Surface operators in $\mathcal{N} = 2$ SQCD and Seiberg duality

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Based on

- S. K. Ashok, S. Ballav, M. Frau and R. R. John , *Surface operators in $\mathcal{N} = 2$ SQCD and Seiberg duality*, [arXiv:1901.09630 [hep-th]] Eur. Phys. J. C (2019) 79: 372

- S. K. Ashok, S. Ballav, M. Billo', E. Dell'Aquila, M. Frau, V. Gupta, R. R. John and A. Lerda, *Surface operators, dual quivers and contours*, [arXiv:1807.06316 [hep-th]] Eur. Phys. J. C (2019) 79: 278

Motivations

- Non-local operators in QFT are disturbances supported on submanifolds of space-time that can be used to probe the theory in the bulk.
- They provide us valuable information about the phases, non-perturbative features of the QFT.
- They are classified by the dimension of their support.
  - Line operators(d=1) e.g. Wilson lines: $\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\; L\rangle$
  - Surface operators(d=2): $\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\; S\rangle$
  - Domain walls(d=3)
- In supersymmetric gauge theories, it is possible to compute these exactly using localization methods Nekrasov, Pestun, ...

Goal

- We study half-BPS surface operators in four dimensional $\mathcal{N} = 2$ supersymmetric QCD with gauge group SU(N) and 2N fundamental flavours using equivariant localization and coupled 2d/4d quiver gauge theories.
- We want to understand the precise relationship between the above two descriptions of surface operators.
- This has been done for pure $\mathcal{N} = 2$ gauge theory Sujay, SB et al..
- In addition, we want to understand Seiberg duality in the context of surface operators in $\mathcal{N} = 2$ supersymmetric QCD.

Plan of the talk

- Surface operators as monodromy defects and as flavour defects in pure $\mathcal{N} = 2$ theory. Dictionary between the two approaches
- Seiberg duality in pure $\mathcal{N} = 2$ theory.
- Surface operators in $\mathcal{N} = 2$ supersymmetric QCD.
- Seiberg duality in 2d SQCD
- Generalized Seiberg duality and its application on 3-node quivers.
- Conclusion

Review: monodromy defects in pure $\mathcal{N} = 2$ theory with SU(N) gauge group

Surface operators as monodromy defects

- Surface operators are co-dimension-2 defects in the 4d gauge theory. It is supported on a surface D in $\mathbb{R}^4$.

- In this approach the defect is defined by introducing a singularity structure of the four dimensional gauge field.

- On the plane transverse to the defect D, the gauge field behaves as :
  Gukov, Witten

$$A \sim \text{diag}\,\Big(\underbrace{\alpha_1,\ldots,\alpha_1}_{n_1},\underbrace{\alpha_2,\ldots,\alpha_2}_{n_2},\ldots\underbrace{\alpha_M,\ldots,\alpha_M}_{n_M}\Big)\,d\theta\,,$$

$$\sum_J n_J = N \qquad \sum_J n_J\,\alpha_J = 0\,.$$

- In the path integral, one integrates over all gauge field configurations with this prescribed singular boundary condition.

Surface operators as monodromy defects

- One can also add a phase factor to the path integral :

$$\exp\left(2\pi \mathrm{i} \sum_{l=1}^{M} \eta_l \int_D \mathrm{Tr}\, F_{\mathrm{U}(n_l)}\right)$$

- At the defect, the gauge group $SU(N)$ is broken to a Levi subgroup

$$\mathbb{L} = S[U(n_1) \times U(n_2) \times \ldots \times U(n_M)]$$

- For every partition $N = n_1 + \ldots + n_m$, there is a surface operator.
- A surface operator is specified by :
    - Discrete labels : $[n_1, n_2, \ldots, n_M]$
    - Continous labels : $(\alpha_1, \ldots, \alpha_M)$ and $(\eta_1, \ldots, \eta_M)$

The twisted superpotential

- Our interest is in the low-energy effective action of such theories on the Coulomb branch, in the presence of a surface operator. This effective action is encoded in two holomorphic functions.

- The prepotential ($\mathcal{F}$) describes the effective 4d dynamics.

- The (twisted) superpotential ($\mathcal{W}$) describes the effective 2d/4d dynamics on the 2d defect.

- The instanton contributions to $\mathcal{F}$ and $\mathcal{W}$ are obtained from the instanton partition function $\mathcal{Z}^{\text{inst}}[\vec{n}]$:

$$\lim_{\epsilon_i \to 0} \log(1 + \mathcal{Z}^{\text{inst}}[\vec{n}]) = -\frac{\mathcal{F}^{\text{inst}}}{\epsilon_1 \hat{\epsilon}_2} + \frac{\mathcal{W}^{\text{inst}}}{\epsilon_1}$$

where $\hat{\epsilon}_2 = \frac{\epsilon_2}{M}$, and ϵ_1 and ϵ_2 are the Ω-deformation parameters.

Instanton partition function

- The partition function is calculated using equivariant localization. Kanno-Tachikawa.
- The Ω-deformation parameters $(\epsilon_1, \hat{\epsilon}_2)$ regulate the volume of $\mathbb{R}^4$ and localize the partition function. Nekrasov.
- The 4d Coulomb vevs $\{a_u\}$ also split according to the Levi subgroup:

$$\langle\Phi\rangle = \left\{a_1, \ldots, a_{r_1}\middle| \ldots \middle| a_{r_{I-1}+1}, \ldots a_{r_I}\middle| \ldots \middle| a_{r_{M-1}+1}, \ldots, a_N\right\} .$$

where $r_J = \sum_{I=1}^{J} n_I$.

- The partition function is calculable (order by order in the instanton counting parameter q_I):

$$Z_{\text{inst}}[\vec{n}] = \sum_{\{d_I\}} (q_I)^{d_I} Z_{\{d_I\}}(a_u, \epsilon_1, \epsilon_2)$$

where d_I's the number of ramified instantons.

Instanton partition function

$$Z_{\text{inst}}[\vec{n}] = \sum_{\{d_I\}} (q_I)^{d_I} Z_{\{d_I\}}[\vec{n}] \quad \text{with} \quad Z_{\{d_I\}}[\vec{n}] = \prod_{I=1}^{M} \left[\frac{(-1)^{d_I}}{d_I!} \int \prod_{\sigma=1}^{d_I} \frac{d\chi_{I,\sigma}}{2\pi \mathrm{i}} \right] z_{\{d_I\}}$$

where

$$z_{\{d_I\}} = \prod_{I=1}^{M} \prod_{\sigma,\tau=1}^{d_I} \frac{(\chi_{I,\sigma} - \chi_{I,\tau} + \delta_{\sigma,\tau})}{(\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_1)} \times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{\rho=1}^{d_{I+1}} \frac{(\chi_{I,\sigma} - \chi_{I+1,\rho} + \epsilon_1 + \hat{\epsilon}_2)}{(\chi_{I,\sigma} - \chi_{I+1,\rho} + \hat{\epsilon}_2)}$$
$$\times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \frac{1}{\prod_{s \in \mathcal{N}_I} \left(a_s - \chi_{I,\sigma} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\right) \prod_{t \in \mathcal{N}_{I+1}} \left(\chi_{I,\sigma} - a_t + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\right)} \ .$$

Contour prescription

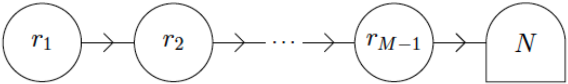
- The ramified instanton partition function is a multi-dimensional contour integral.
- So far we have not been very precise about the contour of integration on the localization side.
- One has to specify the contour of integration to evaluate the integral.
- Which poles contribute to the partition function?
- Assign $\mathrm{Re}(a_u) = 0$ and $\mathrm{Im}(\epsilon_1) >> \mathrm{Im}(\hat{\epsilon}_2) >> 0$. Then, contour amounts to closing in the upper (+) or lower (-) half plane for each set of χ_I.
- An elegant way to fully specify the contour of integration for all variables is using the Jeffrey-Kirwan (JK) residue prescription.

Flavour defects in pure $\mathcal{N} = 2$ theory

- To describe a surface operator of Levi type $\mathbb{L}$ in SU(N) theory, one considers a σ-model with target space: Gukov-Witten, Gadde-Gukov

$$\mathcal{M} = \frac{SU(N)}{\mathbb{L}}$$

- The defect is 1/2-BPS and preserves $(2,2)$ supersymmetry in two dimensions.
- Such sigma models have a gauged linear sigma model (GLSM) description whose gauge and matter content can be summarized in the quiver diagram :Witten '93



- This is a U(r_1)$\times$ U(r_2) $\times \ldots$ gauge theory in 2d with bi-fundamental matter and SU(N) flavour group.
- The ranks $r_J = n_1 + n_2 + \ldots n_J$. The flavour group or global symmetry group of the 2d theory is identified with the 4d gauge group: flavour defects Gaiotto-Gukov-Seiberg

2d/4d quivers: low energy physics

- We integrate out the massive chiral multiplets and write an effective action for the vector multiplets of the 2d theory.

- $(2,2)$ supersymmetry on the defect ensures that the low energy effective action is completely specified by a twisted chiral superpotential $\mathcal{W}(\sigma_s^{(I)})$:

$$\mathcal{W} = 2\pi\mathrm{i}\sum_{I=1}^{M-1}\sum_{s=1}^{r_I}\tau_I\,\sigma_s^{(I)} - \sum_{I=1}^{M-2}\sum_{s=1}^{r_I}\sum_{t=1}^{r_{I+1}}\varpi\left(\sigma_s^{(I)} - \sigma_t^{(I+1)}\right) - \sum_{s=1}^{r_{M-1}}\left\langle \mathrm{Tr}\,\varpi\left(\sigma_s^{(M-1)} - \Phi\right)\right\rangle$$

where

$$\varpi(x) = x\left(\log\frac{x}{\mu} - 1\right),$$

μ is the UV cut-off scale, and τ_I is the complexified FI parameter of the I^{th} node at the scale μ, $\tau_I = \frac{\theta_I}{2\pi} + \mathrm{i}\,\zeta_I$.

Twisted chiral ring equations

- The term in angular bracket corresponds to the double-integral of the resolvent of the 4d gauge theory :

$$\left\langle \mathrm{Tr}\,\frac{1}{\sigma - \Phi} \right\rangle = \sum_{\ell=1}^{\infty} \frac{1}{\sigma^{\ell+1}} \left\langle \mathrm{Tr}\,\Phi^{\ell} \right\rangle$$

- The massive vacua are obtained by extremizing $\mathcal{W}(\sigma_s^{(I)})$:

$$\exp\left(\frac{\partial \mathcal{W}}{\partial \sigma_s^{(I)}}\right) = 1$$

- twisted chiral ring equations.

- Solve the twisted chiral ring equations order by order in the dynamically generated scales $\Lambda_I^{b_I}$ and Λ_{4d} (via the resolvent) to find the massive vacua $\sigma_\star^{(I)}$.

$$b_I = n_I + n_{I+1}\,.$$

The match

- Evaluate the twisted chiral superpotential on the solution :

$$\mathcal{W}(\sigma_*) = \mathcal{W}(a_u, \Lambda_I, \Lambda_{4d})$$

where Λ_I is the strong coupling scale of the I-th 2d gauge node.

- $\mathcal{W}(\sigma_\star^{(I)})$ matches $\mathcal{W}_{\text{inst.}}$ calculated using localization provided :

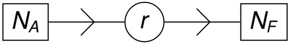
$$q_1 = \Lambda_1^{b_1}\,, \quad q_2 = \Lambda_2^{b_2}\,, \quad \ldots \quad q_{M-1} = \Lambda_{M-1}^{b_{M-1}}\,, \quad q_M = \frac{\Lambda_{4d}^{2N}}{q_1 q_2 \ldots q_{M-1}}\,.$$

Seiberg duality in pure $\mathcal{N} = 2$ gauge theory

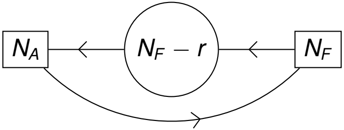
- There is a duality in the two dimensional gauge theory that allows one to write distinct 2d/4d quivers that all have the same infrared behaviour.
- All these quivers provide different realizations of the same surface operator.
- For dual quivers the low energy effective superpotentials, evaluated in particular vacua, match exactly.

Duality: the basic move

- We begin with a 2d U(N) gauge theory with N_f fundamental flavours and N_f anti-fundamental flavours and assume $N_F > N_A$:

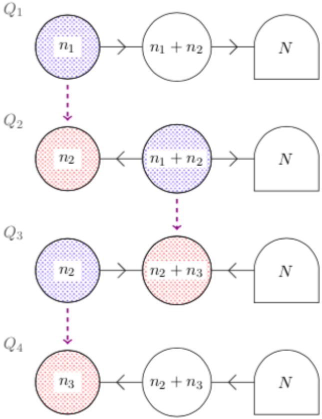


- We now perform a Seiberg duality operation on the 2d gauge node, and obtain the quiver diagram :



- There is an ordinary superpotential induced by the loop in the diagram.

Seiberg duality on 3-node quivers in pure $\mathcal{N} = 2$ theory



Comments

- All these quivers provide different realizations of the same surface operator:
$$SU(N) \longrightarrow S\left[U(n_1) \times U(n_2) \times U(n_3)\right] \ .$$
- $\mathcal{W}$ evaluated in their respective vacua are all equal.
- How do we understand these dual quivers from the localization point of view?
- On the localization side, each Seiberg dual realization of the surface operator is associated to a contour prescription.
- Residue theorems guarantee the equality of the low energy effective superpotentials.
- For pure gauge theory, distinct contour choices are equivalent.

Localization results at one-instanton level

Quiver 1 : Integration contour $(\chi_1|_+, \chi_2|_+, \chi_3|_-)$

$$\mathcal{W}^{Q_1}_{1-\text{inst}} = \sum_{s\in\mathcal{N}_1} \frac{(-1)^{n_1} q_1}{\prod_{r\in\widehat{\mathcal{N}_1}\cup\mathcal{N}_2}(a_s - a_r)} + \sum_{t\in\mathcal{N}_2} \frac{(-1)^{n_2} q_2}{\prod_{r\in\widehat{\mathcal{N}_2}\cup\mathcal{N}_3}(a_t - a_r)} + \sum_{s\in\mathcal{N}_1} \frac{(-1)^{n_3+1} q_3}{\prod_{r\in\mathcal{N}_3\cup\widehat{\mathcal{N}_1}}(a_s - a_r)} .$$

Quiver 2 : Integration contour $(\chi_1|_-, \chi_2|_+, \chi_3|_-)$:

$$\mathcal{W}^{Q_2}_{1-\text{inst}} = \sum_{t\in\mathcal{N}_2} \frac{(-1)^{n_1+1} q_1}{\prod_{r\in\mathcal{N}_1\cup\widehat{\mathcal{N}_2}}(a_t - a_r)} + \sum_{t\in\mathcal{N}_2} \frac{(-1)^{n_2} q_2}{\prod_{r\in\widehat{\mathcal{N}_2}\cup\mathcal{N}_3}(a_t - a_r)} + \sum_{s\in\mathcal{N}_1} \frac{(-1)^{n_3+1} q_3}{\prod_{r\in\mathcal{N}_3\cup\widehat{\mathcal{N}_1}}(a_s - a_r)} .$$

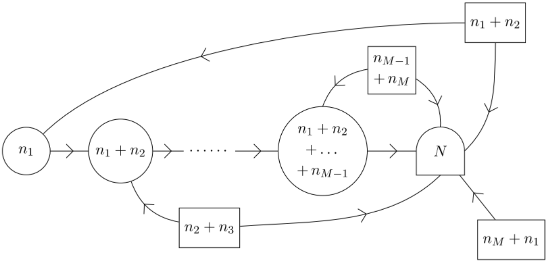
Although there is no term by term match of the superpotentials, the difference vanishes exactly as a consequence of Residue theorem.

Surface operators in $\mathcal{N} = 2$ SQCD

- The main difference with the pure case is that the matter multiplets now provide flavours to the 2d gauge nodes as well.
- A new feature of surface operators in SQCD is that they break the SU(2N) flavour symmetry to the following subgroup:

$$\mathbb{F} = S\left[U(n_1 + n_2) \times U(n_2 + n_3) \times \ldots \times U(n_M + n_1)\right] .$$

- The additional constraint we impose is that for every quiver the complexified FI parameters of the 2d gauge nodes do not run, so that the 2d gauge theories are conformal.



Surface operators in $\mathcal{N} = 2$ SQCD

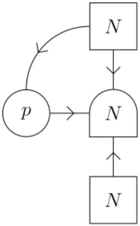
- On the localization side, if we consider the form of Z_{inst} for SQCD, the denominator and its singularity structure is same as that of pure gauge theory. The fundamental flavours only add factors in the numerator of the instanton partition function.

- For a given contour prescription, the set of poles that contribute to the localization integral is identical to those that contribute in the pure gauge theory.

- Therefore, the quiver we may associate to a given integration contour has the same 2d gauge content of the one in the corresponding case without flavour. In particular the ranks of the 2d gauge nodes remain identical.

2-node case: [p, N-p] defect

The quiver Q_0:



- After the massive chiral multiplets are integrated out, the twisted chiral superpotential takes the following form:

$$\mathcal{W}_{Q_0} = \log x \sum_{s\in\mathcal{N}_1} \sigma_s - \sum_{s\in\mathcal{N}_1}\sum_{i\in\mathcal{F}_1} \varpi(m_i - \sigma_s) - \sum_{s\in\mathcal{N}_1} \left\langle \mathrm{Tr}\,\varpi(\sigma_s - \Phi)\right\rangle$$

where x is the exponentiated FI parameter of the 2d theory and the m_i are the masses of the 4d flavours that also act as twisted masses for the 2d chiral multiplets.

2-node case: [p, N-p] defect

- Write down the twisted chiral ring equations.
- Find the vacuum by solving the twisted chiral ring equations.
- Evaluate $\mathcal{W}$ on that particular vacuum.
-

$$\mathcal{W}_{Q_0}(\sigma_\star) = \log x \sum_{s\in\mathcal{N}_1} a_s + (-1)^N x \sum_{s\in\mathcal{N}_1} \frac{B_1(a_s)}{P_1'(a_s)P_2(a_s)} + (-1)^{N+1}\frac{q_0}{x} \sum_{s\in\mathcal{N}_1} \frac{B_2(a_s)}{P_1'(a_s)P_2(a_s)}$$

- It can be easily checked that the 1-instanton terms match the localization result with the $(+-)$ prescription, namely

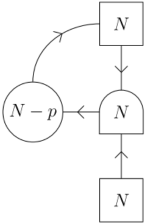
$$\mathcal{W}(\sigma_\star)\Big|_{\text{1-inst}} = \mathcal{W}^{+-}_{\text{1-inst}}\,,$$

  provided we make the following identifications:

$$q_1 = (-1)^{N+p+1}\,x\,, \quad q_2 = (-1)^{p+1}\,\frac{q_0}{x}\,.$$

2-node case: [N-p, p] defect

The quiver Q_1:



- The twisted chiral superpotential takes the following form:

$$\mathcal{W}_{Q_1} = \log y \sum_{s \in \mathcal{N}_2} \sigma_s - \sum_{s \in \mathcal{N}_2} \sum_{i \in \mathcal{F}_1} \varpi(\sigma_s - m_i) - \sum_{s \in \mathcal{N}_2} \left\langle \mathrm{Tr}\, \varpi(\Phi - \sigma_s) \right\rangle$$

2-node case: [N-p, p] defect

- Follow the same prescription and evaluate $\mathcal{W}$ on its vacuum.
-

$$\mathcal{W}_{Q_1}(\sigma_\star) = \log y \sum_{s\in\mathcal{N}_2} a_s + (-1)^{N+1}\frac{1}{y}\sum_{s\in\mathcal{N}_2}\frac{B_1(a_s)}{P_1(a_s)P_2'(a_s)} + (-1)^N q_0\, y \sum_{s\in\mathcal{N}_2}\frac{B_2(a_s)}{P_1(a_s)P_2'(a_s)}$$

- If we now impose that the classical contributions in $\mathcal{W}_{Q_0}$ and $\mathcal{W}_{Q_1}$ match, we find :

$$y = \frac{1}{x}\,.$$

- Using this identification and the (q_1, q_2) vs (q_0, x) map, it can be checked that

$$\mathcal{W}_{Q_1}(\sigma_\star)\Big|_{\text{1-inst}} = \mathcal{W}^{-+}_{\text{1-inst}}$$

Seiberg duality in 2d SQCD theory

- For conformal SQCD, residue theorems include a non-vanishing contribution from infinity : distinct contours are inequivalent.
- For instance, in the 2-nodes case

$$\mathcal{W}_{\text{inst}}^{-+} - \mathcal{W}_{\text{inst}}^{+-} = -\Big[\log(1 + (-1)^p q_1) + \log(1 + (-1)^{N-p} q_2) \Big] \sum_{i \in \mathcal{F}_1} m_i \,.$$

- This has a parallel in the flavour defect :

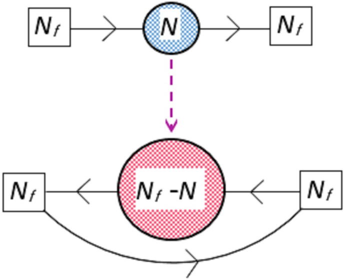
$$\mathcal{W}_{Q_1}(\sigma_\star) - \mathcal{W}_{Q_0}(\sigma_\star) = \left(\log\left(1 - (-1)^N x\right) + \log\left(1 - (-1)^N \frac{q_0}{x}\right)\right) \sum_{i \in \mathcal{F}_1} m_i$$

Seiberg duality in 2d SQCD theory

- We interpret these extra terms as modified Seiberg duality rules that are needed to make the quivers equivalent in the IR.
- The quiver theory that is actually dual to Q_0 (denoted by $\widetilde{Q}_1$), is the one whose superpotential differs from that of Q_1 by non-perturbative corrections according to

$$\mathcal{W}_{\widetilde{Q}_1} = -\log x \sum_{s\in\mathcal{N}_2} \sigma_s - \sum_{i\in\mathcal{F}_1} \sum_{s\in\mathcal{N}_2} \varpi(\sigma_s - m_i) - \sum_{s\in\mathcal{N}_2} \Big\langle \mathrm{Tr}\, \varpi(\Phi - \sigma_s) \Big\rangle$$
$$+ \left(\log\left(1 - (-1)^N x\right) + \log\left(1 - (-1)^N \frac{q_0}{x}\right) \right) \sum_{i\in\mathcal{F}_1} m_i \,.$$

Seiberg duality in 2d SQCD theory: basic rules



Seiberg duality on a 2d conformal gauge node with N_f fundamental and N_f anti-fundamental flavours.

Basic duality rules

- For such a duality move, the exponentiated FI couplings of the pair of dual quivers are related by inversion : $y = 1/x$.

- If the dualized node is only connected to flavour or other 2d gauge nodes, the twisted chiral superpotential of the dual quiver is corrected by a non-perturbative piece given as Benini, Park, Zhao

$$\delta W = \log\left(1 - (-1)^{N_f}\, x\right)\left(\mathrm{Tr}\,\widetilde{m} - \mathrm{Tr}\,m\right) .$$

  where x is the exponentiated FI parameter of the 2d gauge node that is dualized, and $\mathrm{Tr}\,m$ and $\mathrm{Tr}\,\widetilde{m}$ denote respectively the sum of twisted masses for all N_f fundamental and N_f anti-fundamental flavours attached to that node.

- The twisted masses are replaced by the twisted scalars of the vector multiplet in case the flavour is realized by a 2d gauge node.

Generalized Seiberg duality

- We now study Seiberg duality in the 2d/4d quiver realization of the defect in $\mathcal{N} = 2$ SQCD and propose a relation between the twisted superpotentials of dual quivers.

- If the dualized node is connected to the dynamical 4d gauge node, we claim

$$\delta W = \left[\log\left(1 - (-1)^{N_f}\, x\right) + \log\left(1 - (-1)^{N_f} \frac{q_0}{x}\right) \right] \left(\mathrm{Tr}\, \widetilde{m} - \mathrm{Tr}\, m\right)$$

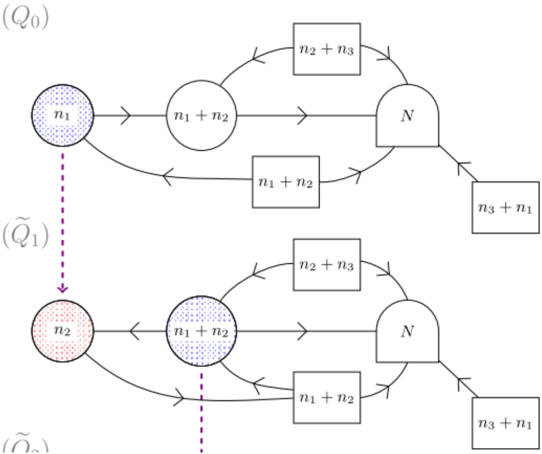
- When dualized node is connected to another 2d gauge node (instead of flavour node) :

$$\mathrm{Tr}\, m \longrightarrow \mathrm{Tr}\, \sigma$$

  This affects the form of the twisted chiral ring equations.

- The nonperturbative terms affect the dynamics in case of a generic quiver.

Case study: 3-node quivers



Case study: 3-node quivers

- The twisted chiral superpotential for the first quiver Q_0 is:

$$\mathcal{W}_{Q_0}(\{x\}) = \log x_1 \operatorname{Tr}\sigma^{(1)} + \log x_2 \operatorname{Tr}\sigma^{(2)} - \sum_{s\in\mathcal{N}_1}\sum_{t\in\mathcal{N}_1\cup\mathcal{N}_2} \varpi(\sigma_s^{(1)} - \sigma_t^{(2)}) - \sum_{s\in\mathcal{N}_1}\sum_{i\in\mathcal{F}_1} \varpi(m_i - \sigma_s^{(1)}) - \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2}\sum_{i\in\mathcal{F}_2} \varpi(m_i - \sigma_s^{(2)}) - \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2} \left\langle \operatorname{Tr}\varpi(\sigma_s^{(2)} - \Phi)\right\rangle.$$

- We now perform a duality on the $U(n_1)$ gauge node in Q_0 to obtain the quiver $\widetilde{Q}_1$ whose twisted superpotential is

$$\mathcal{W}_{\widetilde{Q}_1}(\{x\}) = -\log x_1 \operatorname{Tr}\sigma^{(1)} + \log(x_1 x_2) \operatorname{Tr}\sigma^{(2)} - \sum_{s\in\mathcal{N}_2}\sum_{i\in\mathcal{F}_1} \varpi(\sigma_s^{(1)} - m_i) - \sum_{s\in\mathcal{N}_2}\sum_{t\in\mathcal{N}_1\cup\mathcal{N}_2} \varpi(\sigma_t^{(2)} - \sigma_s^{(1)}) - \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2}\sum_{i\in\mathcal{F}_1\cup\mathcal{F}_2} \varpi(m_i - \sigma_s^{(2)}) - \sum_{s\in\mathcal{N}_1\cup\mathcal{N}_2} \left\langle \operatorname{Tr}\varpi(\sigma_s^{(2)} - \Phi)\right\rangle + \log\left(1 - (-1)^{n_1+n_2} x_1\right)\left(\sum_{i\in\mathcal{F}_1} m_i - \operatorname{Tr}\sigma^{(2)}\right)$$

Comments

- Upon evaluating the respective superpotentials on the resulting solutions of the TCR's, we find a perfect match up to purely q_0-dependent terms.

- We have checked this order by order in instantons for several low rank cases. This agreement is a confirmation of the proposal for 2d Seiberg duality at the level of the low energy effective action.

Conclusion

- Given the localization integrand, one could choose any contour prescription to evaluate the partition function. On the 2d/4d quiver side, this corresponds to choosing a particular quiver Q_k.
- One could then perform a set of Seiberg dualities:

$$\widehat{Q}_0 \overset{\mathcal{D}_1}{\longleftrightarrow} \widehat{Q}_1 \overset{\mathcal{D}_2}{\longleftrightarrow} \widehat{Q}_2 \cdots \longleftrightarrow Q_k \longleftrightarrow \widehat{Q}_{k+1} \longleftrightarrow \cdots$$

- All the others $\widehat{Q}_\ell$ are related to it by Seiberg-dualtiy and their superpotentials would differ from those one would write for the quiver Q_ℓ by non-perturbative pieces determined by the sequence of dualities involved.

Conclusion

- The low energy superpotentials for each quiver in the chain are identical to that obtained for Q_k (up to purely q_0-dependent terms).
- The results match along the rows of dual quiver: these are interpreted as the result of deforming the integration contour from one set of poles to another, keeping into account the residues at infinity.

ありがとうございます