Higher dimensional Virasoro alg's and superconformal symmetry

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Superconformal algebras describe symmetries of certain physical systems in d-dimensional spacetime.

d = 2 Superconformal algebras

- Super enhancements ($N = 1, 2, 4$) of Virasoro Lie algebra.

  Symmetry of states of the superstring.

- $N = 2$ Superconformal index = elliptic genus.

- Symmetry alg of familiar vertex algebras (CDR on CY, Chiral polyvector fields, ...)

d > 2 Superconformal algebras

- Superconformal index still a good invariant.

  (Witten index = $S^3 \times S^1$ partition fn

  Higher dim^l versions)

- All finite dimensional.

②

After a holomorphic twist, $d=4$ superconformal algebras admit ∞-dim'l refinements. (Higher dim'l Virasoro algebras)

- Holomorphic twist: Let $\mathfrak{g}$ = super Lie algebra

Data: An odd element $Q \in \mathfrak{g}$ s.t.

$$[Q, Q] = 0 \in \mathfrak{g}.$$

Given a (dg) rep^n $M \overset{\rho}{\circlearrowleft} \mathfrak{g}$, can twist

$$M \rightsquigarrow (M, \rho(Q))$$

SUSY theory $\rightsquigarrow$ module for the $N=k$ super Lie alg

$$\mathfrak{g}_{N=k} = \mathbb{C}^4 \oplus \Pi \Sigma_{N=k}$$

complexified translations $\{\partial_{x_i}\}_{i=1,\dots,4}$.

Spin(4) - rep.

$$\Sigma_{N=k} = (S_+ \oplus S_-) \otimes \mathbb{C}^k.$$

Fact: For 4d $N=1,2,4$ SUSY $\exists\, Q \in \mathfrak{g}$, $[Q,Q]=0$ so that

$$\mathrm{Im}([Q,-]) \cong \mathbb{C}^2 \subseteq \mathbb{C}^4 .$$

This is called a holomorphic supercharge. Choose complex coord's $(z_1, z_2) \in \mathbb{C}^2$, then for such a Q

$$\frac{\partial}{\partial \bar{z}_i} = [Q, \eta_i], \text{ for some } \eta_i .$$

Theorem [SW]: Let $\mathfrak{L}(N=k)$ be the $d=4$, $N=k$ superconformal Lie algebra. Then $\exists$ embeddings:

$$\underbrace{H^\bullet_Q}_{\text{holomorphic supercharge}}(\mathfrak{L}(N=k)) \hookrightarrow \mathrm{Vect}^{hol}(\underbrace{\mathbb{C}^{2|k-1}}_{\text{graded manifold w/}}) .$$

graded manifold w/ $\mathcal{O}(\mathbb{C}^{2|k-1}) = \mathcal{O}(\mathbb{C}^2)[\underbrace{\varepsilon_1, \dots, \varepsilon_{k-1}}_{\text{odd variables}}]$

Ex: $(\mathfrak{L}(N=2), Q) \simeq \mathfrak{sl}(3|1)$, acts geometrically as:

$$\mathfrak{sl}(3|1) \cong \mathrm{span}\left\{ \frac{\partial}{\partial z_i}, z_i \frac{\partial}{\partial z_j}, z_i \Big(\overbrace{E}^{\text{Euler v.f.}} + \varepsilon \frac{\partial}{\partial \varepsilon}\Big), \frac{\partial}{\partial \varepsilon}, \varepsilon \frac{\partial}{\partial z_i} \right\} .$$

Suggests following generalizations of the $d=2$ super Virasoro algebras, derivations of graded algebra:

$$\mathcal{O}(\mathbb{C}^2 \setminus 0)[\varepsilon_1, \dots, \varepsilon_{k-1}].$$

<u>Problem</u>: $\mathcal{O}(\mathbb{C}^2 \setminus 0) \cong \mathcal{O}(\mathbb{C}^2)$ "Hartog's Lemma"

<u>Better</u> (Kapranov, et. al.) Use a dg model

$$\mathbb{R}\Gamma(\mathbb{C}^2 \setminus 0, \mathcal{O}) =: \underline{\underline{(A_2^\bullet, \bar\partial)}} .$$

cdga

Then, the dg Lie alg of graded vector fields is of the form:

$$\underbrace{A_2^\bullet [\varepsilon_1 \dots \varepsilon_{k-1}]\left\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right\} \oplus A_2^\bullet [\varepsilon_1 \dots \varepsilon_{k-1}]\left\{\frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_{k-1}}\right\}}_{} .$$

$$/\!/ \ \mathrm{Witt}^{N=k}$$

$$\left[\frac{\partial}{\partial \varepsilon_i}, \varepsilon_j\right] = \delta_{ij}$$

$$\left[\frac{\partial}{\partial \varepsilon_i}, A_2^\bullet \frac{\partial}{\partial z_j}\right] = 0$$

- Central extensions

- $k=1$, we just have

$$\mathrm{Witt}^{\bullet} = \mathrm{Der}\left(A_2^{\bullet}\right).$$

Central ext. are parametrized by A, C- cycles in physics

v.f.'s on formal disk

$$H^5\left(W_2 ; \mathbb{C}\right) \cong \mathbb{C}\left\{\widetilde{\varphi_A}, \varphi_C\right\}.$$

Define the following "2-cocycles":

$$\varphi_A\left(\xi_1, \xi_2, \xi_3\right) = \oint_{S^3} \mathrm{Tr}\left(J\xi_1\right) \wedge \partial \mathrm{Tr}\left(J\xi_2\right) \wedge \partial \mathrm{Tr}\left(J\xi_3\right) + \cdots$$

$$\varphi_C\left(\xi_1, \xi_2, \xi_3\right) = \oint_{S^3} \mathrm{Tr}\left(J\xi\right) \wedge \mathrm{Tr}\left(\partial J \xi_2 \wedge \partial J \xi_3\right) + \cdots$$

- $k \geq 2$. Have additional cocycle:

$$\varphi_{vir} \in H^2\left(\mathrm{Witt}^{N=k} ; \mathbb{C}\right)$$

of form:

Residue

$$i = 1,2, \quad \varphi_{vir}^i\left(\xi, \varepsilon \xi'\right) = \oint_{S^3} \mathrm{Tr}\left(J\xi\right) \wedge \partial \mathrm{Tr}\left(J\xi'\right) \wedge dz_i$$

$$\xi, \xi' \in A_2^{\bullet}\left\{\frac{\partial}{\partial z_i}\right\}$$

<u>Def</u> : 1) The higher Virasoro algebra of type $(a, c) \in \mathbb{C} \times \mathbb{C}$ is :

$$\mathbb{C} \longrightarrow \underset{\substack{\uparrow \\ \text{2-cocycle} \quad a\varphi_A + c\varphi_C .}}{\mathrm{Vir}_{a,c}} \longrightarrow \mathrm{Witt}^{\circ}$$

2) The higher (twisted) $N=2$ Virasoro algebra of type (a, c, c_i) is :

$$\mathbb{C} \longrightarrow \underset{\substack{\uparrow \\ \text{2-cocycle} \quad \underbrace{a\varphi_A + b\varphi_B + c_i \varphi^i_{vir}} .}}{\mathrm{Vir}^{N=2}_{(a,c,c_i)}} \longrightarrow \mathrm{Witt}^{N=2}$$

In the second part of the talk, we will see how the $N=2$ virasoro localizes to the ordinary Virasoro Lie algebra.

For now, we indicate some natural representations of the $(N=1)$ higher Virasoro algebra.

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Ex: Let V be a $\mathbb{C}$-vector space. Let $\mathcal{H}_V$ be the following dg Lie alg:

$$\mathbb{C}\cdot\hbar \longrightarrow \mathcal{H}_V \longrightarrow \underbrace{A_2^\bullet \otimes (V \oplus V^*[1])}_{\text{abelian dg Lie alg}}$$

β γ

2-cocycle:

$$(\gamma \otimes v \,,\, \beta \otimes v^*) \longmapsto \langle v, v^* \rangle \oint_{S^3} \gamma \wedge \beta \wedge d^2 z$$

Have (abelian) sub dg Lie alg:

$$\mathcal{H}_{V,+} = A_{2,+}^\bullet \otimes (V \oplus V^*[1]) \oplus \mathbb{C}$$

where

quotient

$$A_{2,+}^\bullet = \ker\left(A_2^\bullet \longrightarrow H^1(A_2)\right)$$

Set:

central term acts by 1.

$$\mathcal{V}_V = \mathrm{Ind}_{\mathcal{H}_{V,+}}^{\mathcal{H}_V}(\mathbb{C}_{\hbar=1}) = U(\mathcal{H}_V) \otimes_{U(\mathcal{H}_{V,+})} \mathbb{C}_{\hbar=1}.$$

☆ This example is motivated by the holomorphic twist of the free $N=1$ chiral multiplet

Virasoro action : Filtration on $\mathcal{U}(\mathcal{H}_V) \rightsquigarrow \mathrm{gr}\,\mathcal{U}(\mathcal{H}_V)$ is Poisson algebra.

By def^n $\mathrm{Witt}^\bullet \circlearrowright A_2^\bullet$, define

$$J : \mathrm{Witt}^\bullet \longrightarrow \mathrm{gr}\,\mathcal{U}(\mathcal{H}_V)$$
$$\xi \longmapsto \oint_{S^3} \langle \beta, \xi\cdot\gamma\rangle_V$$

Check this is map of Lie algebras. Want to lift this to "quantization" $\mathcal{U}(\mathcal{H}_V)$ of $\mathrm{gr}\,\mathcal{U}(\mathcal{H}_V)$

[W]

$$\begin{tikzcd} \mathcal{U}(\mathrm{Vir}_{(a,c)}) \arrow[r, dashed, "\tilde{J}"] \arrow[d] & \mathcal{U}(\mathcal{H}_V) \circlearrowright \mathcal{V}_V \arrow[d] \\ \mathcal{U}(\mathrm{Witt}^\bullet) \arrow[r, "J", "\text{Poisson map}"'] & \mathrm{gr}\,\mathcal{U}(\mathcal{H}_V) \circlearrowright \mathrm{gr}\,\mathcal{V}_V \end{tikzcd}$$

For some combination (a, c) :

$$a\,\varphi_A + c\,\varphi_c = \dim(V) \times \left(\begin{array}{c}\text{"universal} \\ \text{Todd class"}\end{array}\right) \in H^5(W_2).$$

The module $\mathcal{V}_V$ is a "Free field realization" of $\mathrm{Vir}_{(a,c)}$.

- Another example: for $\mathfrak{g}$ any Lie algebra obtain dg Lie alg: $A_2^\bullet \otimes \mathfrak{g}$.

For $\Theta \in \mathrm{Sym}^3(\mathfrak{g})^{\mathfrak{g}}$ have a central extension

$$\mathbb{C} \longrightarrow \hat{\mathfrak{g}}_\Theta \longrightarrow A_2^\bullet \otimes \mathfrak{g}$$

Kapranov et al.: "higher KM algebra".

Have "classical map" of Lie algebras:

$$\mathrm{Witt}^\bullet \xrightarrow{J} \mathrm{gr}\, \mathcal{U}(\hat{\mathfrak{g}}_\Theta)$$

independent of Θ.

Conj: For $\Theta \neq \Theta_{critical} = \#\, \mathrm{Tr}_{adj}(x^3)$

there exists a map of algebras

$$\mathcal{U}\, \mathrm{Vir}_{(a,c)} \xrightarrow{\tilde{J}} \mathcal{U}\, \hat{\mathfrak{g}}_\Theta$$

This is a higher version of the "Segal-Sugawara" construction.

- Superconformal localization

Local enhancement : the Lie algebra of holomorphic vf's on cplx surface X has natural local enhancement

$$\mathrm{Vect}^{hol}(X) \xhookrightarrow{\simeq} \Omega^{0,\bullet}(X, T_X^{1,0}) .$$

Dolbeault resolution

Obtain a factorization algebra on X :

$$\mathcal{V}ir_{(a,c)} := C_\bullet^{\varphi}(\Omega_c^{0,\bullet}(X, T^{1,0}))$$

Twisted Lie alg. homology

$\varphi = a\,\varphi_A + c\,\varphi_c$.

This is analog of "chiral enveloping algebra" of Beilinson-Drinfeld.

(Holomorphic fact alg's on Riem surfaces agree w/ the notions of chiral alg's / vertex alg's. In this case, the construction agrees w/ BD)

Can do same thing for $N=k$ version of Virasoro.

We are most interested in the factorization algebra

$\mathcal{V}ir^{N=2}$.

* **Localization** : Focus on $X = \mathbb{C}^2$. Consider plane :

$$\mathbb{C}_{z_1} \overset{i}{\hookrightarrow} \mathbb{C}^2_{z_1, z_2}.$$

This is a vector field : $z_2 \frac{\partial}{\partial \varepsilon} \in \mathrm{Vect}^{hol}(\mathbb{C}^{2|1})$.

Notice this is a MC element $\rightsquigarrow$ can deform any $\mathrm{Vir}^{N=2}$-module by such an element.

Ex : Higher dim'l KM also has a factorization algebra enhancement :

$\varphi \in \mathrm{Sym}^3(\mathfrak{g})^{\mathfrak{g}} \oplus \mathrm{Sym}^2(\mathfrak{g})^{\mathfrak{g}}$

$$\mathrm{Vir}^{N=2} \circlearrowright \mathcal{Y}^{N=2}_{\mathfrak{g},\varphi} = C^{\varphi}_{\bullet}\left(\Omega^{0,\bullet}_c(X, \mathfrak{g})[\varepsilon]\right)$$

- Have deformation :

Koszul resolution $\mathbb{C}_{z_1} \subseteq \mathbb{C}^2$

$$\left(\mathcal{Y}^{N=2}_{\mathfrak{g},\varphi}, d_{\mathcal{Y}}\right) \rightsquigarrow \left(\mathcal{Y}^{N=2}_{\mathfrak{g},\varphi}, d_{\mathcal{Y}} + z_2 \frac{\partial}{\partial \varepsilon}\right)$$

"Localization" $\qquad \wr\!\wr$

$$\varphi = \underset{S^3(\mathfrak{g})}{\Theta} + \underset{S^2(\mathfrak{g})}{\kappa} \qquad\qquad i_* \mathrm{KM}^{-\kappa/2}_{\mathbb{C}_{z_1}}.$$

where $\mathrm{KM}_{\mathbb{C}_{z_1}}$ is the ordinary KM factorization algebra supported on $\mathbb{C}_{z_1}$.

<u>Thm</u>: [SW] Let $\mathcal{V}ir^{c}_{z_1}$ be the ordinary Virasoro vertex alg thought of as a fact. alg on $\mathbb{C}_{z_1} \overset{i}{\subseteq} \mathbb{C}^2$.

Then, there is an equiv. of fact alg's:

$$\left(\mathcal{V}ir^{N=2}_{\varphi = c\varphi_1^{Vir}} \,,\; \bar{\partial} + \left[z_2 \frac{\partial}{\partial \varepsilon} , - \right] \right)$$

$$\wr\wr$$

$$i_* \mathcal{V}ir^{-12c}_{z_1} .$$

- A similar argument also works at the level of dg Lie algebras.

$$\mathrm{Vir}^{N=2} \rightsquigarrow \left(\mathrm{Vir}^{N=2} \,,\; \left[z_2 \frac{\partial}{\partial \varepsilon} , - \right] \right)$$

$$\wr\wr$$

$$\mathrm{Vir}$$

Usual Virasoro Lie algebra.

Remarks : • Can perform same type of localization for $N=4$ version of higher Virasoro. Get super versions of 2d Virasoro.

• Thm [SW] : Suppose an $N=k$ theory $\mathcal{T}^{N=k}$ has a global gauge symmetry by $\mathfrak{g}$, commuting w/ SUSY. Then, if $\mathcal{T}^{N=k}$ is superconformal

Holomorphic twist

$$\underbrace{\mathrm{Vir}^{N=k}}_{\text{Higher Virasoro}} \ltimes \underbrace{\mathcal{Y}_{\mathfrak{g}}^{N=k}}_{\text{higher KM algebra.}} \circlearrowright \big(\mathcal{T}^{N=2}, \underbrace{Q}_{\text{holomorphic supercharge}}\big)$$

• At the level of holomorphic twist, the superconformal index is a character for $\mathrm{Vir}^{N=k}$.

Eg : 4d $N=1$ SUSY $\rightsquigarrow$ characters are elliptic Γ-functions [SW].

$\parallel$

Partition f^n on Hopf mflds $(\mathbb{C}^2 \setminus 0)/\text{lattice}$.

- Higher <u>vertex structure</u> on $\mathcal{V}ir^{N=k}$, $\mathcal{F}_{\mathfrak{g}}^{N=k}$.

  These factorization algebras define algebras over the (colored) operad of holomorphic 2-disks. $\rightsquigarrow$ higher OPE.

$$\text{Coefficients of holomorphic OPE} \leftrightsquigarrow \text{Higher Virasoro algebras}$$

$$\leftrightsquigarrow \text{holomorphic gravitational anomalies} \leftrightsquigarrow$$

- Holomorphic twist of 6d $N = (2,0)$ theory:

  free theory $\overset{\text{localize}}{\rightsquigarrow}$ $U(1)$-KM of level 1.