD^{\rm 3D})/2)$$

T-anomalous

$$Z_{\rm bulk} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}]\right)$$

T-anomalous

$$Z_{\rm edge}Z_{\rm bulk} \propto (-1)^{\mathfrak{I}} = (-1)^{-\mathfrak{I}}$$

→ T is protected !

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$

[Related works: Metlitski 15, Seiberg-Witten 16, Tachikawa-Yonekura 16&18, Freed-Hopkins 16, Witten 16, Yonekura 16&19, Witten-Yonekura 19…]

What is good with APS index?

Bulk-edge correspondence of T anomaly with APS index is given in position space.

We do not need momentum space, which is often difficult to define when the translational invariance is violated by interaction.

What puzzled us

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   [except for Alvarez-Gaume et al. 1985 (but boundary condition is obscure.)]

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   [except for Alvarez-Gaume et al. 1985 (but boundary condition is obscure.)]

→ We launched a study group reading original APS paper and it took 3 months to translate it into "physics language". Moreover, we found another fermionic quantity, which coincides with the APS index.

A physicist-friendly reformulation using domain-wall fermion

[F, Onogi, Yamaguchi 2017]

$$\Im = \frac{1}{2}\eta(H_{DW})$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

$$H_{DW} = \gamma_5(D_{\mathrm{4D}} + M\epsilon(x_4))$$

⬇ perturbative computation

$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

coincides with APS index, keeping the features of topological insulator.

1. massive Dirac in bulk (massless mode at edge)
2. local boundary cond.
3. SO(2) rotational sym. on boundary is kept.

Mathematician's response

In last August, I gave a talk in a workshop

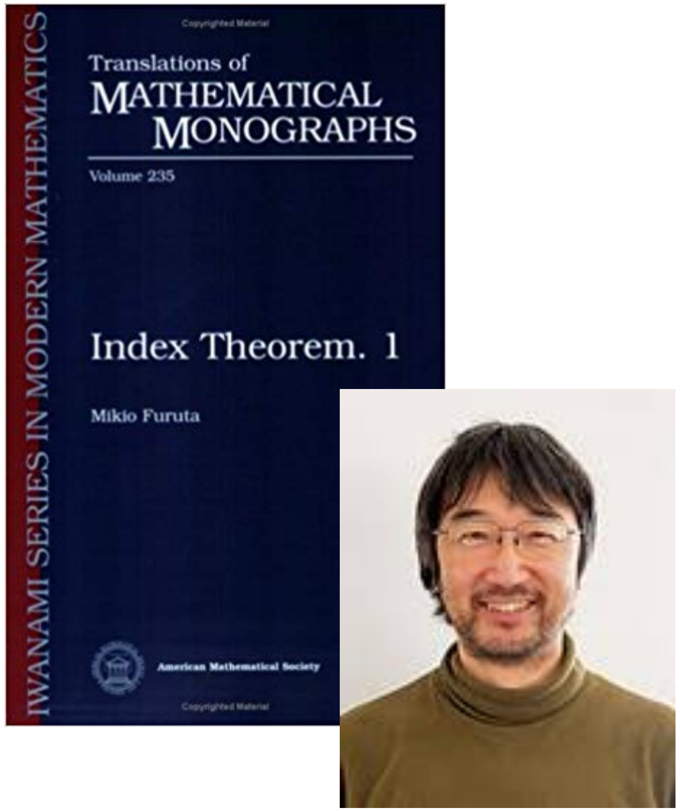
organized by Mikio Furuta (U. Tokyo).

He said "Interesting!"

Moreover, only 1 week later,

he proposed a sketch of proof of

$$\frac{1}{2}\eta(H_{DW}^{reg}) = Ind(D_{\rm APS})$$



[F, Furuta, Matsuo, Onogi,

Yamaguchi, and Yamashita, arXiv:1910.01987]

Overview

$$\mathfrak{J} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

|| [APS 1975]

||

CONJECTURE from perturbation in 4D flat space

[F, Onogi, Yamaguchi 2017]

$Ind(D_{\mathrm{APS}})$

with physicist-unfriendly boundary condition

=

$\frac{1}{2}\eta(H_{DW})$

with physicist-friendly set-up (topological insulator)

Lattice version

[F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]

THEOREM

on any even-dim. curved manifold

[F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, 2019]

Contents

Atiyah-Singer index theorem [1968] on a manifold without boundary

A theorem on the number of solutions of Dirac equation $D\psi = 0$

electric field magnetic field

$\boldsymbol{E} \cdot \boldsymbol{B}$

$$\overbrace{n_+ - n_-}^{\mathrm{Ind}(D)} = \frac{1}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}(F_{\mu\nu} F_{\rho\sigma})$$

#sol with + chirality

#sol with - chirality

we consider U(1) or SU(N) gauge field (connection).

Dirac equation = EOM of electrons

Schrodinger equation (non-relativisitic)

$$\left[i\frac{\partial}{\partial t} + \frac{1}{2m}\frac{\partial^2}{\partial x_i^2}\right]\psi = 0.$$

Klein-Gordon equation (consistent only for bosons)

$$\left[-\partial_t^2 + \partial_i^2 + m^2\right]\psi = 0.$$

$$\left[-i\gamma_\mu\partial^\mu + m\right]\left[i\gamma_\mu\partial^\mu + m\right]\psi = 0.$$

Dirac equation

$$\left[i\gamma_\mu\partial^\mu + m\right]\psi = 0.$$

$i\gamma_\mu\partial^\mu$ = Dirac operator

Gamma matrices and chirality

$$D = \gamma^{\mu}(\partial_{\mu} + iA_{\mu})$$

gamma matrices   space-time derivatives   EM field (connection)

4x4 gamma matrices in Euclidean 4D space

$$\gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}$$

σ_i Pauli matrices

Chirality operator

$$\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Algebra

$$\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu} \qquad \{\gamma_5, \gamma_{\nu}\} = 0$$

$$\mathrm{Tr}\gamma_5\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma} = 4i\epsilon_{\mu\nu\rho\sigma} \qquad \mathrm{Tr}\gamma_5(\text{up to 3 } \gamma\text{'s}) = 0$$

Chirality = spin in moving direction

Left-handed fermion has $\gamma_5 = -1$

Right-handed has $\gamma_5 = 1$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_1^{os} \\ |\uparrow\rangle \\ |\downarrow\rangle \\ \psi_4^{os} \end{pmatrix}$$

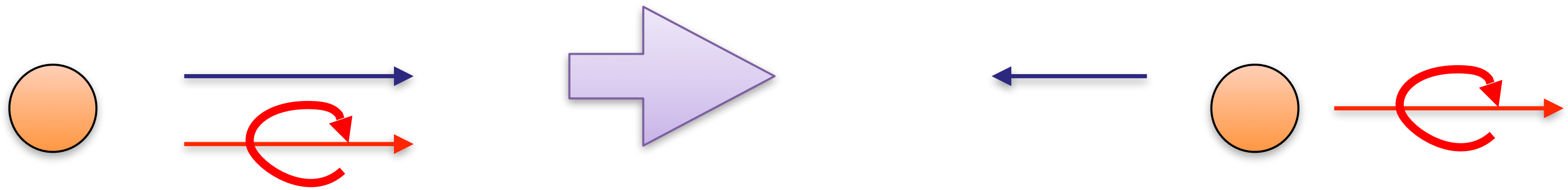
* os=off-shell modes, non-classical, not satisfying

$$E = mc^2\sqrt{1 + \boldsymbol{p}^2/m^2c^2}$$

but this is true only for massless fermion.

Chirality = spin in moving direction

For massive fermion, we can flip the chirality by Lorentz transformation,



but for massless fermion (with speed of light) we cannot.

Naively, for the index theorem, fermion needs to be massless.

Atiyah-Singer index

$$D = \gamma^\mu(\partial_\mu + iA_\mu) \qquad \{D, \gamma_5\} = 0.$$

$$\gamma_5\phi(x) = +\phi(x) \to \gamma_5 D\phi(x) = -D\gamma_5\phi(x) = -D\phi(x)$$

Eigenmodes make ± chirality pairs

except for zero-modes.

$$n_+ - n_- = \mathrm{Tr}\gamma_5^{\mathrm{reg}}.$$

#sol with + chirality #sol with - chirality

Physicist-friendly description
(Fujikawa method 1979)

1. Heat-kernel regularization

$$\mathrm{Tr}\gamma_5^{\mathrm{reg}} = \lim_{M\to\infty} \mathrm{Tr}\gamma_5 e^{\frac{D^2}{M^2}}$$

2. plane-wave complete set

$$= \lim_{M\to\infty} \int d^4x \int d^4k \; e^{-ikx} \mathrm{tr}\gamma_5 e^{D_{\mathrm{4D}}^2/M^2} e^{ikx}$$

3. perturbative expansion $\left(D^2 = D_\mu D^\mu + \frac{g}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\right)$

$$= \frac{g^2}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

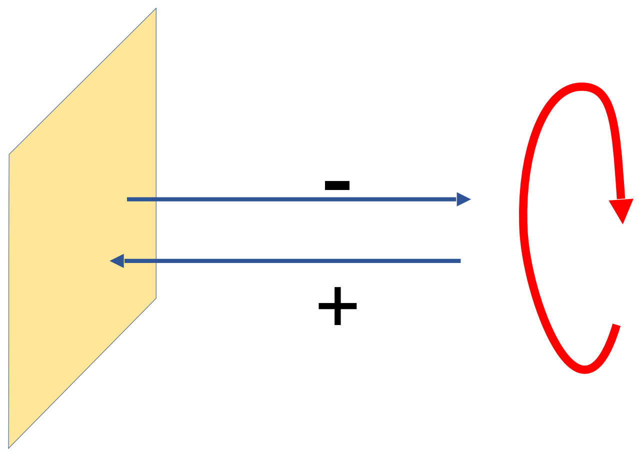
Why "physicist-friendly"?

Fujikawa method is

1. easy to compute (pert. computation),
2. intuitively understandable (appearing from Jacobian in the path-integral),
3. [empirically] correct (in spite of hand-waving approximations and expansions),
4. experimentally confirmed (pion decay).

Difficulty with boundary

If we impose **local** and **Lorentz (rotation)** invariant boundary condition, + and – chirality sectors do not decouple any more.



angular momentum is conserved

n_+, n_- and the index do not make sense.

Atiyah-Patodi-Singer boundary condition [Atiyah, Patodi, Singer 75]

Gives up the locality and rotational symmetry but keeps the chirality.

Eg. 4 dim $x^4 \geq 0$ $A_4 = 0$ gauge

$$D = \gamma^4 \partial_4 + \gamma^i D_i = \gamma^4(\partial_4 + \underbrace{\gamma^4 \gamma^i D_i}_{A})$$

They impose a non-local b.c.

$$(A + |A|)\psi|_{x^4=0} = 0$$

$$\Rightarrow \quad \text{index} = n_+ - n_-$$

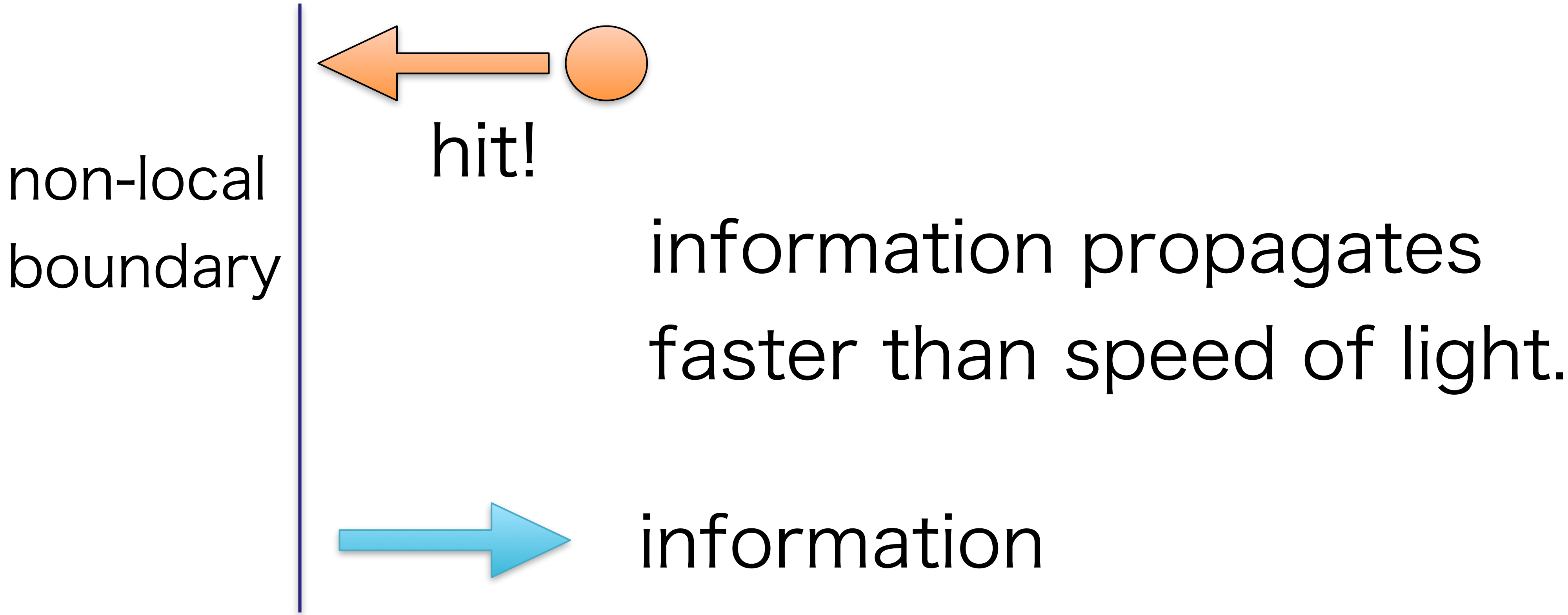
Beautiful!

But physicist-unfriendly.

Locality >> chirality for physicists

Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.



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→ need to give up chirality and consider L/R mixing

(massive case)

$$\sout{n_+ - n_-} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\,\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

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Can we still make a fermionic integer (even if it is ugly)?

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Can we still make a fermionic integer (even if it is ugly)?

Our answer is "Yes, we can".

Different explanation why APS appears [Witten Yonekura 2019]

They rotate the x4 to the “time” direction and introduced the APS boundary condition as intermediate “states”. The unphysical property of APS is canceled between the bra/ket states.

(In our work, we try to remove it.)

Short summary and our goal

Summary of introduction part:

Atiyah-Singer index is physicist-friendly but APS is not.

Massless fermion with boundary is not natural.

Question:

Can we understand the index theorems with massive Dirac operator (without unnatural boundary condition)?

In the rest of my talk, we will show the answer is “YES”.

Contents

Massive Dirac operator

$$H = \gamma_5 (D + M)$$

$$D = \gamma^\mu (\partial_\mu + i A_\mu)$$

$$M \propto identity$$

Zero-modes of D = still eigenstates of H:

$$H\phi_0 = \gamma_5 M \phi_0 = \pm M \phi_0.$$

Non-zero modes make ± pairs

$$H\phi_i = \lambda_i \phi_i$$

$$HD\phi_i = -DH\phi_i = -\lambda_i D\phi_i$$

Eta invariant of massive Dirac operator

$$\eta(H) = \sum_i \mathrm{sgn}\lambda_i$$

$$H = \gamma_5(D + M)$$

$$= \# \text{ of } +M - \# \text{ of } -M$$

coincides with the original AS index?

Eta invariant of massive Dirac operator

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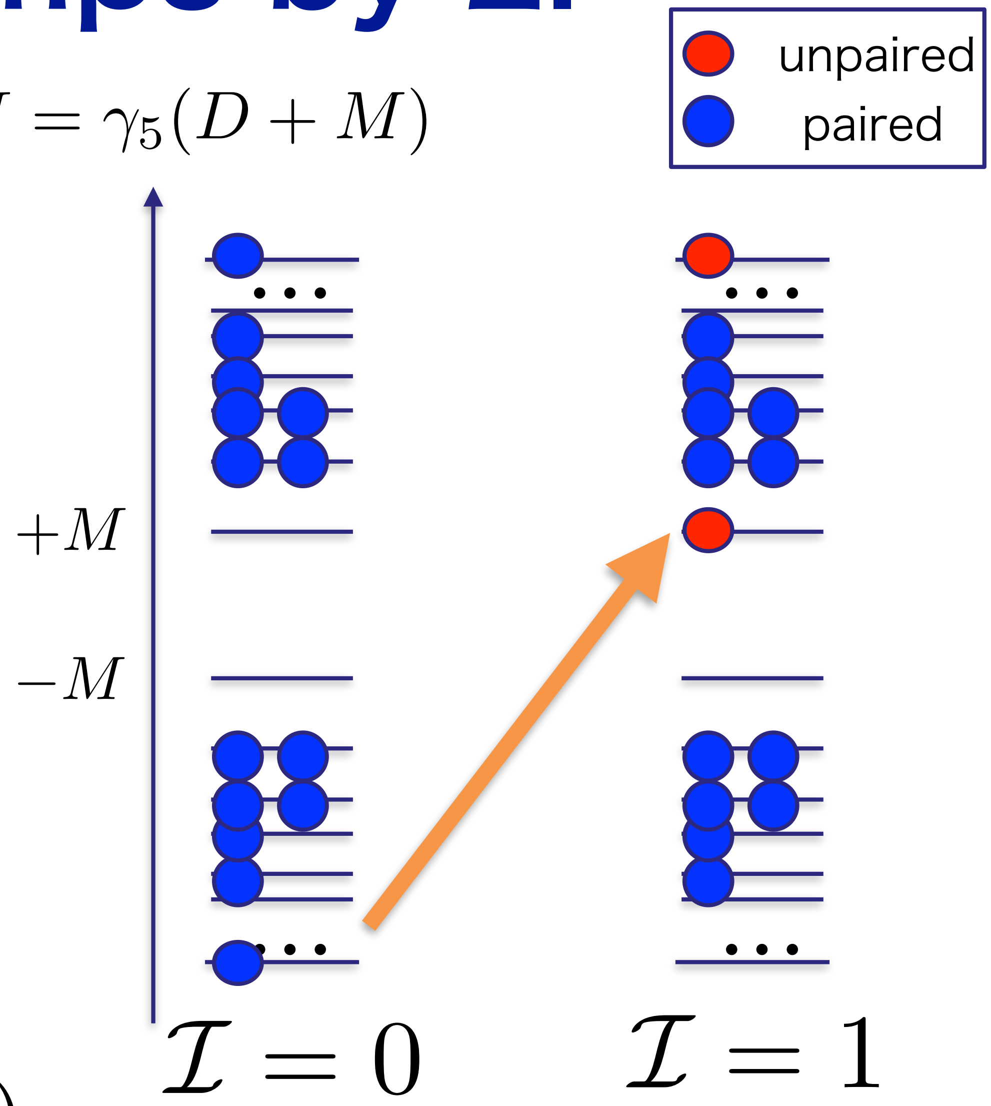
coincides with the original AS index?

In fact, we need a factor 1/2.

$$\mathrm{Index}(D) = \frac{1}{2}\eta(H)^{reg}.$$

$\eta(H)$ always jumps by 2.

$H = \gamma_5(D+M)$



To increase + modes, we have to borrow one from - (UV) modes.

Good regularizations (e.g. Pauli-Villars, lattice) respect this fact.

⇨

$$\mathrm{Index}(D) = \frac{1}{2}\eta(H).$$

Perturbative "proof" (in physics sense)

using Pauli-Villars subtraction

$$\frac{1}{2}\eta(H)^{reg} = \frac{1}{2}\left[\eta(H) - \eta(H_{PV})\right].$$

$$H = \gamma_5(D+M)$$

$$H_{PV} = \gamma_5(D-\Lambda), \quad \Lambda \gg M$$

$$\eta(H) = \lim_{s\to 0} \mathrm{Tr}\frac{H}{(\sqrt{H^2})^{1+s}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt t^{-1/2}\mathrm{Tr}He^{-tH^2}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty dt' t'^{-1/2}\mathrm{Tr}\gamma_5\left(M + \frac{D}{M}\right)e^{-t'D^\dagger D/M^2}e^{-t'},$$

$(t' = M^2 t)$

Fujikawa-method

does not contribute.

$$= \frac{1}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} + \mathrm{UV}.$$

$$-\eta(H_{PV}) = \frac{1}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \mathrm{UV}.$$

*mathematical proof is also shown in our paper.

Contents

More physical set-up?

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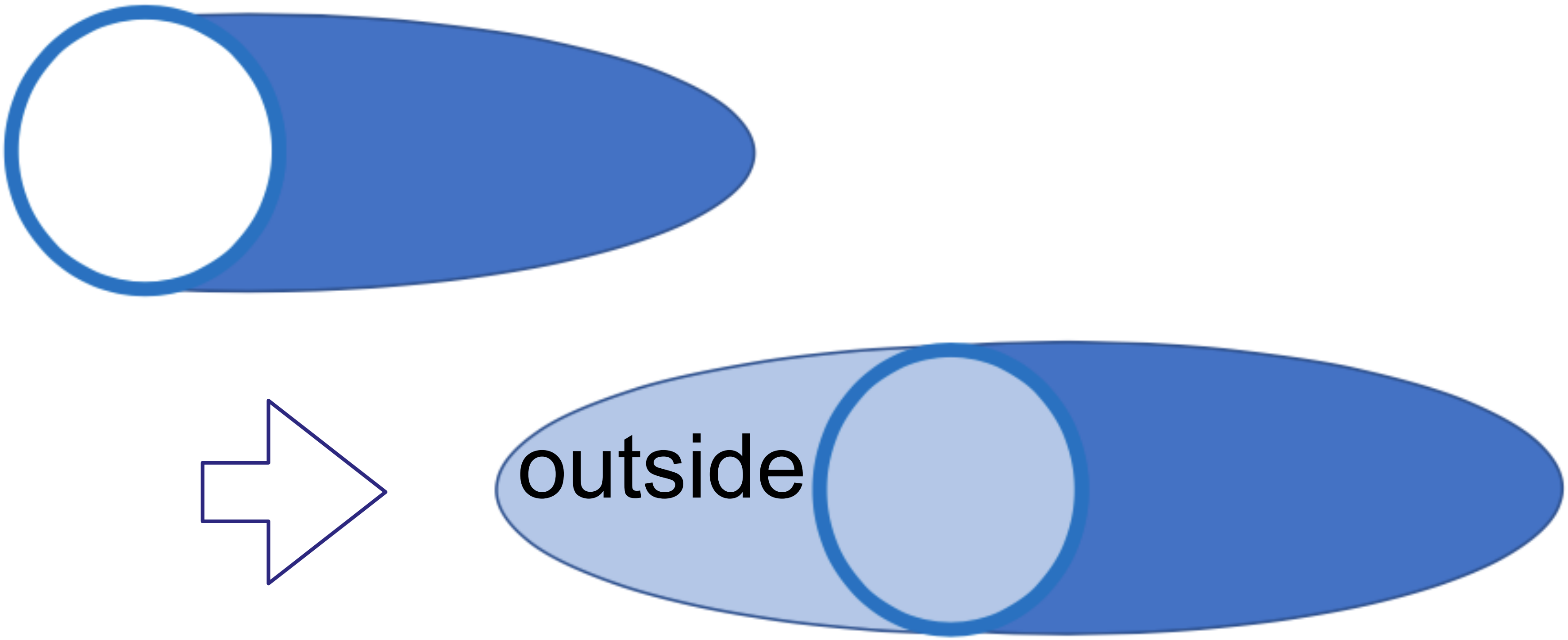
In physics,

1. Any boundary has “outside”: ~~manifold + boundary~~ → domain-wall.



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In physics,

1. Any boundary has “outside”: manifold + boundary → domain-wall.
2. Boundary should not preserve helicity but keep angular-mom: massless → massive (in bulk)
3. Boundary condition should not be put by hand → but automatically chosen.
4. Edge-localized modes play the key role.

Domain-wall Dirac operator

Let us consider

$$D_{\rm 4D} + M\epsilon(x_4), \quad \epsilon(x_4) = {\rm sgn} x_4$$

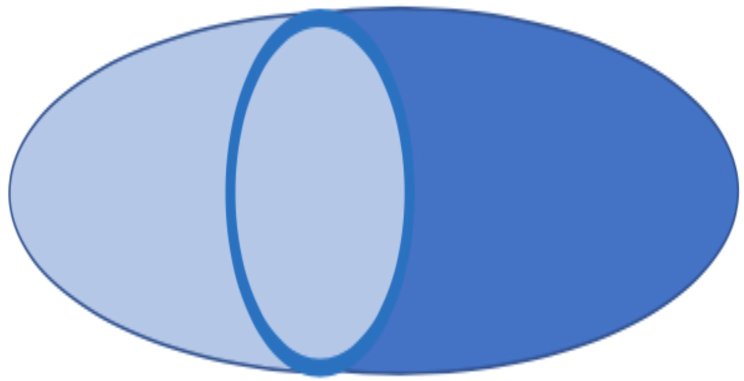
[Jackiw-Rebbi 1976, Callan-Harvery 1985, Kaplan 1992]

on a closed manifold

with sign flipping mass,

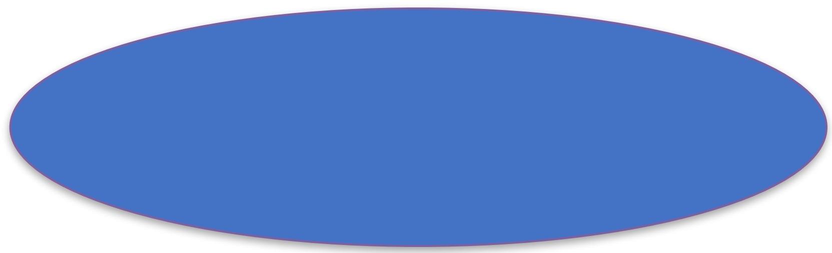
without assuming any boundary condition

(we expect it dynamically given.).



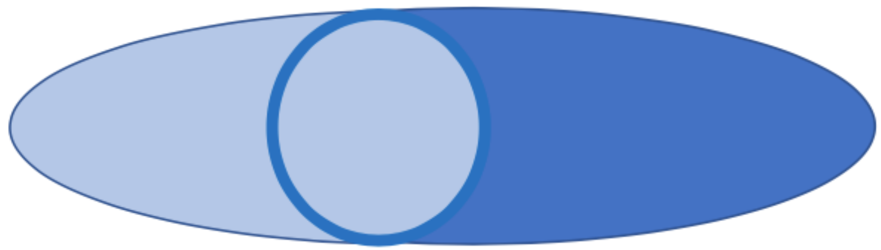
“new” APS index [F-Onogi-Yamaguchi 2017]

$$\frac{1}{2}\eta(\gamma_5(D+M))^{reg}$$ = AS index



$$\Downarrow$$

$$\frac{1}{2}\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}$$



$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

which can be shown by Fujikawa-method.

Fujikawa method:

$$\frac{1}{2}\eta(H_{DW}) = \frac{1}{2}\mathrm{Tr}\frac{\gamma_5(D+M\varepsilon(x_4))}{\sqrt{\{\gamma_5(D+M\varepsilon(x_4))\}^2}}$$

1. choose regularization

2. choose complete set to evaluate trace

3. perturbation

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2. choose complete set to evaluate trace

   eigen set of $\{\gamma_5(D^{\mathrm{free}}+M\varepsilon(x_4))\}^2$

3. perturbation

Complete set in the free case

Solutions to

$$\{\gamma_5(D^{\rm free} + M\varepsilon(x_4))\}^2\phi = \left[-\partial_\mu^2 + M^2 - 2M\gamma_4\delta(x_4)\right]\phi = \lambda^2\phi$$

are $\varphi(x_4)\otimes e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ where

$$\varphi^{\omega}_{\pm,o}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right),$$

$$\varphi^{\omega}_{\pm,e}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega \mp M)e^{i\omega|x_4|} + (i\omega \pm M)e^{-i\omega|x_4|}\right),$$

$$\varphi^{\rm edge}_{+,e}(x_4) = \sqrt{M}e^{-M|x_4|},$$

➡ Edge mode appears !

Here, $\omega = \sqrt{\boldsymbol{p}^2 + M^2 - \lambda_{4D}^2}$ and $\gamma_4\varphi^{\omega,\rm edge}_{\pm,e/o} = \pm\varphi^{\omega,\rm edge}_{\pm,e/o}$

“Automatic” boundary condition

We didn’t put any boundary condition by hand.
But

$$\left[\frac{\partial}{\partial x_4} \pm M\epsilon(x_4)\right] \varphi^{\omega,\mathrm{edge}}_{\pm,e}(x_4)\bigg|_{x_4=0} = 0, \quad \varphi^{\omega}_{\pm,o}(x_4=0) = 0.$$

is automatically satisfied due to the domain-wall. This condition is LOCAL and PRESERVES angular-momentum in x_4 direction but DOES NOT keep chirality.

Fujikawa-method

$$\eta(H_{DW}) = \frac{1}{\Gamma(\frac{1+s}{2})}\int_0^\infty dt' t'^{\frac{s-1}{2}} \mathrm{Tr}\gamma_5 \left(\epsilon(x_4) + \frac{D}{M}\right) e^{-t' H_{DW}^2/M^2} e^{-t'},$$

Perturbative expansion

We insert our complete set $\{\varphi^{\omega,\mathrm{edge}}_{\pm,e/o}(x_4) \times e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\}$

(See our paper for the details.)

100% edge-mode effect

$$= \frac{1}{32\pi^2}\int d^4x\, \epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \eta(iD^{3\mathrm{D}})$$

$$\epsilon(x_4) = \mathrm{sgn} x_4 \qquad (CS \text{ mod integer})$$

Total index

$$
\begin{aligned}
\mathfrak{I} &= \frac{\eta(H_{DW}))}{2} - \frac{\eta(H_{PV})}{2} \\
&= \frac{1}{2}\left[\frac{1}{32\pi^2}\int d^4x\ \epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \eta(iD^{\mathrm{3D}})\right. \\
&\left. + \frac{1}{32\pi^2}\int d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma}\right] \\
&= \frac{1}{32\pi^2}\int_{x_4>0} d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \frac{1}{2}\eta(iD^{\mathrm{3D}})
\end{aligned}
$$

What is “physicist-friendly”?

Fujikawa method for the new index is

1. easy to compute (1-loop computation),
2. intuitively understandable

   (bulk -> FFtilder, edge -> eta-invariant),
3. [empirically] correct (in spite of hand-

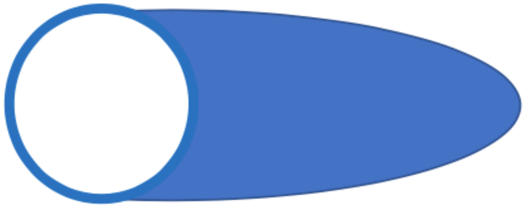
   waving approximations and expansions),
4. experimentally confirmed (topological insulator).

Contents

Just a coincidence?

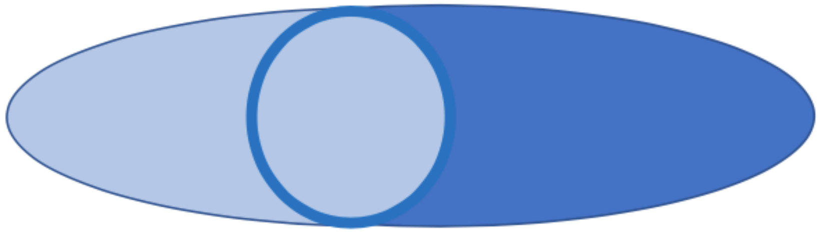
$$Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW}^{reg})$$

on general even-dimensional manifolds ?



APS

1. massless Dirac
   (even in bulk)
2. non-local boundary cond.
   (depending on gauge fields)
3. SO(2) rotational sym. on boundary is lost.
4. no edge mode appears.
5. manifold + boundary



Domain-wall fermion

1. massive Dirac in bulk (massless mode at edge)
2. local boundary cond.
3. SO(2) rotational sym. on boundary is kept.
4. Edge mode describes eta-invariant.
5. closed manifold + domain-wall

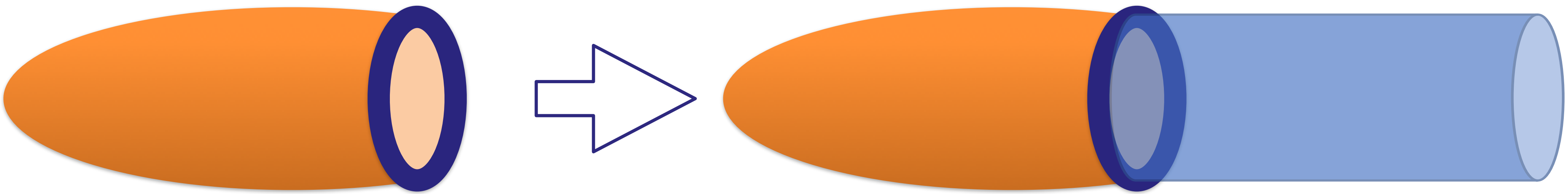
Theorem

(F-Furuta-Matsuo-Onogi-Yamaguchi-Yamashita 2019)

For any APS index of a massless Dirac operator on a even-dim. manifold X+ with boundary, there exists a massive (domain-wall) Dirac operator on a closed manifold, sharing its half with X+, and its eta invariant is equal to the original index.

Theorem 1:
APS index = index with infinite cylinder

In original APS paper, they showed



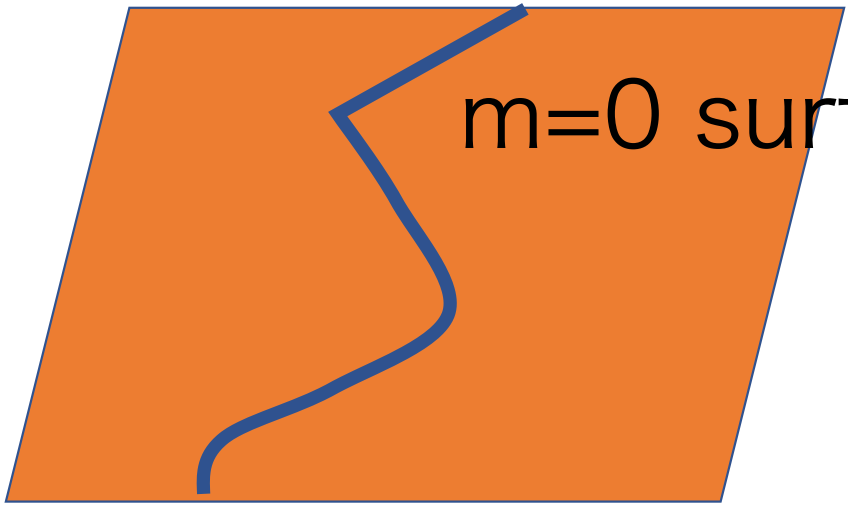
Index w/ APS b.c. = Index with infinite cylinder attached to the original boundary (w.r.t. square integrable modes).

* On cylinder, gauge fields are constant in the extra-direction.

Theorem 2:
Localization (& product formula)

By giving position-dependent "mass", we can localize the zero modes to "massless" lower-dimensional surface and the index is given by the product:

$$Ind(\gamma_s(D^d + \partial_s + i\gamma_s M(s))) =$$
$$Ind(D^d) \times Ind(\gamma_s\partial_s + M(s))$$



= generalization of domain-wall fermion

Theorem 3:
In odd-dim, APS index = boundary eta-invariant

$$\int F \wedge F \wedge \cdots$$

exists only in even-dim.

$$Ind(D_{\rm APS}^{odd-dim}) = \frac{1}{2}\left[\eta(D^{\rm boundary1}) - \eta(D^{\rm boundary2})\right]$$

5-dimensional Dirac operator

we consider

$$D^{5\mathrm{D}} = \begin{pmatrix} 0 & \partial_5 + \gamma_5 (D^{4\mathrm{D}} + m(x_4, x_5)) \\ -\partial_5 + \gamma_5 (D^{4\mathrm{D}} + m(x_4, x_5)) & 0 \end{pmatrix}$$

where

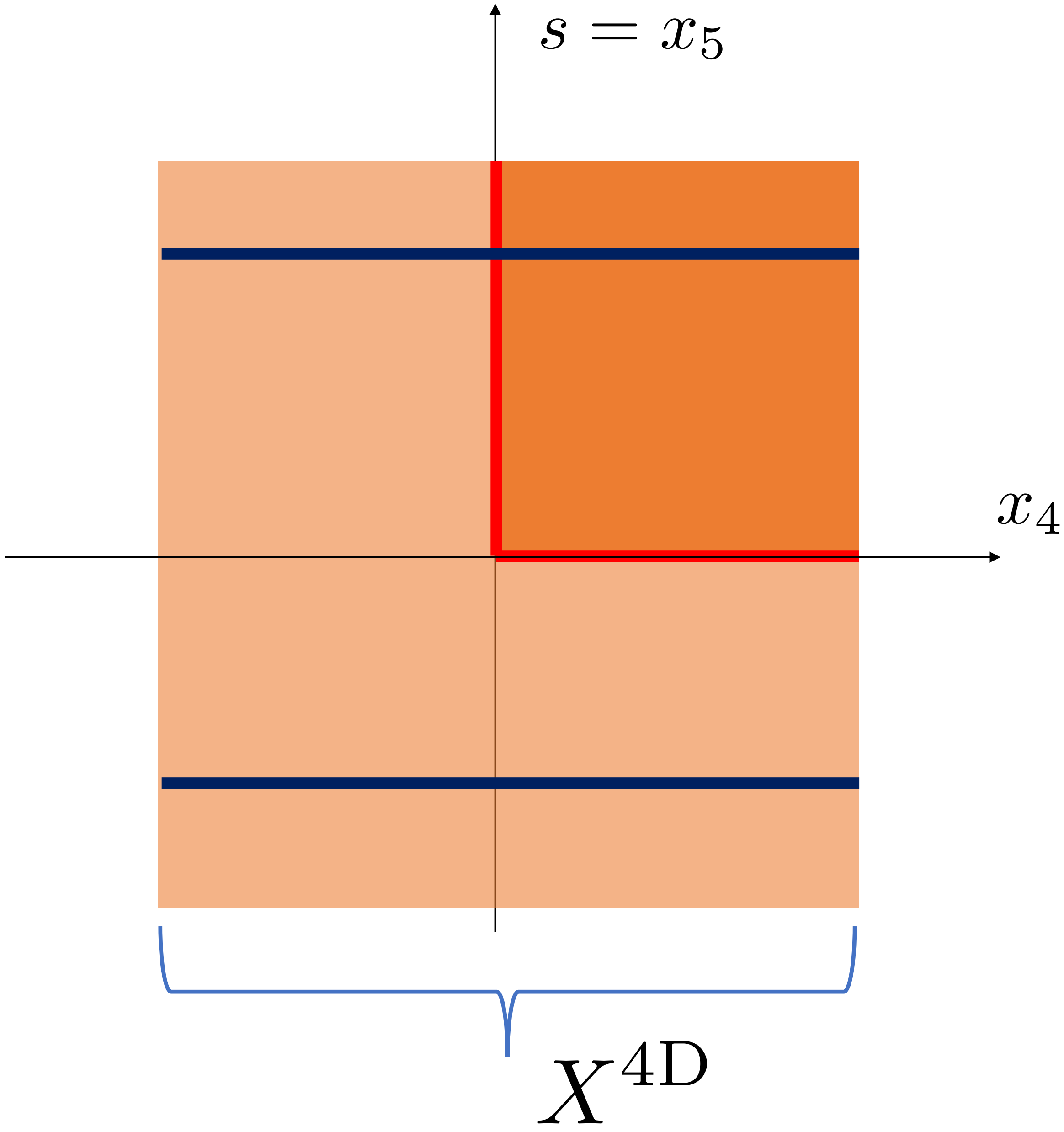
$$m(x_4, x_5) = \begin{cases} M & \text{for } x_4 > 0 \ \& \ x_5 > 0 \\ 0 & \text{for } x_4 = 0 \ \& \ x_5 = 0 \\ -M_2 & \text{otherwise} \end{cases}$$

and A_μ is

independent of x_5 .

* Application is straightforward to
any 2n+1 dimensions.

On X^{4D} x R,



we compute

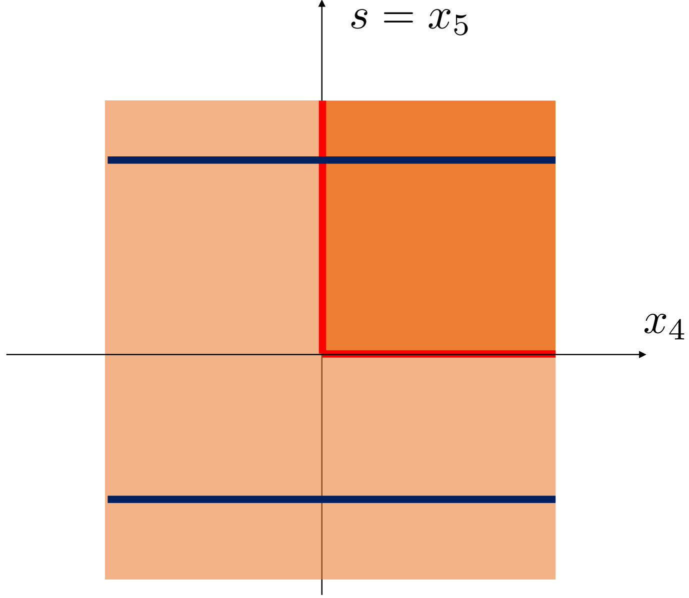
$$Ind(D^{5\mathrm{D}})$$

in two different ways:

1. localization

2. eta-inv. at $x_5 = \pm 1.$

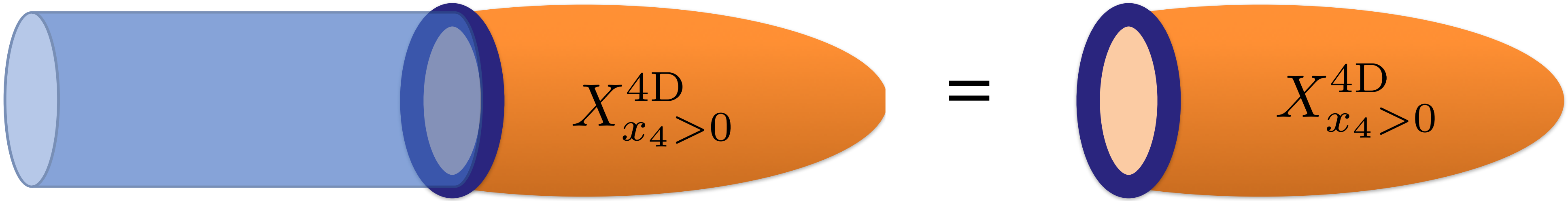
Localization



Theorem 2 tells us

$$Ind(D^{5D})|_{M,M_2\to\infty} = Ind(D^{4D}_{m=0\mathrm{surface}}) \times \underbrace{IndD^{1D}_{normal}}_{=1}$$

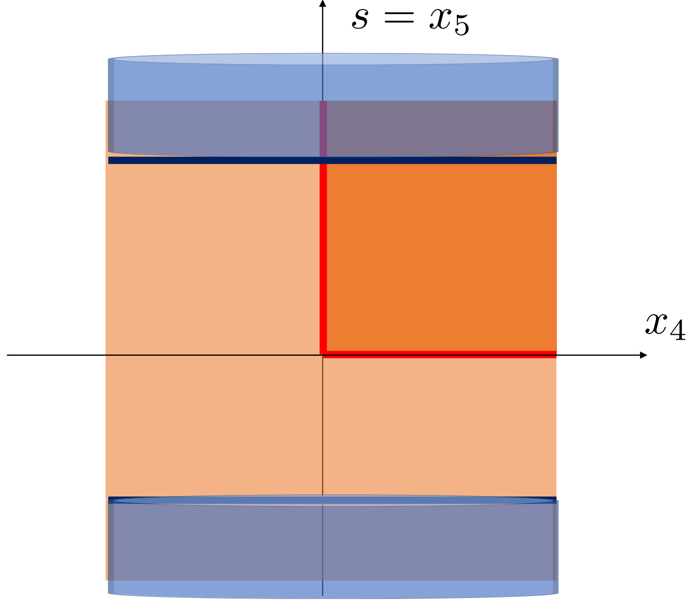
and on the massless surface



theorem 1 indicates

$$Ind(D^{4D}_{m=0\mathrm{surface}}) = Ind(D^{X^{4D}_{x_4>0}}_{\mathrm{APS}})$$

Boundary eta invariants



Theorem 1 tells us

$$Ind(D^{5\mathrm{D}}) = Ind(D^{5\mathrm{D}}_{\mathrm{APS\ b.c.}ats=\pm 1})$$

and from theorem 3, we obtain

$$Ind(D^{5\mathrm{D}}_{\mathrm{APS\ b.c.}ats=\pm 1}) = \frac{1}{2}\left[\eta(D^{4\mathrm{D}}_{s=1}) - \eta(D^{4\mathrm{D}}_{s=-1})\right]$$

$$= \frac{1}{2}\left[\eta(\gamma_5(D^{4\mathrm{D}} + M\epsilon(x_4)) - \eta(\gamma_5(D^{4\mathrm{D}} - M_2)\right] = \frac{1}{2}\eta^{PVreg.}(\gamma_5(D^{4\mathrm{D}} + M\epsilon(x_4))$$

therefore,

$$Ind(D^{5D}) = Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW})$$

Q.E.D.

Contents

Chiral symmetry on the lattice

Nielsen-Ninomiya theorem [1981]:

if $\gamma_5 D + D\gamma_5 = 0,$ we have unphysical modes.

Ginsparg-Wilson relation [1982]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D.$$

a :lattice spacing

indicated a solution to avoid NN theorem.

Overlap Dirac operator [Neuberger 1998]

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right) \quad H_W = \gamma_5(D_W - M). \quad M = 1/a.$$

satisfies the GW relation.

Chiral symmetry on the lattice

Overlap fermion action $S = \sum_x \bar{q}(x) D_{ov} q(x)$
is invariant under

$$q \to e^{i\alpha\gamma_5(1-aD_{ov})}q, \quad \bar{q} \to \bar{q}e^{i\alpha\gamma_5}.$$

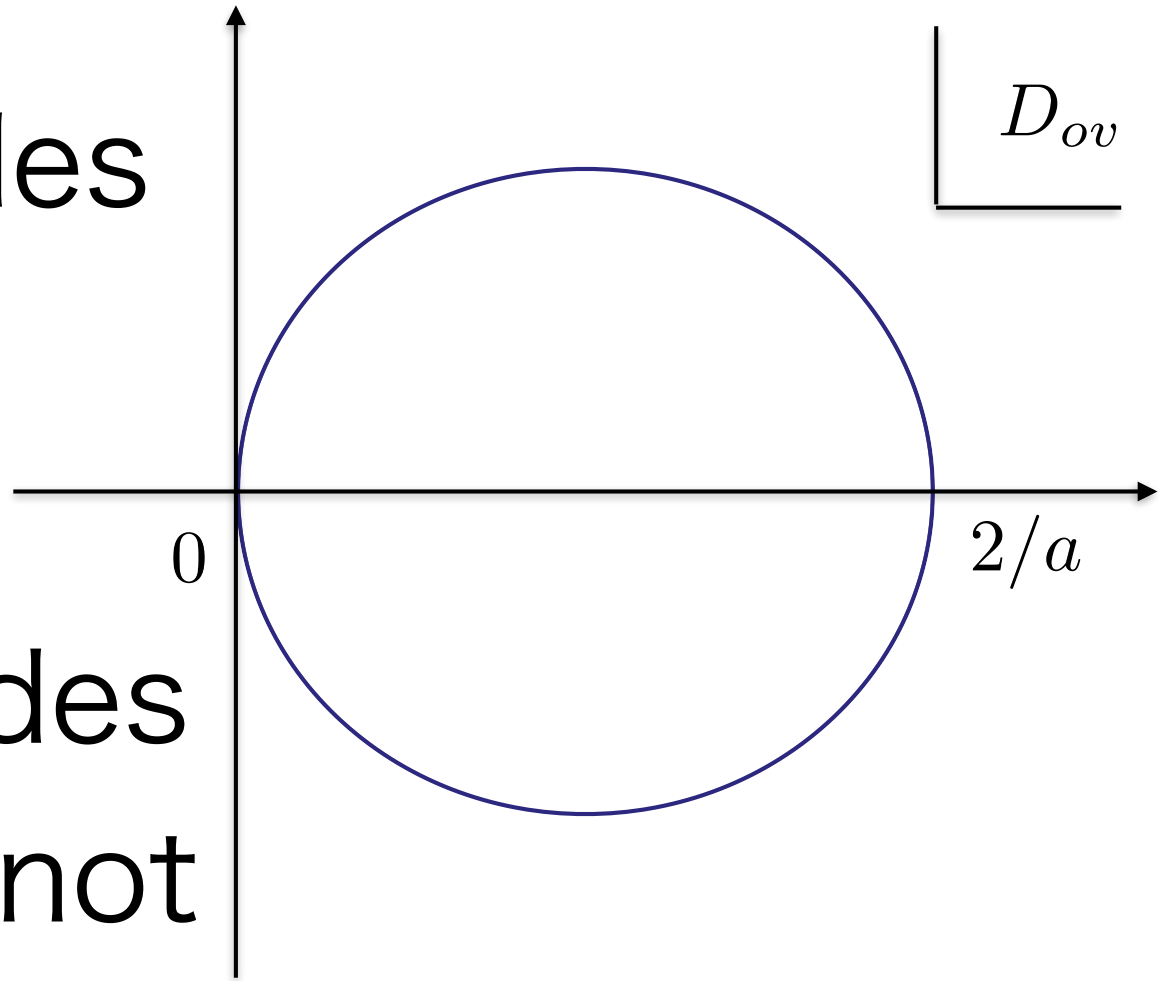
but fermion measure transforms
as $Dq\bar{q} \to \exp\left[2i\alpha \mathrm{Tr}(\gamma_5 + \gamma_5(1 - aD_{ov}))/2\right] Dq\bar{q}$

which reproduces U(1)A anomaly.
Moreover, $\mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$ is AS index !

Overlap Dirac operator spectrum

complex modes make ± pairs of $\gamma_5\left(1-\frac{aD_{ov}}{2}\right)$.

Real $2/a$ modes (doubles) do not contribute.



$$\mathrm{Tr}\gamma_5\left(1-\frac{aD_{ov}}{2}\right)=\mathrm{Tr}_{\mathrm{zeros}}\gamma_5 .$$

On the lattice, AS is O.K. but APS is not.

Atiyah-Singer index can be formulated by overlap Dirac operator, but APS is not known.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right)$$

1. Lattice version of APS condition impossible, as it does not have a form $N + B$

2. Any boundary condition breaks chiral sym.

But the lattice AS index theorem "knew" our work !

$$Ind(D_{ov}) = \frac{1}{2}\mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) \qquad D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right)$$

$$= -\frac{1}{2}\mathrm{Tr}\frac{H_W}{\sqrt{H_W^2}} = -\frac{1}{2}\eta(\gamma_5(D_W - M))!$$

The lattice index theorem "knew"

1. index can be given with massive Dirac.
2. chiral symmetry is not important.

   Wilson Dirac operator is enough.

Unification of index theorems

index theorem with massless Dirac

	continuum	lattice
AS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$	$\mathrm{Tr}\gamma^5(1-aD_{ov}/2)$
APS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$ w/ APS b.c.	not known.

index theorem with massive Dirac

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))?$

YES !

[F, Kawai, Matsuki, Mori, Onogi, Yamaguchi, arXiv:1910.09675.]

APS index on the lattice

F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, arXiv:1910.09675

On 4-dimensional Euclidean lattice with periodic boundaries (T^4), we have shown

$$-\frac{1}{2}\eta(\gamma_5(D_W - M\epsilon(x_4 - a/2)))$$

$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2} + O(a).$$

Note that LHS is always an integer.

See our paper for the details or please invite N. Kawai to your seminar.

Contents

✔ 1. Introduction

✔ 2. Physicist's view of index theorems

AS index is physicist-friendly but APS is not.

✔ 3. Massive Dirac operator index without boundary

$\mathfrak{I} = \eta(\gamma_5(D+M))^{reg}/2$ coincides with the AS index.

✔ 4. New index with boundary [F, Onogi, Yamaguchi 2017]

$\mathfrak{I} = \eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ coincides with the APS index.

✔ 5. Mathematical proof [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, 2019]

$Ind(D_{\text{APS}})$ and $\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ are different expressions of the same 5D Dirac index.

✔ 6. APS index on a lattice [F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]

can be defined by $-\eta(\gamma_5(D_W - M\epsilon(x_4+a/2)))/2$

7. Summary

Summary

Q. Can we understand the index theorems with massive Dirac operator?

Our Answer = YES (even on a lattice) !

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))$

Higher-order topological insulator?

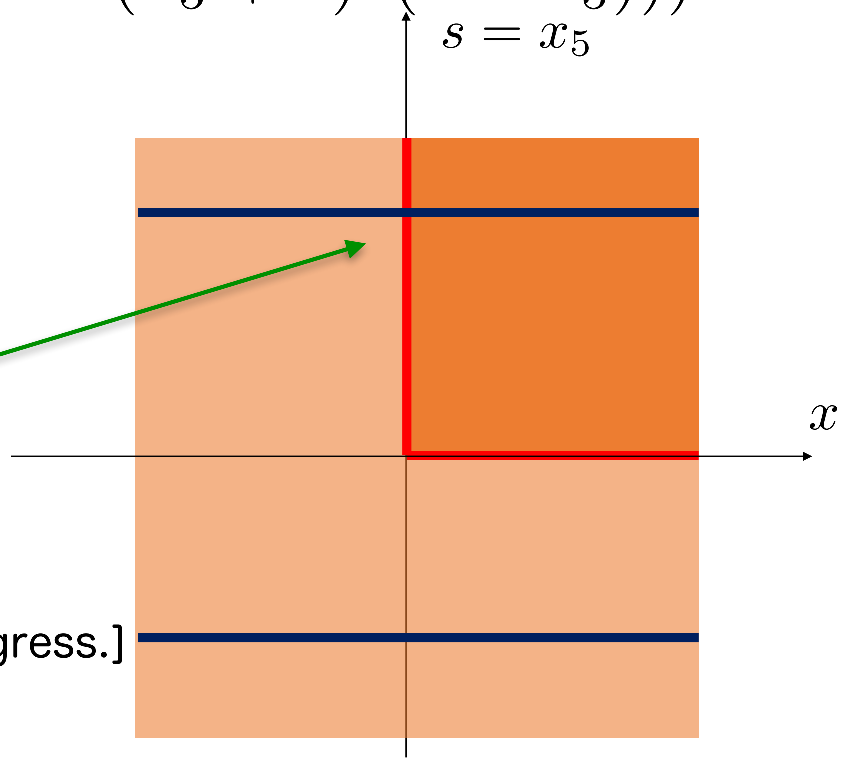
We have proved

$$Ind_{\mathrm{APS}}(D^{5D}_{[-1,1]\times X}) = {\color{red}Ind(D_{\mathrm{APS}})} = {\color{blue}\frac{1}{2}\eta(H_{DW})}$$

What about

$$Ind_{\mathrm{APS}}(D^{5D}_{[-1,1]\times X}) = \frac{1}{2}\eta(\gamma_7(D^{5D} - M\varepsilon(x_5+1)\varepsilon(1-x_5)))?$$

If this is correct, the edge-of-edge states appear at the junction of the two domain-walls.



[F, Furuta,Matsuo, Onogi, Yamaguchi,Yamashita, in progress.]

Backup slides

Eta invariant = Chern Simons term + integer (non-local effect)

$$\frac{\eta(iD^{3\mathrm{D}})}{2} = \frac{CS}{2\pi} + \text{integer}$$

$$CS \equiv \frac{1}{4\pi}\int_Y d^3x\, \mathrm{tr}_c \left[\epsilon_{\nu\rho\sigma}\left(A^\nu \partial^\rho A^\sigma + \frac{2i}{3}A^\nu A^\rho A^\sigma\right)\right],$$

= surface term.

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$