Unifying Lattice Models, Links and Quantum Geometric Langlands via Branes in String Theory

Meng-Chwan Tan

National University of Singapore

Kavli IPMU, 17th December 2019

Scope of Presentation

- Introduction/Motivation
- Summary of results
- 4d Chern-Simons theory from partial twist of D4-NS5 system
- Relationship with 3d Chern-Simons theory and the Geometric Langlands Program
- Conclusion and Future Work

Introduction/Motivation

- 4d Chern-Simons theory has the action*

$$S = \frac{1}{\hbar}\int_{Y\times\Sigma} C\wedge \mathrm{Tr}\Big(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big), \qquad (1.1)$$

  where $\mathcal{A}$ is a complex-valued gauge field, Y is a framed 2-manifold, and Σ is $\mathbb{C}$, $\mathbb{C}^\times$ or $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ endowed with a meromorphic one-form $C = C(z)dz$ (with no zeros).

- Within the realm of **perturbation theory**, this gauge theory is well-defined, and realizes the **Yang-Baxter equation** with spectral parameter.

∗. K. Costello, *Supersymmetric gauge theory and the Yangian*, arXiv:1303.2632
K. Costello, E. Witten, M. Yamazaki, *Gauge Theory and Integrability, I, II*, arXiv:1709.09993, 1802.01579

- Outside of perturbation theory, 4d CS is not well-understood - path integral is **exponentially divergent**.
- Suggestion† - **Nonperturbative definition** comes from the **D4-NS5 system** of string theory, similar to how the D3-NS5 system realizes the nonperturbative 3d analytically-continued Chern-Simons theory.‡
- Such a string realization could also allow us to relate 4d CS to 3d CS via T-duality, as well as to the geometric Langlands correspondence and Gaitsgory-Lurie conjecture via S-duality.
- This would furnish **a novel bridge** between the mathematics of integrable systems, geometric topology, geometric representation theory, and quantum algebras.

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†. E. Witten, *Integrable Lattice Models From Gauge Theory*, arXiv:1611.00592
‡. E. Witten, *Fivebranes and Knots*, Quantum Topology **3** (1) (2012) 1–137

This talk is based on

- M. Ashwinkumar, M.-C. Tan, *Unifying Lattice Models, Links and Quantum Geometric Langlands via Branes in String Theory*, arXiv:1910.01134

Summary of results

	$\tilde{V}$									
	Y		$\mathbb{R}$	Σ		$N\tilde{V}\subset T^*\tilde{V}$				
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D4	$\times$	$\times$	$\times$	$\times$	$\times$					
NS5	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$			

- We begin with this brane configuration in type IIA string theory, where we have a stack of N D4-branes.

- Here, the D4-brane worldvolume is $Y \times \mathbb{R}_+ \times \Sigma$, with boundary conditions determined by an NS5-brane.

- Moreover, the worldvolume theory is partially twisted along $Y \times \mathbb{R}_+$.

- This twisting gives us 4 supercharges that are scalar along V. We take a linear combination of 2 of them, denoted $\mathcal{Q} = \kappa Q + \lambda Q'$ (for $\kappa, \lambda \in \mathbb{C}$), to define our theory.

- We have a $\mathcal{Q}$-invariant action

$$
\begin{aligned}
S =& \{\mathcal{Q}, \widetilde{V}\} \\
&+ \frac{w - \bar{w}}{4} \frac{i\widetilde{\Psi}}{2\pi} \int_{\partial M} dz_w \wedge \mathrm{Tr}\left(\mathcal{A}_w \wedge d\mathcal{A}_w + \frac{2}{3}\mathcal{A}_w \wedge \mathcal{A}_w \wedge \mathcal{A}_w\right),
\end{aligned} \tag{2.1}
$$

  that is $\mathcal{Q}$-exact up to a 4d Chern-Simons action, where $\widetilde{\Psi}$ and w are variable parameters depending on (g_5, κ, λ) and (κ, λ), respectively.

- This action can be written as that of a 1d gauged A-model, with target space the space of all possible $\mathcal{A}_w$ fields, and the 4d Chern-Simons action as superpotential.

- This 1d A-model was shown by Witten[§] to reduce exactly to a path integral over the boundary action, with integration cycle, $\widetilde{\Gamma}$, determined by localization equations.

- In other words, we end up (after a change of variables) with

$$\int_{\widetilde{\Gamma}} D\mathcal{A} \exp\left(\frac{\widetilde{\Psi}\mathrm{Im}(w)}{4\pi} \int_{\partial M} dz \wedge \mathrm{Tr}\left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \right), \tag{2.2}$$

which for $\frac{\widetilde{\Psi}\mathrm{Im}(w)}{2} = \frac{i}{\hbar}$, is the path integral for 4d Chern-Simons theory.

§. A New Look at the Path Integral of Quantum Mechanics, arXiv:1009.6032

- The $\mathcal{Q}$-invariant localization equations can be written as a **gradient flow equation** associated with the 1d model, i.e.,

$$\frac{dx^i}{d\tau} = -g^{i\bar{j}}\frac{\partial \overline{W}}{\partial x^{\bar{j}}} \tag{2.3}$$

for

$$W \sim i\int_{Y\times\Sigma} dz \wedge \mathrm{Tr}\Big(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big), \tag{2.4}$$

where x^i is a coordinate in target space, and $g_{i\bar{j}}$ is its metric.

- Such a gradient flow equation defines an integration cycle $\widetilde{\Gamma}$ for the path integral over (2.2) that ensures its **convergence** because the real part of the action is fixed along the cycle. That is, the path integral for 4d Chern-Simons theory is convergent **for all** $\hbar$.

- The D4-NS5 system we have used is T-dual to Witten's D3-NS5 system that realizes 3d analytically-continued Chern-Simons theory.
- This holds at the level of gauge theory as well, due to scale-invariance of the 5d topological-holomorphic theory along Σ, if we choose $\Sigma = \mathbb{R} \times S^1$.

- The lattice in 4d Chern-Simons theory is furnished by Wilson lines realized by the ends of fundamental strings. Therefore, the partition function of the lattice model is equivalent to link invariants of analytically-continued 3d Chern-Simons theory.
- Replacing $\mathbb{R}_+$ with an interval I that ends on appropriate branes and further shrinking $Y = C$, we obtain a 2d A-model on $\mathbb{R} \times I$ with target Hitchin's moduli space, $\mathcal{M}_H(G, C)$.
- Under IIB S-duality, the effective 2d A-model with $\mathcal{M}_H(G, C)$ as target space is dualized to another A-model, with target $\mathcal{M}_H({}^LG, C)$, i.e., Hitchin's moduli space for the dual gauge group ${}^LG$.
- We are thus able to relate integrable lattice models, link invariants in analytically-continued 3d CS theory, and the **quantum geometric Langlands correspondence** that maps twisted D-modules to twisted D-modules.

- In fact, due to the presence of the lattice of Wilson lines, we have a **generalized version of the quantum geometric Langlands correspondence**, whereby twisted D-modules involve representations of **quantum groups**.
- An additional consequence of type IIB S-duality will turn out to be S-duality of 3d analytically-continued Chern-Simons theory.
- We also discuss how the **Gaitsgory-Lurie conjecture** (that relates representations of $U_q(G)$ to Whittaker D-modules on the affine Grassmannian of ${}^L G$) and its vertex algebra incarnation are realized in our setup.
- Finally, we speculate on how this could lead to the realization of the quantum q-Langlands correspondence via string dualities.

$$
\begin{array}{ccccc}
 & & \mathrm{KL}_{\Psi}(G)/\widehat{\mathfrak{g}}_{\Psi-h^{\vee}}\text{-}mod_C^0 & \xrightarrow{\mathcal{S}'} & \mathrm{Whit}_{\frac{1}{\Psi}}({}^LG)/\mathcal{W}_{\frac{1}{\Psi}}({}^L\mathfrak{g})\text{-}mod_C^0 \\
 & & \uparrow{\scriptstyle \mathcal{T}^{-1}} & & \uparrow{\scriptstyle \mathcal{S}'\mathcal{T}\mathcal{S}'^{-1}} \\
4\mathrm{d}\ \mathrm{CS}_{\widetilde{\Psi}}(G_{\mathbb{C}}) & \xrightarrow{\mathcal{T}\mathrm{T}} & 3\mathrm{d}\ \mathrm{CS}_{\Psi+1}(G_{\mathbb{C}}) & \xrightarrow{\mathcal{S}'} & 3\mathrm{d}\ \mathrm{CS}_{\frac{1}{\Psi+1}}({}^LG_{\mathbb{C}}) \\
 & & \downarrow{\scriptstyle C\to 0} & & \downarrow{\scriptstyle C\to 0} \\
 & & D\text{-}mod_{\Psi+1}^{U_{q'}(G_{\rho^i}),\rho^i}(\mathcal{M}(G,C)) & \xrightarrow{\mathcal{S}'} & D\text{-}mod_{\frac{1}{\Psi+1}}^{U_{{}^Lq'}({}^LG_{{}^L\rho^i}),{}^L\rho^i}(\mathcal{M}({}^LG,C))
\end{array}
$$

A relationship between 4d Chern-Simons theory, 3d S-dual Chern-Simons theories, the quantum group modification of quantum geometric Langlands, and the Gaitsgory-Lurie conjecture. Here, $q' = \exp(\frac{\pi i}{\Psi+1})$.

Now to explain our results

4d Chern-Simons theory from partial twist of D4-NS5 system

D4-brane worldvolume theory with NS5 boundary conditions

The low energy worldvolume theory of N coincident D4-branes on a flat manifold, $\mathcal{M}$, involves fields which transform as reps. of $SO_{\mathcal{M}}(5) \times SO_R(5)$:

$$\begin{aligned} A_M &: (\mathbf{5}, \mathbf{1}) \\ \phi_{\widehat{M}} &: (\mathbf{1}, \mathbf{5}) \\ \rho_{A\widehat{A}} &: (\mathbf{4}, \mathbf{4}) \end{aligned} \tag{4.1}$$

with the classical action of 5d $\mathcal{N} = 2$ SYM:

$$S = -\frac{1}{g_5^2} \int_{\mathcal{M}} d^5x \, \mathrm{Tr} \Big(\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} D_M \phi_{\widehat{M}} D^M \phi^{\widehat{M}} + \frac{1}{4} [\phi_{\widehat{M}}, \phi_{\widehat{N}}][\phi^{\widehat{M}}, \phi^{\widehat{N}}] + i\rho^{A\widehat{A}} (\Gamma^M)_A{}^B D_M \rho_{B\widehat{A}} + \rho^{A\widehat{A}} (\Gamma^{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}} [\phi_{\widehat{M}}, \rho_{A\widehat{B}}] \Big).$$

It is invariant under the SUSY transformations

$$\begin{aligned}\delta A_M =& 2\zeta^{A\widehat{A}}(\Gamma_M)_A{}^B\rho_{B\widehat{A}}\\ \delta\phi^{\widehat{M}} =& -i2\zeta^{A\widehat{A}}(\Gamma^{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}}\rho_{A\widehat{B}}\\ \delta\rho_{A\widehat{A}} =& (\Gamma^M)_A{}^B D_M\phi^{\widehat{M}}(\Gamma_{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}}\zeta_{B\widehat{B}} - \frac{i}{2}(\Gamma_{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}}(\Gamma_{\widehat{N}})_{\widehat{B}\widehat{C}}[\phi^{\widehat{M}},\phi^{\widehat{N}}]\zeta_A{}^{\widehat{C}}\\ & - \frac{i}{2}F^{MN}(\Gamma_{MN})_{AB}\zeta^B_{\widehat{A}}.\end{aligned} \tag{4.2}$$

The stack of D4-branes shall be taken to end on an NS5-brane in the following type IIA brane configuration in flat Euclidean space

	1	2	3	4	5	6	7	8	9	10
D4	$\times$	$\times$	$\times$	$\times$	$\times$					
NS5	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$			

where, e.g., an empty entry under '3' indicates that the brane is located at $x^3 = 0$. The scalar fields $\{\phi_{\widehat{1}}, \phi_{\widehat{2}}, \phi_{\widehat{3}}, \phi_{\widehat{4}}, \phi_{\widehat{5}}\}$ are understood to parametrize the $\{6, 7, 8, 9, 10\}$ directions, respectively.

The NS5-brane provides **boundary conditions** for the D4-brane worldvolume theory.

Partial twist

4d Chern-Simons theory on $Y \times \Sigma$ is topological-holomorphic:

- It has **diffeomorphism invariance** along the 2-manifold denoted Y.
- It has **holomorphic dependence** on the Riemann surface, Σ.

We can obtain it from the D4-NS5 system, where it affords a **partial twist** along the D4 worldvoume that leads to a 4d theory with the above properties at its boundary intersection with the NS5 brane.

To this end, we shall take the D4 worldvolume to be $\mathcal{M} = Y \times \mathbb{R}_+ \times \Sigma$, whereby we wish to realize a topological twist of the D4-brane worldvolume theory along $Y \times \mathbb{R}_+$.

This amounts to redefining the $SO_V(3)$ rotation group of $V = Y \times \mathbb{R}_+$ to be the diagonal subgroup

$$SO_V(3)' \subset SO_V(3) \times SO_R(3),$$

where $SO_R(3) \subset SO_R(5)$ rotates $\{\phi_{\widehat{1}}, \phi_{\widehat{2}}, \phi_{\widehat{3}}\}$.

Specifically, we are studying the following type IIA configuration:

		$\tilde{V}$								
	Y		$\mathbb{R}$	Σ		$N\tilde{V}\subset T^*\tilde{V}$				
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D4	$\times$	$\times$	$\times$	$\times$	$\times$					
NS5	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$			

The twist arises in this configuration because $V \subset \tilde{V} = Y \times \mathbb{R}$, where $\tilde{V}$ is the zero section of the cotangent bundle $T^*\tilde{V}$, and 'coordinates' normal to $\tilde{V}$ in $T^*\tilde{V}$ must be components of one-forms, as we shall obtain via twisting.¶

¶. M. Bershadsky, C. Vafa, V. Sadov, *D-branes and topological field theories*, *Nuclear Physics B* **463** (2-3) (1996) 420-434

Let us now compute the partial twist. Having performed the reductions $SO_{\mathcal{M}}(5) \to SO_V(3) \times SO_\Sigma(2)$ and $SO_R(5) \to SO_R(3) \times SO_R(2)$, we denote the relevant indices as

	$SO_V(3)$	$SO_R(3)$	$SO_\Sigma(2)$	$SO_R(2)$
Vector	$\alpha, \beta, \gamma, \dots$	$\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \dots$	$m, n, p, \dots$	$\widehat{m}, \widehat{n}, \widehat{p}, \dots$
Spinor	$\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \dots$	$\widehat{\bar{\alpha}}, \widehat{\bar{\beta}}, \widehat{\bar{\gamma}}, \dots$	$\bar{m}, \bar{n}, \bar{p}, \dots$	$\widehat{\bar{m}}, \widehat{\bar{n}}, \widehat{\bar{p}}, \dots$

Partial twisting amounts to setting the hatted $SO_R(3)$ indices to unhatted indices.

As a result, the scalar fields $\{\phi_{\widehat{1}}, \phi_{\widehat{2}}, \phi_{\widehat{3}}\}$ now transform as the components $\{\phi_1, \phi_2, \phi_3\}$ of a one-form on $Y \times \mathbb{R}_+$.

In addition, the spinor fields $\rho_{A\widehat{A}} = \rho_{\bar{\alpha}\bar{m}\widehat{\bar{\alpha}}\widehat{\bar{m}}}$ can be expanded after twisting as

$$\rho_{\bar{\alpha}\bar{m}\bar{\beta}\widehat{\bar{m}}} = \epsilon_{\bar{\alpha}\bar{\beta}}\eta_{\bar{m}\widehat{\bar{m}}} + (\sigma^{\alpha})_{\bar{\alpha}\bar{\beta}}\psi_{\alpha\bar{m}\widehat{\bar{m}}}, \tag{4.3}$$

where $\eta_{\bar{m}\widehat{\bar{m}}}$ and $\psi_{\alpha\bar{m}\widehat{\bar{m}}}$ transform as **1** and **3** under $SO_V(3)'$.

Here we have used the antisymmetric matrix $\epsilon_{\bar{\alpha}\bar{\beta}}$ and the symmetric matrix $(\sigma^{\alpha})_{\bar{\alpha}\bar{\gamma}} = (\sigma^{\alpha})_{\bar{\alpha}}{}^{\bar{\beta}}\epsilon_{\bar{\beta}\bar{\gamma}}$, where ϵ is the Levi-Civita symbol and σ^{α} are the Pauli matrices.

Likewise, we can expand the SUSY transformation parameters $\zeta_{A\widehat{A}} = \zeta_{\bar{\alpha}\bar{m}\widehat{\bar{\alpha}}\widehat{\bar{m}}}$ as

$$\zeta_{\bar{\alpha}\bar{m}\bar{\beta}\widehat{\bar{m}}} = \epsilon_{\bar{\alpha}\bar{\beta}}\zeta_{\bar{m}\widehat{\bar{m}}} + (\sigma^{\alpha})_{\bar{\alpha}\bar{\beta}}\zeta_{\alpha\bar{m}\widehat{\bar{m}}}. \tag{4.4}$$

Substituting these expansions into the SUSY transformations, we can obtain the partially twisted SUSY transformations.

However, we wish to pick a supercharge, $\mathcal{Q}$, **that is scalar along** V, w.r.t. which we shall eventually localize the theory.

We shall consider the transformations associated with only ζ_{11} and ζ_{21}, and take a linear combination of the corresponding supercharges to be $\mathcal{Q}$.

This choice will eventually lead to localization equations that define an integration cycle for 4d Chern-Simons theory such that its **path integral is convergent**.

Let $\zeta_{11} = \kappa$ and $\zeta_{21} = \lambda$, where $\kappa, \lambda \in \mathbb{C}$. The supercharge, $\mathcal{Q}$, generates the SUSY transformations

$$\delta A_\alpha = -2i\kappa\psi_{\alpha 22} + 2i\lambda\psi_{\alpha 12} \qquad \delta\eta_{11} = i\kappa\left(F_{45} + \left[\phi_{\hat{4}}, \phi_{\hat{5}}\right] + D_\beta\phi^\beta\right)$$

$$\delta\phi_\alpha = 2\kappa\psi_{\alpha 22} + 2\lambda\psi_{\alpha 12} \qquad \delta\eta_{12} = -i\lambda\left(D_4 - iD_5\right)\left(\phi_{\hat{4}} + i\phi_{\hat{5}}\right)$$

$$\delta A_4 = 2i\kappa\eta_{12} + 2i\lambda\eta_{22} \qquad \delta\eta_{21} = -i\lambda\left(F_{45} - \left[\phi_{\hat{4}}, \phi_{\hat{5}}\right] + D_\beta\phi^\beta\right)$$

$$\delta A_5 = -2\kappa\eta_{12} + 2\lambda\eta_{22} \qquad \delta\eta_{22} = -i\kappa\left(D_4 + iD_5\right)\left(\phi_{\hat{4}} + i\phi_{\hat{5}}\right)$$

$$\delta\phi_{\hat{4}} = 2\kappa\eta_{21} + 2\lambda\eta_{11} \qquad \delta\psi_{\alpha 12} = \kappa\left(\left[\phi_\alpha, \phi_{\hat{4}} + i\phi_{\hat{5}}\right] - iD_\alpha\left(\phi_{\hat{4}} + i\phi_{\hat{5}}\right)\right)$$

$$\delta\phi_{\hat{5}} = 2i\kappa\eta_{21} + 2i\lambda\eta_{11} \qquad \delta\psi_{\alpha 22} = \kappa\left(\left[\phi_\alpha, \phi_{\hat{4}} + i\phi_{\hat{5}}\right] + iD_\alpha\left(\phi_{\hat{4}} + i\phi_{\hat{5}}\right)\right)$$

$$\delta\psi_{\alpha 11} = \kappa\varepsilon_{\alpha\beta\gamma}\left(\frac{i}{2}F^{\beta\gamma} - \frac{i}{2}\left[\phi^\beta, \phi^\gamma\right] - D^\beta\phi^\gamma\right) + \lambda\left(F_{\alpha 4} - iF_{\alpha 5} + i\left(D_4 - iD_5\right)\phi_\alpha\right)$$

$$\delta\psi_{\alpha 21} = \kappa\left(-F_{\alpha 4} - iF_{\alpha 5} + i\left(D_4 + iD_5\right)\phi_\alpha\right) + \lambda\varepsilon_{\alpha\beta\gamma}\left(\frac{i}{2}F^{\beta\gamma} - \frac{i}{2}\left[\phi^\beta, \phi^\gamma\right] + D^\beta\phi^\gamma\right)$$

We perform the following convenient redefinitions:

$$\sigma = \frac{1}{\sqrt{2}}\left(\phi_{\widehat{5}} - i\phi_{\widehat{4}}\right), \quad \bar{\sigma} = \frac{1}{\sqrt{2}}\left(\phi_{\widehat{5}} + i\phi_{\widehat{4}}\right), \tag{4.5}$$

$$\begin{aligned}
\chi_\alpha &= \frac{(1-i)}{2^{5/4}}\psi_{\alpha 11} + \frac{(-1-i)}{2^{5/4}}\psi_{\alpha 21}, & \widetilde{\chi}_\alpha &= \frac{(-1-i)}{2^{5/4}}\psi_{\alpha 11} + \frac{(1-i)}{2^{5/4}}\psi_{\alpha 21}\\
\eta &= \frac{(1+i)}{2^{1/4}}\eta_{11} + \frac{(1-i)}{2^{1/4}}\eta_{21}, & \widetilde{\eta} &= \frac{(-1+i)}{2^{1/4}}\eta_{11} + \frac{(-1-i)}{2^{1/4}}\eta_{21}\\
\psi_\alpha &= \frac{(1+i)}{2^{3/4}}\psi_{\alpha 12} + \frac{(-1+i)}{2^{3/4}}\psi_{\alpha 22}, & \widetilde{\psi}_\alpha &= \frac{(-1+i)}{2^{3/4}}\psi_{\alpha 12} + \frac{(1+i)}{2^{3/4}}\psi_{\alpha 22}\\
\Upsilon &= \frac{(1-i)}{2^{3/4}}\eta_{12} + \frac{(1+i)}{2^{3/4}}\eta_{22}, & \widetilde{\Upsilon} &= \frac{(-1-i)}{2^{3/4}}\eta_{12} + \frac{(-1+i)}{2^{3/4}}\eta_{22},
\end{aligned} \tag{4.6}$$

$$u = \frac{1}{2^{1/4}}\left[(1+i)\kappa + (1-i)\lambda\right], \quad v = \frac{1}{2^{1/4}}\left[(-1+i)\kappa + (-1-i)\lambda\right]. \tag{4.7}$$

The supersymmetry transformations are then (upon rescaling δ)

$$\begin{aligned}
\delta_t A_\alpha &= i\psi_\alpha + it\widetilde{\psi}_\alpha & \delta_t \eta &= t\left(F_{45} + D_\alpha \phi^\alpha\right) + [\bar{\sigma}, \sigma] \\
\delta_t \phi_\alpha &= it\psi_\alpha - i\widetilde{\psi}_\alpha & \delta_t \widetilde{\eta} &= -\left(F_{45} + D_\alpha \phi^\alpha\right) + t\,[\bar{\sigma}, \sigma] \\
\delta_t A_4 &= i\Upsilon + it\widetilde{\Upsilon} & \delta_t \psi_\alpha &= D_\alpha \sigma + t\,[\phi_\alpha, \sigma] \\
\delta_t A_5 &= it\Upsilon - i\widetilde{\Upsilon} & \delta_t \widetilde{\psi}_\alpha &= tD_\alpha \sigma - [\phi_\alpha, \sigma] \\
\delta_t \sigma &= 0 & \delta_t \Upsilon &= D_4 \sigma + tD_5 \sigma \\
\delta_t \bar{\sigma} &= i\eta + it\widetilde{\eta} & \delta_t \widetilde{\Upsilon} &= tD_4 \sigma - D_5 \sigma
\end{aligned} \tag{4.8}$$

$$\delta_t \chi_\alpha = \frac{1}{2}\left[F_{\alpha 4} + D_5 \phi_\alpha + \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\left(F^{\beta\gamma} - \left[\phi^\beta, \phi^\gamma\right]\right)\right] + \frac{1}{2}t\left[F_{\alpha 5} - D_4 \phi_\alpha + \varepsilon_{\alpha\beta\gamma} D^\beta \phi^\gamma\right]$$

$$\delta_t \widetilde{\chi}_\alpha = \frac{1}{2}t\left[F_{\alpha 4} + D_5 \phi_\alpha - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\left(F^{\beta\gamma} - \left[\phi^\beta, \phi^\gamma\right]\right)\right] - \frac{1}{2}\left[F_{\alpha 5} - D_4 \phi_\alpha - \varepsilon_{\alpha\beta\gamma} D^\beta \phi^\gamma\right]$$

so we now have $\mathcal{Q} = \mathcal{Q}_L + t\mathcal{Q}_R$, $t = v/u$. Henceforth, we write $\delta\chi_\alpha = \mathcal{V}_\alpha(t)$ and $\delta\widetilde{\chi}_\alpha = t\widetilde{\mathcal{V}}_\alpha(t)$.

The transformations now take a form very similar to those of GL-twisted $\mathcal{N} = 4$ SYM, as considered by Kapustin and Witten.

In fact, taking $\Sigma = \mathbb{C}^\times$, whereby the x^5 direction is S^1, we can dimensionally reduce along the latter to obtain precisely the transformations of Kapustin and Witten via $A_5 \to \phi_4$, $\chi_\alpha \to \chi^+_{\alpha 4}$, $\widetilde{\chi}_\alpha \to \chi^-_{\alpha 4}$, $\psi_4 \to \Upsilon$, $\widetilde{\psi}_4 \to \widetilde{\Upsilon}$.

To construct an action suitable for localization, we require that it is $\mathcal{Q}$-exact up to some metric-independent term.

To this end we require that the rescaled supersymmetry variation

$$\delta_t = \delta_L + t\delta_R \tag{4.9}$$

is nilpotent up to gauge transformations. This is achieved by introducing auxiliary fields $(H_\alpha, \widetilde{H}_\alpha, P)$ that modify the SUSY variations to

$$\begin{aligned}
\delta_t \chi_\alpha &= H_\alpha & \delta\bar{\sigma} &= i\eta + it\widetilde{\eta} \\
\delta_t \widetilde{\chi}_\alpha &= \widetilde{H}_\alpha & \delta\eta &= tP + [\bar{\sigma}, \sigma] \\
\delta_t H_\alpha &= -i\left(1+t^2\right)[\sigma, \chi_\alpha] & \delta\widetilde{\eta} &= -P + t\,[\bar{\sigma}, \sigma] \\
\delta_t \widetilde{H}_\alpha &= -i\left(1+t^2\right)[\sigma, \widetilde{\chi}_\alpha] & \delta P &= -it[\sigma, \eta] + i\left[\sigma, \widetilde{\eta}\right]
\end{aligned} \tag{4.10}$$

We shall require that our action be invariant under the original transformations on-shell.

As a result, for any field Φ, we have the SUSY algebra

$$\delta_t^2 \Phi = -i(1+t^2)\mathcal{L}_\sigma(\Phi), \tag{4.11}$$

where $\mathcal{L}_\sigma(\Phi)$ is the change in Φ due to a gauge transformation generated by σ, to first order.

We shall define the $\mathcal{Q}$-exact part of our action to be $\delta_t \widetilde{V}$, where $\widetilde{V} = \widetilde{V}_1 + \widetilde{V}_2$.

Here,

$$\widetilde{V}_1 = \frac{2}{{g_5}^2} \int_{\mathcal{M}} d^5x \left(\frac{4}{1+t^2} \right) \mathrm{Tr} \left(\chi_\alpha \left(\frac{1}{2} H^\alpha - \mathcal{V}^\alpha \right) + \widetilde{\chi} \left(\frac{1}{2} \widetilde{H}^\alpha - t \widetilde{\mathcal{V}}^\alpha \right) \right),$$

while

$$\widetilde{V}_2 = -\frac{1}{2t} (\delta_L - t\delta_R) \widetilde{V}_2'$$

with

$$\widetilde{V}_2' = \frac{2}{{g_5}^2} \int_{\mathcal{M}} d^5x \ \mathrm{Tr} \left(-\frac{1}{2} \eta \widetilde{\eta} - i\bar{\sigma} \left(F_{45} + D_\alpha \phi^\alpha \right) \right).$$

The $\mathcal{Q}$-exact action, upon integrating out auxiliary fields, takes the form (suppressing fermions)

$$S_1 = \frac{1}{{g_5}^2}\int_{\mathcal{M}} d^5x \ \mathrm{Tr}\bigg(\frac{-4}{1+t^2}\left(\mathcal{V}^{\alpha}\mathcal{V}_{\alpha} + t^2\widetilde{\mathcal{V}}^{\alpha}\widetilde{\mathcal{V}}_{\alpha}\right) - (F_{45} + D_{\alpha}\phi^{\alpha})^2 - 2D_m\bar{\sigma}D^m\sigma + [\bar{\sigma},\sigma]^2 - 2[\phi_{\alpha},\sigma][\phi^{\alpha},\bar{\sigma}] + 2\partial_{\alpha}(\bar{\sigma}D^{\alpha}\sigma) + \dots\bigg).$$

The first line is just

$$-\frac{1}{{g_5}^2}\int_{\mathcal{M}} d^5x \ \mathrm{Tr}\bigg(F_{\alpha m}F^{\alpha m} + F_{45}F^{45} + \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + D_m\phi_{\alpha}D^m\phi^{\alpha} + D_{\alpha}\phi_{\beta}D^{\alpha}\phi^{\beta} + \frac{1}{2}[\phi_{\alpha},\phi_{\beta}][\phi^{\alpha},\phi^{\beta}] + \partial_{\alpha}\left(\phi^{\alpha}D_{\beta}\phi^{\beta}\right) - \partial_{\gamma}\left(\phi_{\delta}D^{\delta}\phi^{\gamma}\right) + 2\partial_{\alpha}(F_{45}\phi^{\alpha})\bigg) + S_t$$

Apart from the t-dependent term S_t and total derivative terms, we have the standard terms of 5d $\mathcal{N}=2$ SYM (partially twisted).

S_t takes the form

$$S_t = \frac{1}{{g_5}^2}\int_{\mathcal{M}} d^5x\ \varepsilon^{\alpha\beta\gamma}\mathrm{Tr}\left(2\left(\frac{t-t^{-1}}{t+t^{-1}}\right)\left(\frac{1}{2}F_{\alpha 4}F_{\beta\gamma}+\frac{1}{2}\partial_\alpha\left(\phi_\beta D_4\phi_\gamma\right)+\partial_\alpha\left(F_{\beta 5}\phi_\gamma\right)\right)\right.$$
$$\left. -\left(\frac{4}{t+t^{-1}}\right)\left(\frac{1}{2}F_{\alpha 5}F_{\beta\gamma}+\frac{1}{2}\partial_\alpha\left(\phi_\beta D_5\phi_\gamma\right)+\partial_\alpha\left(F_{\beta 4}\phi_\gamma\right)\right)\right). \tag{4.12}$$

We choose to cancel this term by adding $-S_t$ to the action.

Boundary conditions/action

We may obtain the explicit (deformed) NS5 boundary data at the origin of $\mathbb{R}_+$ ($x^3 = 0$) by lifting them from GL-twisted 4d $\mathcal{N} = 4$ SYM. Firstly, we obtain the Dirichlet boundary conditions

$$\phi_3 = 0|_{\partial\mathcal{M}}, \quad \sigma = 0|_{\partial\mathcal{M}}, \quad \overline{\sigma} = 0|_{\partial\mathcal{M}}, \qquad (4.13)$$

whereby the total derivative terms in the $\mathcal{Q}$-exact action are just zero.

The fields $\{\phi_1, \phi_2\}$ and $\{A_1, A_2, A_4\}$ obey generalized Neumann boundary conditions, while A_3 obeys Dirichlet boundary condition.

These conditions are implied by including the boundary action

$$S_{\partial\mathcal{M}} = \frac{1}{{g_5}^2}\int_{\partial M} d^4x \,\mathrm{Tr}\left(\left(t + t^{-1}\right)\left(\frac{1}{2}\varepsilon^{\tilde{\alpha}\tilde{\beta}} D_5\phi_{\tilde{\alpha}}\phi_{\tilde{\beta}}\right) \quad + \left(\frac{t+t^{-1}}{t-t^{-1}}\right)\varepsilon^{ijk}\left(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k\right)\right),$$

where $\tilde{\alpha}, \tilde{\beta} = 1, 2$ and $i, j, k = 1, 2, 4$.

In addition, the boundary conditions on the fermionic fields are projection conditions.

Also, the 4d boundary conditions were shown to imply that $\delta(A_i + w\phi_i) = 0$ for $w = \frac{t-t^{-1}}{2}$. The lift of this to 5d gives

$$\delta(A_{\tilde{\alpha}} + w\phi_{\tilde{\alpha}}) = 0$$

and

$$\delta(A_4 + wA_5) = 0.$$

Finally, the boundary conditions restrict the complex parameter t such that $|t| = 1$.

Localization to 4d Chern-Simons theory

Our total action now takes the form

$$S = \delta_t \widetilde{V} - S_t + S_{\partial\mathcal{M}}. \qquad (4.14)$$

In fact,

$$-S_t + S_{\partial\mathcal{M}} = \frac{w-\bar{w}}{4}\frac{i\widetilde{\Psi}}{2\pi}\int_{\partial M} dz_w \wedge \mathrm{Tr}\left(\mathcal{A}_w \wedge d\mathcal{A}_w + \frac{2}{3}\mathcal{A}_w \wedge \mathcal{A}_w \wedge \mathcal{A}_w\right)$$

where the real parameter

$$\widetilde{\Psi} = \frac{4\pi i}{{g_5}^2}\left(\frac{t-t^{-1}}{t+t^{-1}} - \frac{t+t^{-1}}{t-t^{-1}}\right).$$

Here, we have defined the complex coordinates z_w, $\bar{z}_w$ with corresponding derivatives

$$\partial_{z_w} = \frac{1}{2}(\partial_4 + \overline{w}\partial_5)$$
$$\partial_{\bar{z}_w} = \frac{1}{2}(\partial_4 + w\partial_5), \qquad (4.15)$$

and the complexified gauge fields

$$\mathcal{A}_{w\tilde{\alpha}} = A_{\tilde{\alpha}} + w\phi_{\tilde{\alpha}} \qquad (4.16)$$

(for $\tilde{\alpha} = 1, 2$) and

$$\mathcal{A}_{w\bar{z}_w} = \frac{1}{2}\left(A_4 + wA_5\right) \qquad (4.17)$$

that are $\mathcal{Q}$-invariant along the boundary. Hence, the non-$\mathcal{Q}$-exact 4d CS term **is $\mathcal{Q}$-invariant**, and we have a $\mathcal{Q}$-invariant **5d topological-holomorphic theory.**

In the localization analysis, we exclude $t = \pm 1$ (as $w \neq 0$ is necessary for to obtain 4d CS) and $t = \pm i$ (whereby t can be eliminated from supersymmetry transformations) at the classical level.

Scaling up the $\mathcal{Q}$-exact terms in the action, the path integral localizes to

$$\begin{aligned} \mathcal{V}_\alpha(t) &= 0 \\ \widetilde{\mathcal{V}}_\alpha(t) &= 0 \\ \mathcal{V}_0 &= 0, \end{aligned} \tag{4.18}$$

where $\mathcal{V}_0 = F_{45} + D_\alpha \phi^\alpha$. The remaining localization equations (for σ) are trivial.

In fact, the 5d partially twisted theory can be interpreted as a **1d gauged A-model**, with target space $\mathfrak{A}$, the space of all $\mathcal{A}_w$ fields, and gauge group H, the space of maps from $Y \times \Sigma$ to $U(N)$.

Let

$$\mathcal{A}_\alpha = A_\alpha + i\phi_\alpha, \quad \mathcal{A}_{\bar{z}} = \frac{1}{2}(A_4 + iA_5). \tag{4.19}$$

With the metric

$$g = -\frac{1}{2g_5^2}\int_{Y\times\Sigma} d^2z d^2x \, \mathrm{Tr}(\delta\mathcal{A}^{\widetilde{\alpha}} \otimes \overline{\mathcal{A}}_{\widetilde{\alpha}} + \delta\overline{\mathcal{A}}^{\widetilde{\alpha}} \otimes \mathcal{A}_{\widetilde{\alpha}} + 4\delta A_{\bar{z}} \otimes \delta A_z + 4\delta A_z \otimes \delta A_{\bar{z}}),$$

moment map

$$\mu = -\frac{1}{g_5^2}(D_{\widetilde{\alpha}}\phi^{\widetilde{\alpha}} + F_{45}),$$

and superpotential

$$W = -\frac{e^{i\alpha'}}{g_5^2}\int_{Y\times\Sigma} dz \wedge \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right),$$

the partially-twisted 5d action can be put in the form of the **1d H-gauged A-model**. In particular, the 5d bulk boson action is equivalent to

$$\begin{aligned} S_{1d}^{Bose} = \int d\tau \Big(& g_{i\bar{\jmath}}\partial_\tau^A x^i \partial_\tau^A x^{\bar{\jmath}} + g_{i\bar{\jmath}} V_a^i \widetilde{\sigma}^a \overline{V}_b^{\bar{\jmath}} \overline{\widetilde{\sigma}}^b + g_{i\bar{\jmath}} V_a^i \overline{\widetilde{\sigma}}^a \overline{V}_b^{\bar{\jmath}} \widetilde{\sigma}^b + g_{i\bar{\jmath}} V_a^i \widetilde{\phi}^a \overline{V}_b^{\bar{\jmath}} \widetilde{\phi}^b \\ & - g_{i\bar{\jmath}} F^i \overline{F}^{\bar{\jmath}} + \frac{1}{2} F^i \partial_i W + \frac{1}{2}\overline{F}^{\bar{\jmath}} \partial_{\bar{\jmath}} \overline{W} \Big) \\ - \frac{1}{e^2}\int d\tau \ \mathrm{Tr'} \Big(& D_\tau \widetilde{\phi} D_\tau \widetilde{\phi} + 2 D_\tau \widetilde{\sigma} D_\tau \overline{\widetilde{\sigma}} + [\widetilde{\sigma}, \overline{\widetilde{\sigma}}][\overline{\widetilde{\sigma}}, \widetilde{\sigma}] + 2[\widetilde{\phi}, \widetilde{\sigma}][\widetilde{\phi}, \overline{\widetilde{\sigma}}] \\ & - D^2 + 2e^2 \mu D \Big), \end{aligned} \tag{4.20}$$

where $\partial_\tau^A x^i = \partial_\tau x^i + A_\tau^a V_a^i$. Here, x is a map from $\mathbb{R}_+$ to $\mathfrak{A}$, while V_a, $a = 1, \ldots, \dim\, H$ are the Killing vector fields generating the H-action.

Such a 1d model localizes to its boundary action.[‖]

Hence, our 5d theory is equivalent (after $x^5 \to \mathrm{Im}(w)x^5$) to

$$\int_{\widetilde{\Gamma}} D\mathcal{A} \exp\left(\frac{\widetilde{\Psi}\mathrm{Im}(w)}{4\pi}\int_{\partial M} dz \wedge \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)\right), \tag{4.21}$$

where $\widetilde{\Gamma} \subset \mathfrak{A}$ is defined by solutions of $\mathcal{V}_\alpha = \widetilde{\mathcal{V}}_\alpha = \mathcal{V}_0 = 0$.

We have considered the case that there is no fermion number anomaly, so the partition function is nonvanishing.

‖. E. Witten, A New Look at the Path Integral of Quantum Mechanics, arXiv:1009.6032

For $\frac{\widetilde{\Psi}\mathrm{Im}(w)}{2} = \frac{i}{\hbar}$, this is the path integral for 4d Chern-Simons theory, **defined beyond perturbation theory** with integration cycle $\widetilde{\Gamma}$.

Convergence is ensured by $\widetilde{\Gamma}$ if we tune t to ± 1 by adding irrelevant $\mathcal{Q}$-exact terms to the action prior to localization. To see this, note that $\mathcal{V}_\alpha = 0$ and $\widetilde{\mathcal{V}}_\alpha = 0$ can be rewritten (for any $t \in \mathbb{R}$) via

$$t = \frac{\cos\,\alpha' - 1}{\sin\,\alpha'} \tag{4.22}$$

as

$$\boxed{\begin{aligned} \mathcal{F}_{3\widetilde{\gamma}} &= -e^{-i\alpha'} 2\varepsilon_{\widetilde{\gamma}}^{\ \widetilde{\alpha}} \overline{\mathcal{F}}_{\widetilde{\alpha} z} \\ \mathcal{F}_{3\bar{z}} &= -\frac{1}{4} e^{-i\alpha'} \varepsilon^{\widetilde{\beta}\widetilde{\gamma}} \overline{\mathcal{F}}_{\widetilde{\beta}\widetilde{\gamma}}, \end{aligned}} \tag{4.23}$$

where $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma} = 1, 2$.

These are gradient flow equations; in the gauge $A_3 = 0$ (with $x^3 = \tau$) they are

$$\frac{dx^i}{d\tau} = -g^{i\bar{j}} \frac{\partial \overline{W}}{\partial x^{\bar{j}}}. \tag{4.24}$$

Now, $\mathrm{Im}(W)$ is **conserved** along any gradient flow. For $t = \pm 1$,

$$W = \pm \frac{i}{g_5^2} \int_{Y \times \Sigma} dz \wedge \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \Big). \tag{4.25}$$

Hence, the real part of the argument of the exponent of the 4d CS path integral we derived is conserved along gradient flows. Also, by setting $\mathcal{A} \in \mathrm{Crit}\ W$ at infinity, means that this real part is finite. Thus, the path integral **converges**, since we integrate over the **Lefschetz thimble**, $\widetilde{\Gamma}$.

To obtain lattice, we use **F-strings** ending on D4-brane boundary to realize Wilson lines. The worldlines of the endpoints of these strings realize the desired $\mathcal{Q}$-invariant Wilson lines, given by

$$W = \mathrm{Tr}(P\ e^{\int_L \mathcal{A}_w}), \tag{4.26}$$

where L is a line along $Y \subset \partial\mathcal{M}$.

This means that we may reproduce **R-matrices** and **the YBE with spectral parameter** from a type IIA string theory configuration involving branes and fundamental strings!

Relationship with 3d Chern-Simons Theory and the Geometric Langlands Program

T-duality and 3d Chern-Simons Theory

Recall our brane configuration:

$$\overbrace{\overbrace{1\quad 2}^{Y}\quad \overbrace{3}^{\mathbb{R}}}^{\tilde{V}}\quad \overbrace{4\quad 5}^{\Sigma}\quad \overbrace{6\quad 7\quad 8}^{N\tilde{V}\subset T^*\tilde{V}}\quad 9\quad 10$$

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D4	×	×	×	×	×					
NS5	×	×		×	×	×	×			

Let $\Sigma = \mathbb{R} \times S^1$, with S^1 (parametrized by x^5) having infinitesimal radius. Taking T-duality along this infinitesimal S^1 decompactifies it to $\mathbb{R}$.

In this way, we arrive at the following D3-NS5 configuration

$$\overbrace{\underbrace{\;Y\;}_{}\;\;\mathbb{R}\;\;\mathbb{R}}^{\tilde{V}'}\quad \overbrace{\mathbb{R}\;\;N\tilde{V}\subset T^*\tilde{V}}^{N\tilde{V}'\subset T^*\tilde{V}'}$$

	1	2	3	4	5	6	7	8	9	10
D3	×	×	×	×						
NS5	×	×		×	×	×	×			

(Directions 1–2: Y; directions 3, 4, 5: $\mathbb{R}$ each; directions 6–7: $N\tilde{V}\subset T^*\tilde{V}$; directions 1–4: $\tilde{V}'$; directions 5–7: $N\tilde{V}'\subset T^*\tilde{V}'$.)

This is precisely the system studied by Witten that realizes analytically-continued 3d Chern-Simons theory on $Y \times \mathbb{R}$, with an appropriate integration cycle defined by the 4d localization equations.**

. E. Witten, *Fivebranes and Knots*, Quantum Topology **3 (1) (2012) 1–137

At the gauge theory level, the relationship between our 5d topological-holomorphic theory and the 4d GL-twisted theory studied by Witten follows from **scale invariance** along Σ, which amounts to rescaling of S^1.

Rescaling S^1 to be infinitesimally small, the theory is effectively the dimensionally reduced theory, i.e., the 4d GL-twisted theory.

In this manner, we find a relationship between lattice models realized by 4d Chern-Simons theory and link invariants of analytically-continued 3d Chern-Simons theory.

The Geometric Langlands Program

The geometric Langlands correspondence is realized via 4d GL-twisted SYM on a product of Riemann surfaces, $C \times (I \times \mathbb{R})$.

To relate our 5d theory on $Y \times \mathbb{R}_+ \times \mathbb{R} \times S^1$ to the geometric Langlands program, we first identify Y with C, which we take to be of genus $g > 1$ (following Kapustin-Witten).

Secondly, we replace $\mathbb{R}_+$ by the interval, I, with the boundary condition at infinity replaced by a suitable one at $x^3 = s$ (equivalently, the D4-NS5 system is modified by an additional D-brane).

Upon T-duality, we obtain a modification of Witten's D3-NS5 setup. Using the topological invariance along C to shrink it, we are led to a sigma model on $I \times \mathbb{R}$ with $\mathcal{M}_H(G, C)$ as target.

As before, we may tune $t = \pm 1$ by adding $\mathcal{Q}$-exact terms, implying that the sigma model is an A-model in symplectic structure ω_K of $\mathcal{M}_H(G, C)$.

Now, the NS5-brane boundary condition on one end of the interval descends to a space-filling coisotropic brane, $\mathcal{B}_c$, of the sigma model, of type (B, A, A).

At the other end, the boundary condition picked is that of an (A, B, A) brane, $\mathcal{B}'$, that is Lagrangian with respect to the symplectic structure ω_K.

In this manner, we arrive at a **twisted D-module**. Sheaf of $(\mathcal{B}_c, \mathcal{B}_c)$ strings = sheaf $\mathcal{D}_{K_{\mathcal{M}}^{1/2} \otimes \mathcal{L}^{\Psi}}$ of holomorphic differential operators acting on sections of $K_{\mathcal{M}}^{1/2} \otimes \mathcal{L}^{\Psi}$, where $\mathcal{M} = \mathcal{M}(G, C)$ and where $\mathcal{L}$ is the determinant line bundle on $\mathcal{M}(G, C)$. The $(\mathcal{B}_c, \mathcal{B}')$ strings = sections of a tensor product bundle that includes $K_{\mathcal{M}}^{1/2} \otimes \mathcal{L}^{\Psi}$.

Under type IIB S-duality, the parameter $\Psi \to -\frac{1}{\Psi}$. The parameter t is not relevant, and is considered to be invariant under S-duality.

Thus, the ω_K A-model with target $\mathcal{M}(G, C)$ is dualized to the ω_K A-model with target $\mathcal{M}(^L G, C)$. Moreover, twisted D-modules get mapped to twisted D-modules, realizing the **quantum Geometric Langlands correspondence**, in a manner similar to Kapustin-Witten.

Why do the coisotropic branes map to each other under type IIB S-duality?

Under the $SL(2,\mathbb{Z})$ transformation that maps τ to $\tau+1$, $\theta \to \theta + 2\pi$, and NS5-brane $\to$ (1,1)-brane. Since the effect is just a shift of θ, the sigma model description of the boundary condition as a B_c brane is unaffected.

Under the combination of orientation reversal and S-duality, we have $(1,1) \to (1,-1) \to (1,1)$, implying that the B_c brane is mapped to another B_c brane.

Note that performing an orientation reversal before S-duality also gives $\Psi \to \frac{1}{\Psi}$, as required.

S-duality of 3d Analytically-continued Chern-Simons Theory

Now, we have a duality of the 4d D3-brane worldvolume theory that maps G to ${}^LG$ and (1,1) fivebranes to themselves.

Moreover, both systems localize to 3d analytically-continued Chern-Simons theory at the boundary.

This can be interpreted as an S-duality for 3d analytically-continued Chern-Simons theory, which has been predicted previously by Terashima-Yamazaki and Dimofte-Gukov.

Note that the $SL(2,\mathbb{Z})$ transformation that gave $NS5 \rightarrow (1,1)$ also gives $F1 \rightarrow (1,1)$-string. The latter are Wilson-'t Hooft lines.

G Wilson-'t Hooft lines are thus mapped to ${}^LG$ Wilson-'t Hooft lines under S-duality of 3d analytically-continued CS.

Modification of the QGL Correspondence

The **geometric Langlands correspondence** is actually **generalized** in our case, as follows. In the **integrable lattice models** realized by 4d Chern-Simons theory, we ought to include a network of Wilson lines along $x^3 = 0$ in our setup above.

These Wilson lines (in reps. of the quantum affine algebra) are located at points on $\Sigma = \mathbb{R} \times S^1$, and upon shrinking S^1, they are located on points along $\mathbb{R}$.

As before, mapping $NS5 \to (1,1)$ via an $SL(2,\mathbb{Z})$ transformation, the Wilson lines become Wilson-'t Hooft lines. Upon shrinking Y, they then become local operators located at $x^3 = 0$ and along $\mathbb{R}$, and therefore correspond to points on the coisotropic brane.

The aforementioned twisted D-modules are thus modified to involve data of Wilson-'t Hooft lines (labelled by i), i.e., homomorphisms $\rho^i : U(1) \to G$ ('t Hooft) and representations of the quantum deformation of a subgroup of G (Wilson), which is the commutant of $\rho^i(U(1))$.

Thus, QGL is **modified** in our present setup by reps. of quantum groups.

Gaitsgory-Lurie Conjecture

Correspondence between (i) Kazhdan-Luzstig category, $\mathrm{KL}_\Psi(G)$, of finitely generated modules over the affine Kac-Moody algebra $\widehat{\mathfrak{g}}_{\Psi - h^\vee}$ on which the action of $\mathfrak{g}[[z]]$ integrates to an action of the group $G[[z]]$, and (ii) category of Whittaker D-modules on the affine Grassmannian $\mathrm{Gr}(^LG)$ of ^LG, denoted $\mathrm{Whit}_{\frac{1}{\Psi}}(^LG)$.††

(i) is equivalent to the category of representations of quantum groups $U_q(G)$, where q is related to Ψ via $q = \exp(\frac{\pi i}{\Psi})$, with $\Psi = k + h^\vee$ (where k is the level of the affine Kac-Moody algebra).

††. The Whittaker category Whit_c consists of c-twisted D-modules on Gr_G that are $N((z))$-equivariant with respect to a non-degenerate character, where N is the maximal unipotent subgroup of G.

The deformed NS5-brane boundary condition leads to 3d analytically-continued Chern-Simons theory on the boundary $C \times \mathbb{R}$.

Now, we may deform the integration cycle of 3d analytically-continued Chern-Simons theory to one that parametrizes real gauge fields, and upon making the identification $\Psi = k + h^\vee$, we obtain ordinary 3d Chern-Simons theory, where $k \in \mathbb{Z}_+$ is the level.

Upon doing so, the Wilson lines will be in representations of $U_q(G)$, where $q = \exp(\frac{\pi i}{k+h^\vee})$.

Moreover, the Wilson lines can be taken to lie along $\mathbb{R}$ and on points on C, via topological invariance of Chern-Simons theory. As a result, there are local operators of the corresponding WZW theory on C associated with the quantum group representations, giving rise to the aforementioned finitely generated module over $\widehat{\mathfrak{g}}_k$, denoted $\widehat{\mathfrak{g}}_{\Psi - h^\vee}\text{-}mod^0_C$.

Under an orientation reversal and S-duality that sends $\Psi \to \frac{1}{\Psi}$, the NS5-brane becomes a D5-brane that realizes the maximal Nahm pole boundary condition on the ϕ_1, ϕ_2 and ϕ_4 fields.

Moreover, the Wilson lines realized by fundamental strings become 't Hooft lines realized by D1-branes. The category of these 't Hooft lines is precisely the Whittaker category.[‡‡]

‡‡. D. Gaiotto and E. Frenkel, Quantum Langlands dualities of boundary conditions, D-modules, and conformal block, [arXiv:1805.00203]

Vertex algebra version of the conjecture - the Whittaker category is replaced by a certain subcategory of the category of modules of the affine W-algebra, $\mathcal{W}_{\frac{1}{\Psi}}({}^L\mathfrak{g})$, whose objects arise from "magnetic" vertex operators.

We can understand how this arises physically on C. Firstly, the D5-brane boundary condition was shown by Gaiotto and Witten to have a description in terms of ${}^L\mathfrak{g}$ opers (with singularities due to the 't Hooft lines),* which describe $\mathcal{W}_{\frac{1}{\Psi}}({}^L\mathfrak{g})$ conformal blocks on C in the classical limit.

*. D. Gaiotto, E. Witten, *Knot invariants from four-dimensional gauge theory, Advances in Theoretical and Mathematical Physics* **16** (3) (2012) 935-1086

Furthermore, with 't Hooft line knots/links, braiding along $\mathbb{R}$ of the $\mathbb{R}$-independent *quantum* BPS states of the D3-D5 system was identified with braiding of complex integration cycles for 3d analytically-continued Chern-Simons theory on $C \times I$ with coupling $\frac{1}{\Psi}$.

This in turn is related via deformation of its complex integration cycle to one that parametrizes real gauge fields, as well as the oper boundary condition, to $\mathcal{W}_{\frac{1}{\Psi}}({}^L\mathfrak{g})$ conformal blocks on C.

$$
\begin{array}{ccccc}
 & & \mathrm{KL}_{\Psi}(G)/\widehat{\mathfrak{g}}_{\Psi-h^\vee}\text{-}mod^0_C & \xrightarrow{\mathcal{S}'} & \mathrm{Whit}_{\frac{1}{\Psi}}({}^LG)/\mathcal{W}_{\frac{1}{\Psi}}({}^L\mathfrak{g})\text{-}mod^0_C \\
 & & \uparrow{\scriptstyle \mathcal{T}^{-1}} & & \uparrow{\scriptstyle \mathcal{S}'\mathcal{T}\mathcal{S}'^{-1}} \\
4\mathrm{d}\ \mathrm{CS}_{\widetilde{\Psi}}(G_{\mathbb{C}}) & \xrightarrow{\mathcal{T}\mathrm{T}} & 3\mathrm{d}\ \mathrm{CS}_{\Psi+1}(G_{\mathbb{C}}) & \xrightarrow{\mathcal{S}'} & 3\mathrm{d}\ \mathrm{CS}_{\frac{1}{\Psi+1}}({}^LG_{\mathbb{C}}) \\
 & & \downarrow{\scriptstyle C\to 0} & & \downarrow{\scriptstyle C\to 0} \\
 & & D\text{-}mod^{U_{q'}(G_{\rho^i}),\rho^i}_{\Psi+1}(\mathcal{M}(G,C)) & \xrightarrow{\mathcal{S}'} & D\text{-}mod^{U_{{}^Lq'}({}^LG_{{}^L\rho^i}),{}^L\rho^i}_{\frac{1}{\Psi+1}}(\mathcal{M}({}^LG,C))
\end{array}
$$

A relationship between 4d Chern-Simons theory, 3d S-dual Chern-Simons theories, the quantum group modification of quantum geometric Langlands, and the Gaitsgory-Lurie conjecture. Here, $q' = \exp(\frac{\pi i}{\Psi+1})$.

Quantum q-Langlands Correspondence

The quantum q-Langlands correspondence relates modules of the quantum deformation of the affine algebra, $\widehat{\mathfrak{g}_{\mathbb{C}}}$, and the q-deformed affine W-algebra for ${}^L\mathfrak{g}_{\mathbb{C}}$. We speculate that this could be realized in our setup by T-dualizing both the D3-NS5 and D3-D5 systems.

The former leads to the D4-NS5 system studied in previous sections that gave us 4d Chern-Simons theory on $C \times \mathbb{R} \times S^1$, with Wilson lines in representations of the quantum deformation of $\widehat{\mathfrak{g}_{\mathbb{C}}}$.

On the other hand, T-dualizing the D3-D5 system in an appropriate direction gives us a D4-D6 system, which is also described by Nahm Pole boundary conditions.

We expect that braiding along $\mathbb{R}$ of the $\mathbb{R}$-independent quantum BPS states of the D4-D6 system can be identified with braiding of complex integration cycles for 4d Chern-Simons theory on $C \times I \times S^1$.

This in turn ought to be related via the Nahm pole boundary condition to a 3d boundary theory on $C \times S^1$ that realizes modules of the q-deformed affine W-algebra for ${}^L\mathfrak{g}_{\mathbb{C}}$

Conclusion and Future Directions

- We have shown that integrable lattice models, link invariants, quantum geometric Langlands and the Gaitsgory-Lurie conjecture, are related via dualities in string theory which have field-theoretic incarnations on the respective worldvolumes.

- The crucial ingredient is the fact that the 5d $\mathcal{N} = 2$ SYM theory admits a partial twist that is topological-holomorphic, and analogous to the GL-twist of 4d $\mathcal{N} = 4$ SYM.

- Future work involves defining GL type twists for maximally supersymmetric Yang-Mills theory in 6d and 7d as well, which would lead to 5d and 6d Chern-Simons theories once NS5-type boundary conditions are imposed.

- Moreover, further T-dualities applied to the S-dual D3-D5 system ought to furnish higher analogues of the quantum geometric Langlands correspondence, just as the D4-NS5 and D4-D6 systems ought to realize the quantum q-Langlands correspondence via TST-duality.

- Namely, we expect the quantum q, v-geometric Langlands correspondence involving elliptic affine W-algebras to follow from T-duality to the D5-D7 system, and a further generalization to be furnished by T-duality to the D6-D8 system.

Thank you for your attention!